A DEGREE PHYSICS

Electricity and Magnetism

C. J. Smith

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A Degree Physics

PART V

ELECTRICITY AND MAGNETISM

By

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' Le temps se venge de ce qui a ete. fait sans lui '

Acc No- 16783

This book is dedicated
to the memory of Joseph Ernest Boyt, Esq., M.A., B.Sc.,
late Headmaster of King Edward VI Grammar School, Stourbridge,
for it was he who first revealed to me the
beauties of mathematical physics.

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PREFACE

In the new edition of this book, which is one of a series to be published under the title 'A Degree Physics', an endeavour is made to give an up-to-date account of the fundamental parts of classical electricity and magnetism which is both experimental and theoretical in its outlook. The ground covered is rather more than that usually required by candidates taking physics at the B.Sc. General Degree. It is wellnigh impossible to give advice concerning those parts which may be omitted or, rather, not committed to memory, since all candidates have different likes and dislikes and, in addition, the syllabus varies at different universities; requirements at some of these necessitate the addition of a chapter on electromagnetic waves.

Throughout the book I have tried to give proofs of all equations that are used; in particular, reference may be made to Laplace's law and the expressions for the forces and couples on rigid circuits. Perhaps candidates at General Degree level should not be required to reproduce the proofs, yet I maintain that in a course of lectures such proofs should be studied. Some of the proofs are not general, i.e. they do not refer to *n*-objects or systems, but are limited to two to reduce the necessary mathematics to a minimum. Throughout the work attention is drawn to the use of equations which are correct dimensionally. I have again made use of a vector notation which may be written as well as printed. Many worked examples have been added since the ability of all students to solve similar problems is of paramount importance.

In dealing with the various parts of the subject I have, perhaps unfortunately, given less attention than in former editions to the historical background. By doing this, room has been made for an account of modern work on the validity of the inverse square law. the structure of high potential electrostatic generators, etc.

The subjects of electrostatics and magnetism are first discussed and these are followed by current electricity. In dealing with direct currents, which some prefer to call currents of zero frequency, methods for solving calculations for meshes are now discussed in some detail. The chapter on thermoelectricity has been considerably enlarged as the subject is no longer of academic interest only;

a brief reference is made to the use of semi-conductors. The chapters on magnetism have been subjected to extensive revision in view of the facts revealed by recent researches.

The subject of alternating currents is of such importance today that I have ventured to treat it yet more fully than is usual in textbooks of this standard. In doing so I have first given a very elementary treatment of the usual R-L-C series circuit ; then follows a more rigorous discussion in which the existence of transients is established and this in turn is followed by a discussion of the vector method for dealing with such problems when all transients have disappeared. In so doing I have made use of a unit vector, for I consider that such treatment keeps the physics of the problem in view and that it is better than the purely mathematical use of complex quantities. Here, too, I have differentiated between an impedance $Z = \sqrt{R^2 + X^2}$ and an impedance operator $\mathbf{Z} = R + jX$; the symbol $\mathbf{Y}$ is now used for an admittance operator. Many more a.c. bridges have been discussed and usually reference has been made to the related vector diagram.

The measurement of power in a.c. circuits is discussed and the structure of a compensated wattmeter is described.

Several new pieces of apparatus are described and I should like to place on record my appreciation of the help I have received in this connexion from A. Grant, Esq., the senior technician in the physics department at this college, and his staff.

It will be found that I have made no use of a certain system of units which finds favour with some, for while I agree that such a system may have advantages in the particular subject of electricity, yet it has no such advantages in other and equally important subjects. I cannot see that for physicists the new system of units has any academic value other than a negative one; now that the ampère has been added to this system, the very idea of a fundamental set of units seems to have been violated, for the ampère can only be defined in terms of the other units.

At the end of each chapter a set of examples, mainly numerical, has been appended. Some of these have been set at University of London Examinations, and I again wish to put on record my appreciation of the ready permission given to me by the Senate to publish them.

In order to help those who have little guidance at hand in their studies, I have given answers to all the numerical questions. My wife has checked all the solutions, and to her I again express my best thanks.

The work now presented to my readers has occupied me for many years ; the process of collecting data has been to me most interesting, and while the labour has been colossal, yet, if the result gives satisfaction to those for whom the book was specially written, I shall be more than content.

Finally, I wish to place on record my appreciation of the help I have received from both past and present colleagues at the Royal Holloway College and from co-examiners. In particular I must mention Dr. D. Owen, J. Nicol, Esq., B.A., B.Sc., and K. A. Macfadyen, Esq., M.Sc., for the great help they have given; also Sir Frank E. Smith, F.R.S., Prof. L. F. Bates, F.R.S., Prof. C. A. Coulson, F.R.S., Dr. F. W. Chapman, Dr. H. J. Goldsmid, Dr. L. Hartshorn, Dr. A. G. Foster, Dr. V. I. Little, Dr. J. G. Powles, Dr. D. H. Smith, Dr. R. W. B. Stephens, Dr. A. R. Stokes, F. M. Colebrook, Esq., B.Sc., A. S. Edmondson, Esq., M.Sc., and D. Orson Wood, Esq., M.Sc. My thanks are also tendered to R. H. Humphry, Esq., M.Sc., who has again read the proofs most diligently. Without his help many obscurities would have remained; his criticisms have always been most friendly and sincere and it is a great joy to acknowledge this help so freely given.

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PART V

ELECTRICITY AND MAGNETISM

CHAPTER I

THE FUNDAMENTAL PRINCIPLES OF ELECTROSTATICS

Qualitative phenomena of electrostatics.—The beginnings of the science of electricity are lost in the dim mists of antiquity. Probably the first known fact emerged when it was discovered that amber, a mineralized yellowish resin, acquired the remarkable property of attracting light objects after it had been rubbed. The Greek word for amber is *ἤλεκτρον*, and it is from this that the word electricity is derived. When amber or any other material is in such a state that it has the property of attracting small light objects, it is said to be electrified. It was not until about 1600 that it was observed that electrified bodies may repel as well as attract. Such observations lead to the conclusion that there are two kinds of electricity produced by frictional rubbing; the type of electricity found on glass when rubbed with silk is now known as ***positive electricity***, while that on the silk is known as ***negative electricity***.

Towards the end of the nineteenth century scientific attention had become somewhat concentrated on the phenomena observed when an electric discharge takes place through a rarefied gas; the tube containing the gas is called a 'Crookes tube' and it was in 1897 that J. J. Thomson, using such a tube, showed that electrified particles emanated from the cathode of a discharge tube. These particles were negatively electrified and travelled with high velocities in straight lines unless they passed through magnetic or electric fields. Each such particle has a mass about 1/1840 part of the mass of the hydrogen atom and the same charge as is carried by a chlorine ion in electrolysis. These particles are now known as ***electrons***. A charge equal to the electronic charge but opposite in sign was found to be carried by a hydrogen ion—the so-called ***proton***—and for many years it was thought that all matter was composed of protons and electrons. In 1932, however, CHADWICK discovered the ***neutron,*** a particle without charge but having a mass almost equal to that of the proton. The same year saw the discovery by ANDERSON of the ***positron*** or ***positive electron.*** The electron, the proton, the neutron and the positron are only some of the fundamental particles found in nature.

The inverse square law in electrostatics.—In 1785 COULOMB discovered the following law: ***the mutual force existing between two point charges varies directly as the product of the charges and inversely as the square of the distance between them.*** Coulomb carried out his experiments with air as the intervening medium and so was not acquainted with the fact that the magnitude of the force depends upon the nature of the intervening medium. This discovery was made later. For point charges, q_1 and q_2 separated by a distance r in air (strictly in a vacuum) Coulomb's law is expressed mathematically by the equation

$$F = \lambda \frac{q_1 q_2}{r^2},$$

where λ is a constant.

The c.g.s. unit of electrostatic charge.—Since there are two kinds of electricity, positive and negative, we must select one of them when defining the unit electrostatic charge. It is natural to select the positive kind and define the unit as follows. ***The unit of electrostatic charge (or quantity of electricity) is that point charge, of the type of electricity found on glass when this is rubbed with silk, which when placed one centimetre away from an equal point charge in a vacuum repels it with a force of one dyne.***

With such a definition as the above, λ becomes unity and we write

$$F = \frac{q_1 q_2}{r^2}.$$

Strictly speaking, this equation only gives us the numerical magnitude of the force for, dimensionally, cf. p. 689, it is incorrect. The correct expression is

$$F = \frac{q_1 q_2}{\kappa_0 r^2},$$

where κ_0 is the relative permittivity of empty space; its numerical value is unity.

The electronic charge e is such that $e = -4{\cdot}8027 \times 10^{-10}$ e.s.u.

Strength of an electric field.—When an electric field is explored by placing a small positive charge at different points in turn, it is observed that the charge experiences a small but definite force at every point; in exploring a field in this way it is tacitly assumed that the configuration of the field is not disturbed by the exploring charge. If $\overrightarrow{\delta F}$ is the force which a charge δq experiences in an electric field at a point P, then $\lim \dfrac{\overrightarrow{\delta F}}{\delta q}$ is known as the ***strength of the electric field*** at P; it is denoted by $\overrightarrow{E}$.

Field strength due to a point charge in a vacuum (air).—Let q be a point charge situated at O, Fig. 1·01; let us calculate the

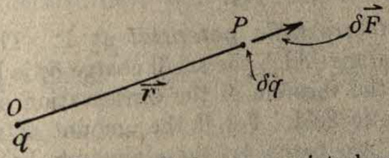


FIG. 1·01.—Electric field strength due to a point charge in a vacuum.

strength of the electric field at a point P at a distance r from O. To do this a small charge δq is placed at P. Then, assuming that the insertion of the charge does not alter the distribution of the charges giving rise to the field, as it may do if the charges are on conductors, the force, $\overrightarrow{\delta F}$, on δq is given in magnitude by the equation

$$\delta F = \frac{q\,\delta q}{r^2}.$$

$$\therefore\ E = \lim \frac{\delta F}{\delta q} = \frac{q}{r^2}.$$

To obtain a more complete expression for the field strength we must write

$$\overrightarrow{\delta F} = \hat{r}\frac{q\,\delta q}{r^2},$$

where $\hat{r}$ is a unit vector at O and along OP. Then

$$\overrightarrow{E} = \hat{r}\frac{q}{r^2}.$$

Again, strictly speaking, if κ_0 is the permittivity of a vacuum,

$$E = \frac{q}{\kappa_0 r^2}, \quad \text{and} \quad \overrightarrow{E} = \hat{r}\frac{q}{\kappa_0 r^2}.$$

Electrostatic potential.—If δW is the work done against the electric field in moving a charge δq from A to B, then

$$\lim \frac{\delta W}{\delta q}$$

is called the increase in electric potential, δV, in passing from A to B; again it is assumed that the field is not disturbed by the small charge. This increase in potential, and hence the potential itself, must be a ***scalar quantity.*** To measure potential we must have a zero of potential. Mathematically this is taken to be at infinity but in practice the earth is taken as a convenient body at zero potential. Thus if δW is the work done against the field in bringing a small charge δq from infinity to a point P in the field, which is assumed to be undisturbed by the small charge, then

$$\lim \frac{\delta W}{\delta q} = V \text{ (say)},$$

is called the ***electrostatic potential*** at P. This is independent of the path along which the small charge δq is brought up from infinity to P if the theorem of the conservation of energy is true for an electrostatic field. For if the amount of work done were different on the two paths, let these amounts be $V\,\delta q$ and $V'\,\delta q$ and let us assume $V > V'$. Then by taking the charge ***from* P *to infinity*** by the former path and bringing it back by the latter, an amount of work $(V - V')\delta q$ would be gained, and this is contrary to the principle of the conservation of energy. Thus V and V′ must be equal and the potential at P is the same no matter by what path we reach P. The potential at P depends only on the position of P.

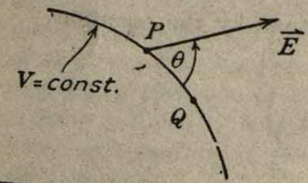


FIG. 1·02.—Equipotential surfaces and lines of force are orthogonal.

A surface in an electric field such that at every point on it the potential has the same value is called an ***equipotential surface.***

To show that equipotential surfaces cut lines of force at right angles or that equipotential surfaces and lines of force are orthogonal.—Let P and Q, Fig. 1·02, be two neighbouring points on an equipotential surface (V = constant). If a charge δq is taken from P to Q, then δW, the work done against the field, is given by

$$\delta W = \delta q \times \text{increase in potential} = \delta q \times 0 = 0.$$

But if $\overrightarrow{E}$ is the field strength at P, the force on δq is given by

$$\overrightarrow{\delta F} = \overrightarrow{E}\,\delta q,$$

so that the work done under the conditions stated is

$$\delta W = (\overrightarrow{\delta F} \cdot \overrightarrow{PQ}) = \delta F(PQ \cos \theta),$$

if θ is the angle between $\vec{E}$ and $\overrightarrow{PQ}$. But it has been shown that $\delta W = 0$.

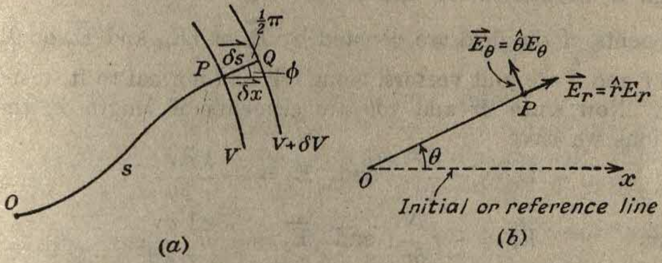
$$\therefore \cos\theta = 0, \quad \text{i.e. } \theta = \tfrac{1}{2}\pi,$$

which establishes the theorem.

The c.g.s. electrostatic unit of potential difference.—If when a small charge δq is taken from A to B, two points in an electrostatic field, the work done against the field is δq erg., then, provided the field is not disturbed, the potential at B exceeds that at A by one c.g.s. electrostatic unit of potential difference.

[N.B.—1 c.g.s. electrostatic unit of potential difference $\equiv$ 300 volts.]

To prove that $E = -\dfrac{\partial V}{\partial s}$ or $\vec{E} = -\hat{s}\dfrac{\partial V}{\partial s}$, where $\dfrac{\partial}{\partial s}$ denotes differentiation along a line of force.—Let P and Q, Fig. 1·03, be two neighbouring points on a line of force, the distances of P and Q from some convenient origin, O, on the line of force being s and $s + \delta s$, respectively. Let V and V + δV be the potentials at P and Q. Let $\vec{E}$ be the electric field strength at P [it should be noted that $\vec{E}$ is in the direction of $\overrightarrow{\delta s}$]. If δq is a small charge at P, the work done against the field in taking this charge from P to Q is $-(\vec{E}\,\delta q)\cdot\overrightarrow{\delta s} = -(E\,\delta s)\delta q$, the negative sign being introduced since the field does work on the charge during the displacement. An alternative expression for the work done against the field is $\delta V(\delta q)$. Hence



$\frac{1}{2}\pi$
$\overrightarrow{\delta s}$
Q
P
ϕ
$\overrightarrow{\delta x}$
V
V+δV
s
O
(a)
$\vec{E}_\theta = \hat{\theta}E_\theta$
$\vec{E}_r = \hat{r}E_r$
P
θ
O
x
Initial or reference line
(b)

FIG. 1·03.—Field strength and potential gradient.

$$-E\,\delta s = \delta V, \quad \text{or} \quad E = -\frac{\partial V}{\partial s},$$

i.e.

$$\vec{E} = -\hat{s}\frac{\partial V}{\partial s},$$

where $\hat{s}$ is a unit vector at P in the direction of $\overrightarrow{\delta s}$.

The quantity $\frac{\partial V}{\partial s}$ is termed the ***potential gradient*** at the point considered; it is independent of the medium.

If $\overrightarrow{\delta x}$ is the distance between the equipotential surfaces which pass through P and Q, and measured in a direction making an angle ϕ with $\overrightarrow{\delta s}$, then the field strength in this direction is given by

$$E_x = E\cos\phi = -\lim\frac{\delta V}{\delta s \div \cos\phi} = -\lim\frac{\delta V}{\delta x} = -\frac{\partial V}{\partial x}.$$

Note on polar coordinates.—Let (r, θ) be the polar coordinates of a point in an electric field, the origin being at O, Fig. 1·03 (*b*) and Ox being the reference line corresponding to $\theta = 0$. If V is the electrostatic potential at P, the strength of the electric field at P, so long as our attention is confined to the plane of the diagram, is completely determined when two components of the field along directions mutually perpendicular are known. For convenience we take one direction along OP and the other along a straight line through P, normal to OP and in the sense of θ increasing. The components of the field are denoted by $\overrightarrow{E_r}$ or $\hat{r}E_r$, and $\overrightarrow{E_\theta}$ or $\hat{\theta}E_\theta$, where $\hat{r}$ and $\hat{\theta}$ are unit vectors, along OP and normal to it, respectively. Now since δr and $r\delta\theta$ are elements of length in these directions we have

$$E_r = -\frac{\partial V}{\partial r}, \quad \text{and} \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta},$$

so that

$$\overrightarrow{E_r} = -\hat{r}\frac{\partial V}{\partial r} \quad \text{and} \quad \overrightarrow{E_\theta} = -\hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta}.$$

Hence

$$\overrightarrow{E} = \overrightarrow{E_r} + \overrightarrow{E_\theta} = -\hat{r}\frac{\partial V}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta}.$$

Field strength when the field is uniform.—An electric field is said to be uniform when $\overrightarrow{E}$ is constant at all points in it, i.e. $\overrightarrow{E}$ is constant in magnitude and direction. If Ox is this direction, then for a uniform field

$$E = -\frac{\partial V}{\partial x} = \text{constant}. \qquad \therefore E\,\delta x = -\delta V.$$

If A and B are two points along Ox such that the coordinates and potentials are x_1, V_1 and x_2, V_2, respectively, integrating the above equation, we get

$$E\int_{x_1}^{x_2} dx = -\int_{V_1}^{V_2} dV.$$

$$\therefore E(x_2 - x_1) = -(V_2 - V_1),$$

or

$$E = \frac{V_1 - V_2}{x_2 - x_1} = \text{the space rate of fall of potential.}$$

This is often written $E = \dfrac{V_1 - V_2}{t}$, where $t = x_2 - x_1$. Such a field is found between the plates of an infinite parallel plate condenser.

Electrostatic potential due to a point charge in air.—Let there be a point charge q at a point O in air (strictly, in a vacuum), cf. Fig. 1·04. Let Q_1 and Q_2 be neighbouring points on the path of a small charge δq which is brought from infinity to a point in the electric field. Let us calculate the electrostatic potential at P due to the point charge q at O, a being the distance OP. The mean value of the field strength along the element of path Q_1Q_2 is $\dfrac{q}{r^2}$, where r is the distance indicated; the field is assumed to be undisturbed by the insertion of δq. [N.B.—q and δq are not related.] Then δW, the work done against the field in moving δq from Q_2 to Q_1, is given by

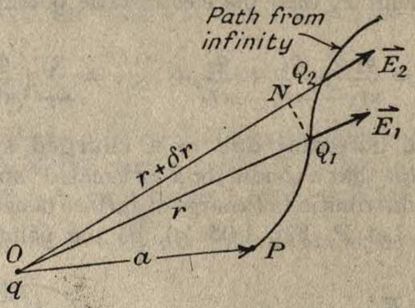


FIG. 1·04.—Electrostatic potential due to a point charge in air.

δW = force on δq × the projection of Q_2Q_1 on the line of action of the force

$$= \left(\frac{q\,\delta q}{r^2}\right)Q_2N = \frac{q\,\delta q}{r^2}(-\delta r), \quad \because\ NQ_2 = \delta r \text{ or } Q_2N = -\delta r.$$

$$\therefore\ \delta V = \lim \frac{\delta W}{\delta q} = -\frac{q}{r^2}\delta r.$$

$$\therefore\ V = -\int_\infty^a \frac{q}{r^2}dr = \frac{q}{a}.$$

This important equation shows that in air (strictly, a vacuum) the electrostatic potential, which is a scalar quantity, depends only upon the charge q and the distance of the point considered from the charge; it is independent of the path followed in bringing the small charge to P. Later on, cf. p. 27, it will be revealed that potential does depend upon the nature of the surrounding medium; the equation then becomes

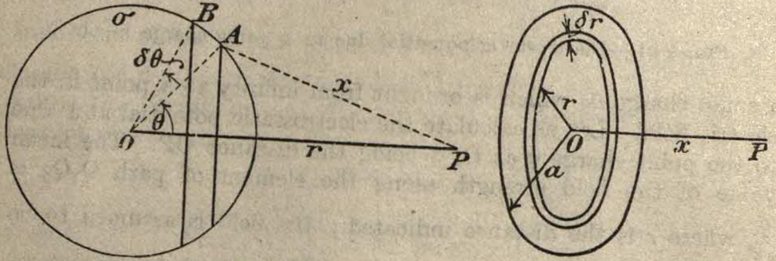
$$[V]_{r=a} = \frac{q}{\kappa a},$$

where κ is the permittivity of the medium.

More generally, if the electric field arises from the presence of point charges, $q_1, q_2, q_3 \ldots$ at distances $r_1, r_2, r_3 \ldots$ respectively from a point P, then the electrostatic potential at P is the scalar sum

$$V = \frac{q_1}{\kappa r_1} + \frac{q_2}{\kappa r_2} + \frac{q_3}{\kappa r_3} + \ldots = \sum\left(\frac{q}{\kappa r}\right).$$

Electrostatic potential due to a charged spherical conductor.—Suppose that a conductor is an isolated* sphere of radius a with a uniform distribution of charge of surface density σ, and it lies in a vacuum. Let P, Fig. 1·05 (a), be the point at which the potential is required. Let O be the centre of the sphere and let $OP = r$. If A and B are two points on the surface such that the angles AOP and BOP are θ and $\theta + \delta\theta$ respectively, the area of the ring traced out by AB when the figure is rotated about OP is $2\pi a \sin\theta\, a\, \delta\theta$. The charge on this is $2\pi a^2 \sigma \sin\theta\, \delta\theta$, and since each point on this ring is at the same distance x from P, the potential at P due to this charge is $2\pi a^2\sigma \sin\theta\, \delta\theta \div x$, where



(a) (b)

FIG. 1·05.—Electrostatic potential in a vacuum due to (a) a charged spherical conductor and (b) a uniformly charged metal disc.

$$x^2 = a^2 + r^2 - 2ar\cos\theta.$$

Hence $x . \delta x = ar \sin\theta . \delta\theta$.

$$\therefore V = \frac{2\pi a\sigma}{r}\int_{x_1}^{x_2} dx,$$

---

* Only by considering an isolated sphere can a uniform distribution of electricity on its surface be obtained.

where x_1 and x_2 are the values of x when θ is 0 and π respectively. For a point outside the sphere $x_1 = (r - a)$, $x_2 = (r + a)$.

$$\therefore V_0 = \frac{2\pi a\sigma}{r}.2a = \frac{4\pi a^2\sigma}{r},$$

or if Q is the total charge on the sphere, $V_0 = \frac{Q}{r}$.

For a point inside the sphere $x_1 = a - r$, $x_2 = a + r$,

$$\therefore V_i = \frac{2\pi a\sigma}{r}.2r = 4\pi a\sigma = \frac{Q}{a},$$

i.e. the potential inside a charged sphere is everywhere constant and equal to that of the sphere itself. This last result is very important for it is true for all closed conductors [cf. p. 39].

Since, in general, $\vec{E} = -\hat{r}\frac{\partial V}{\partial r}$, it follows that at a point outside the sphere $(r > a)$,

$$\vec{E}_{r>a} = -\hat{r}\frac{\partial}{\partial r}\left(\frac{4\pi a^2\sigma}{r}\right) = -\hat{r}\frac{\partial}{\partial r}\left(\frac{Q}{r}\right) = \hat{r}\frac{Q}{r^2}.$$

At points inside the sphere $(r < a)$

$$\vec{E}_{r<a} = -\hat{r}\frac{\partial}{\partial r}\left(\frac{Q}{a}\right) = 0.$$

Thus the field strength is zero at all points within the sphere. Later on (cf. p. 39) it will be shown that this result is true for any closed conductor, provided there is no charge within it.

It will be noticed that the potential and field strength due to a charged spherical conductor are, for points outside the sphere, identical with those which would occur if the total charge were concentrated at the centre of the sphere.

Finally, it should be noted that if κ is the permittivity of the surrounding isotropic medium, then

$$V_0 = \frac{Q}{\kappa r},\ V_i = \frac{Q}{\kappa a},$$

and

$$\vec{E}_{r>a} = \hat{r}\frac{Q}{\kappa r^2}.$$

The electrostatic potential at a point on the axis of a circular metal disc in a vacuum and having a uniform surface charge of density σ.—Let O, Fig. 1·05 (*b*), be the centre of the disc; with radii r and $r + \delta r$ describe two circles on the surface of the disc. The charge on the ring, or annulus, thus formed, is $2\pi\sigma r.\delta r$. Then if OP = x, the potential at P due to this charge is $\dfrac{2\pi\sigma r\,.\,\delta r}{\kappa_0(r^2 + x^2)^{\frac{1}{2}}}$, since all points on the ring are equidistant from P; κ_0 is the permittivity of free space. If a is the radius of the disc, we have

$$V = \frac{2\pi\sigma}{\kappa_0}\int_0^a \frac{r\,dr}{(r^2 + x^2)^{\frac{1}{2}}} = \frac{2\pi\sigma}{\kappa_0}.[(x^2 + a^2)^{\frac{1}{2}} - x].$$

Since the diagram is symmetrical about the x-axis, the resultant electric field strength must be directed along that axis. Hence

$$E = -\frac{\partial V}{\partial x} = -\frac{2\pi\sigma}{\kappa_0}\left[\frac{x}{(x^2 + a^2)^{\frac{1}{2}}} - 1\right].$$

When the disc becomes very large we have $[E]_{a\to\infty} = \dfrac{2\pi\sigma}{\kappa_0}$, i.e. the electric field strength is constant.

In this problem we have assumed that the electricity is confined to one side of the disc. We know that it is on both sides so that the total field strength due to a uniform distribution on a very large metal disc is $\dfrac{4\pi\sigma}{\kappa_0}$ at all external points.

On the potential and field due to a uniform distribution of electric charge bounded by a spherical surface.—For a uniform volume distribution of electricity, of density ρ, throughout a region bounded by a spherical surface, the conditions in the field are symmetrical with respect to the centre, O, of the surface. To calculate the potential at a point P distant r from O, the sphere is divided into small concentric spherical shells each of which is charged uniformly. Two cases must be discussed.

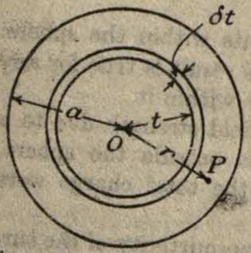


FIG. 1·06.—A uniform distribution of electricity throughout a region bounded by a spherical surface.

(*a*) The potential and field at points outside the spherical surface and in free space with permittivity κ_0 are the same as if the whole charge were concentrated at the centre of the sphere, for this is true for each spherical shell into which the sphere may be divided—cf. p. 9. Thus if a is the radius of the spherical surface and $Q = \frac{4}{3}\pi a^3\rho$, the total charge in the spherical distribution,

$$V_0 = \frac{Q}{\kappa_0 r}, \quad \text{and} \quad \vec{E}_0 = \hat{r}\frac{\partial V}{\partial r} = \hat{r}\frac{Q}{\kappa_0 r^2}.$$

(*b*) When P is within the spherical surface, cf. Fig. 1·06, and the radii of a shell are t and $t + \delta t$, then the point P is an external point for the shells with radii $t < r$, and an internal point when $t > r$. Since the total charge in the shell is $4\pi\rho t^2\,\delta t$, i.e. $\sigma = \rho\,\delta t$, we have

$$\mathrm{V}_i = \int_0^r \frac{4\pi\rho t^2\,dt}{\kappa_0 r} + \int_r^a \frac{4\pi\rho t^2\,dt}{\kappa_0 t}$$

$$= \tfrac{4}{3}\frac{\pi\rho r^2}{\kappa_0} + \frac{4\pi\rho}{\kappa_0}\left[\frac{a^2}{2} - \frac{r^2}{2}\right] = \tfrac{4}{3}\frac{\pi\rho}{\kappa_0}\left[\frac{3a^2}{2} - \frac{r^2}{2}\right].$$

$$\therefore\ \vec{\mathrm{E}}_i = -\hat{r}\frac{\partial \mathrm{V}_i}{\partial r} = \hat{r}\tfrac{4}{3}\frac{\pi\rho r}{\kappa_0} = \hat{r}\frac{\mathrm{Q}r}{\kappa_0 a^3}.$$

This result is due to Laplace (1784).

At the surface of the sphere we have, by writing $r = a$ in the above equations,

$$\left[\vec{\mathrm{E}}_i\right]_{r=a} = \hat{r}\frac{\mathrm{Q}}{\kappa_0 a^2} = \left[\vec{\mathrm{E}}_0\right]_{r=a},$$

and

$$\left[\mathrm{V}_i\right]_{r=a} = \tfrac{4}{3}\frac{\pi\rho a^2}{\kappa_0} = \frac{\mathrm{Q}}{\kappa_0 a} = \left[\mathrm{V}_0\right]_{r=a}.$$

Thus both the field strength and the electrostatic potential are continuous at the surface of the spherical distribution.

These examples demonstrate how, in general, it is very much easier to calculate the potential and derive the field strength rather than to work in a reverse way.

Lines of force in an electric field.—When the vector $\vec{\mathrm{E}}$ is known for all points in an electric field, the field is completely specified. The direction and sense of the vector at any point indicate how a small charge δq would begin to move if put there. By following this direction from point to point curves are obtained and these are termed ***lines of force.*** Thus a line of force in an electric field is a curve such that the tangent at any point on it indicates the direction of the field strength at that point.

The capacitances * of some simple forms of condenser in air.—***(a) An isolated spherical conductor of radius a and situated in free space.***—Let Q be the charge on the conductor. At external points the field is the same as if the charge Q were a point charge at the centre of the sphere. At a point distance r from the centre we therefore have, if κ_0 is the relative permittivity of free space,

$$\mathrm{V} = \frac{\mathrm{Q}}{\kappa_0 r},$$

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* The capacitance of a condenser is defined as the quantity of electricity on the positive plate of the condenser per unit potential difference between its plates [cf. p. 16].

and hence, at the surface of the conductor,

$$V_{r=a} = \frac{Q}{\kappa_0 a}.$$

$$\therefore\ C = \frac{Q}{V_{r=a}} = \kappa_0 a.$$

Numerically, $\kappa_0 = 1$, so that we frequently say that the capacitance of an isolated spherical conductor in free space, is equal to its radius; the statement is only true numerically.

(b) A concentric spherical condenser.—Let a and b $(a < b)$ be the radii of the inner sphere and of the inner surface of the outer sphere; it is assumed that the space between these surfaces is empty. If the outer sphere is earthed and the inner one charged, the charge on the outer sphere lies entirely on its inner surface. Let these charges be Q and Q_1 respectively. If V is the potential of the inner sphere

$$V = \frac{Q}{\kappa_0 a} + \frac{Q_1}{\kappa_0 b},$$

and since the outer sphere is earthed,

$$0 = \frac{Q}{\kappa_0 b} + \frac{Q_1}{\kappa_0 b}.$$

$$\therefore\ Q_1 = -Q.$$

$$\therefore\ V = \frac{Q}{\kappa_0 a} - \frac{Q}{\kappa_0 b},$$

and hence

$$C = \frac{Q}{V} = \frac{\kappa_0 ab}{b - a}.$$

(c) A parallel plate condenser.—When the radii of the above spheres are nearly equal, i.e. $b = a + t$, where $t \rightarrow 0$, the capacitance becomes $\frac{\kappa_0 ab}{t} = \frac{\kappa_0 a(a+t)}{t} = \frac{\kappa_0 a^2}{t}$ (approximately). The capacitance per unit area of the condenser is therefore

$$\frac{\kappa_0 a^2}{t} \div 4\pi a^2 = \frac{\kappa_0}{4\pi t},$$

a result independent of the radii of the spheres provided they are large compared with t.

When the radii of the spheres become very large compared with unit length and we consider unit area of the condenser, we may regard this as a portion of an infinite parallel plate condenser. Hence the capacitance of unit area of an infinite parallel plate condenser is $\frac{\kappa_0}{4\pi t}$; for a parallel plate condenser with plates of area A, the capacitance (in free space) is given by

$$C = \frac{\kappa_0 A}{4\pi t}.$$

This result cannot apply to an actual parallel plate condenser with plates of area A on account of the non-uniformity of the field near the edges of the plates.

If we consider unit area of an infinite parallel plate condenser in a vacuum and having a potential difference across it equal to V, then the charge on unit area of the positive plate is given by

$$\sigma = \left(\frac{\kappa_0}{4\pi t}\right)V.$$

If E is the field strength, necessarily uniform, we have

$$E = \frac{V}{t} = \frac{4\pi\sigma}{\kappa_0}.$$

This important result is obtained here as it will be required in the immediate sequel.

EXAMPLES I

Where necessary it may be assumed that for electrons $e = -4{\cdot}803 \times 10^{-10}$ *e.s.u.*, $\frac{e}{m} = -5{\cdot}28 \times 10^{17}$ *e.s.u. gm.*$^{-1}$, $m = 9{\cdot}11 \times 10^{-28}$ *gm.*; *also, mass of proton* $= 1{\cdot}67 \times 10^{-24}$ *gm.*

1·01. Point charges of electricity equal to + 120, + 60, + 60 and − 120 e.s.u. are placed respectively at the corners of a square ABCD of side 12 cm. If the surrounding medium is air, find (*a*) the field strength at O, the point of intersection of the diagonals, (*b*) the work done in taking an electron from O to E, the mid-point of BC.
[(*a*) 2·63 e.s.u. at $\tan^{-1} 3$ to CA, $-2{\cdot}83 \times 10^{-9}$ erg.]

1·02. Point charges of 1, 2, 3 and 4 e.s.u. are placed respectively at the corners P, Q, R, S of a rectangle in which PQ = 4 cm. and QR = 6 cm. If the medium is air, find the work done in taking a charge of 10^{-3} e.s.u. from the mid-point of QR to (*a*) the mid-point of SP, (*b*) the mid-point of RS. [(*a*) 0, (*b*) $0{\cdot}26 \times 10^{-3}$ erg.]

1·03. Two equal point charges q lie at two opposite corners of a square of side a, and a dipole of moment $2\sqrt{2}\,qa$ is at one of the other corners, its axis pointing towards one of the charges q. Show that the electric field strength at the fourth corner is of magnitude $\sqrt{\frac{17}{2}}\cdot\frac{q}{a^2}$ and that its direction makes an angle $\tan^{-1}\frac{4}{3}$ with the axis of the dipole.

1·04. Two equal point charges q are situated in air at opposite corners A and C of a square ABCD of side a, and an electric dipole of moment $\mu = \sqrt{2}\,qa$ lies at D with its positive charge pointing towards B. Show that the field strength at B is $(1 + \sqrt{2})\dfrac{q}{\kappa_a{}^2}$.

1·05. (a) In the previous problem what charge must be placed at the centre of the square in order that the potential at the corner B shall be zero?

(b) Find the new value of the field strength if the axis of the dipole lies along the side DC of the square.

$$\left[-\left(\frac{1 + 2\sqrt{2}}{2}\right)q,\ \sqrt{\frac{37}{8}}\cdot\frac{q}{\kappa_a{}^2}.\right]$$

1·06. Each side of a thin circular metal disc in air carries a surface charge of uniform density σ. Prove that the potential at a point on the normal to the disc through its centre O and distant x from O is

$$4\pi\sigma[(a^2 + x^2)^{\frac{1}{2}} - x] \div \kappa_a.$$

Hence show that the field strength at the same point is

$$\frac{4\pi\sigma}{\kappa_a}\left[1 - \frac{x}{(a^2 + x^2)^{\frac{1}{2}}}\right].$$

Verify this result by an independent method.

1·07. A linear distribution of electricity, density $\bar{q}$, lies along the Ox axis between the points $(a, 0)$ and $(b, 0)$, where $b > a$. Find the potential and field strength at the point $(0, c)$. Show that the vector $\vec{\mathrm{E}}$ at $(0, c)$ bisects the angle which the line distribution subtends at that point. [Cf. Vol. I, p. 209 *et seq.*]

1·08. A charge of 20 e.s.u. is at the point (0, 0) and a charge − 10 e.s.u. at the point (5, 0) in the xy plane. Calculate the magnitude of the field strength at the points (10, 0), (0, 10), and (3, 4), the coordinate distances being expressed in cm.

Also write down an expression for the potential at the point (x, y) due to the above charges. Evaluate $\hat{\imath}\mathrm{E}_x$ and $\hat{\jmath}\mathrm{E}_y$, and hence verify the results obtained directly in the first part of this problem.

[− 0·2, 0·42, 0·73, e.s.u.]

1·09. Calculate a value for an electron volt, i.e. for the energy acquired by an electron as it moves unhampered through a potential difference of one volt.

The potential difference across an X-ray tube is 90 kV. Calculate the velocity with which an electron, initially at rest at the cathode, will strike the anticathode.

[$1{\cdot}601 \times 10^{-12}$ erg., $1{\cdot}78 \times 10^{10}$ cm.sec.$^{-1}$.]

1·10. Suppose an atom of hydrogen consists of an electron and a proton separated by a distance of 10^{-8} cm. What is the force due to the electrical charges concerned between the electron and the proton if the inverse square law is valid? If the mass of an electron is $9{\cdot}12 \times 10^{-28}$ gm., what value will classical mechanics assign to the frequency of the orbital motion of the electron?

[$2{\cdot}31 \times 10^{-3}$ dyne (attraction), $2{\cdot}5 \times 10^{15}$ sec.$^{-1}$]

1·11. (*a*) A stream of electrons moves at a constant speed of $0{\cdot}50 \times 10^6$ cm.sec.$^{-1}$ between metal plates in a vacuum. If the current is 240 mA., how many electrons are associated with 1 cm. path of the stream ?

(*b*) If an equal number of hydrogen ions also occur in the path and these move with a speed of 300 cm.sec.$^{-1}$ what would be the increase in current ?

[(*a*) 3×10^{12}; (*b*) $0{\cdot}144 \times 10^{-3}$ amp.]

1·12. An electron, charge ε and mass m, is in empty space and free to move in a plane. $Z\varepsilon$ is a positive charge at a fixed point in the above plane. If the only force acting on the electron arises from the field due to the positive charge, prove that the electron may describe a circular path of any chosen radius r, provided that its speed u satisfies the equation

$$mu^2 + \frac{Z\varepsilon e}{\kappa_0 r} = 0,$$

where κ_0 is the permittivity of free space.

1·13. Two equal point charges q are in free space and lie on an axis Ox at the points $(a, 0)$ and $(-a, 0)$ respectively, where O is the point $(0, 0)$. A material particle of mass m and carrying a charge $-q$ revolves in a plane through O normal to Ox, its motion being determined by the attraction due to the charges. If the charge $-q$ describes a path of constant radius r, show that its orbital speed u, is given by

$$mu^2 = \frac{2q^2 r^2}{\kappa_0 (a^2 + r^2)^{1\cdot 5}}$$

CHAPTER II

CAPACITANCE AND THE THEORY OF ISOTROPIC DIELECTRIC MEDIA

The capacitance of a condenser.—The capacitance, C, of a condenser is defined as the quantity of electricity on the positive plate of a condenser per unit potential difference between its plates. Thus if Q is the quantity of electricity on the positive plate of a condenser when V is the potential difference across it,

$$C = \frac{Q}{V}.$$

Experiment shows that the capacitance of a given condenser is a constant and depends only on the size and disposition of the plates when the medium separating them is air (strictly a vacuum).

The insulating medium which sometimes separates the plates of a condenser is known as a ***dielectric*** and simple experiments show that such substances always increase the capacitance of a condenser of fixed size.

Dielectric constant or permittivity.– If the dielectric inserted between the plates of a condenser completely fills the space between them, then experiment shows that the ratio

$$\frac{\text{Capacitance of a condenser with a dielectric}}{\text{Capacitance of the same condenser in air}} = \text{constant } (\kappa).$$

This constant is known as the ***dielectric constant*** or the ***relative permittivity*** of the insulator with reference to air. If (κ) is the relative permittivity of an isotropic material, its actual permittivity is given by

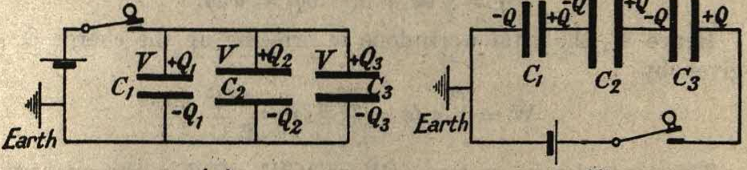
$$\kappa = (\kappa)\kappa_0,$$

where (κ) is a numeric, κ_0 is the permittivity of free space, and $(\kappa_0) = 1$. It is now known that κ is not, strictly speaking, a constant; it varies with temperature and pressure, so that the term permittivity is used to denote this property of an insulator.

Condensers in parallel and in series.—To connect condensers in parallel their positive plates are joined together; likewise their negative plates. If we represent a condenser in the conventional way by two equal and parallel straight lines, then Fig. 2·01 (*a*) shows three condensers in parallel. If C_1, C_2 and C_3 are the capacitances of the condensers, and Q_1, Q_2 and Q_3 the charges on them when they are in parallel, while V is the common potential difference between the plates of each condenser, then

$$V = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_3}{C_3} = \frac{Q_1 + Q_2 + Q_3}{C_1 + C_2 + C_3}.$$

But since $Q_1 + Q_2 + Q_3$ is the total charge on the positive plates of the compound condenser it follows that the capacitance of the latter is given by $C = C_1 + C_2 + C_3$.



(*a*) (*b*)

FIG. 2·01.—Condensers (*a*) in parallel, (*b*) in series.

By referring to Fig. 2·01 (*b*) we see how three condensers are arranged when they are in series. In this instance Q is the numerical value of the charge on each plate. Let V_1, V_2 and V_3 be the potential differences between the plates of the condensers. Then

$$Q = C_1V_1 = C_2V_2 = C_3V_3 = \frac{V_1 + V_2 + V_3}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}.$$

But $V_1 + V_2 + V_3$ is the total potential difference, V, between the extreme plates of the compound condenser. Hence

$$Q = C(V_1 + V_2 + V_3),$$

where C is its effective capacitance, so that

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

Definition.—***When a charge of one e.s.u. on the positive plate of a condenser creates a potential difference of one e.s.u. across the condenser, its capacitance is said to be one e.s.u. of capacitance.***

The potential energy of a charged condenser.—Suppose that when a charge Q is on the positive plate of a condenser of capacitance C, the potential difference across its plates is V. Now the potential energy of a charged condenser is equal to the work done in bringing up a succession of small charges so that the final charge is Q and the final potential difference V. To calculate this work at some instant in the process of charging the condenser let q be the instantaneous charge on the positive plate and v the corresponding potential difference across the plates. There is at the same instant a charge $-q$ on the negative plate which will be assumed to be earthed.

Let a further charge δq be added to the positive plate, when a charge $-\delta q$ will appear simultaneously on the negative plate. The work done against the field in this process is $v\,\delta q$ for the potentials may be considered constant, i.e.

$$\delta W = v\,\delta q + 0(-\delta q) = v\,\delta q.$$

Hence W, the total work done in bringing up the charge Q, is given by

$$W = \int_0^Q v\,dq = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C}.$$

This may also be written $\frac{1}{2}QV$ or $\frac{1}{2}CV^2$. [When the quantities concerned are expressed in e.s.u. the energy is given in ergs.]

Practical unit of capacitance.—The practical unit of capacitance is the ***farad*** and a condenser is said to have a capacitance of one farad if a charge of one coulomb on its positive plate establishes a potential difference of one volt between its plates.

The microfarad, 1 μF, is 10^{-6} farad; the picofarad, 1 pF, is 10^{-12} farad.

On the relation between the e.s.u. of capacitance and the farad.—It will be assumed that one e.s.u. of potential difference is equivalent to 300 volts. Consider a condenser of one farad capacitance with a potential difference of one volt across it. Its energy is

$$W = \tfrac{1}{2}CV^2 = \tfrac{1}{2}.1.1^2 \text{ joule.*} = \tfrac{1}{2} \times 10^7 \text{ erg.}$$

If 1 farad $\equiv (x)$ e.s.u. of capacitance, where (x) is a numeric, the energy of the above condenser is

$$W = \tfrac{1}{2}(x)(\tfrac{1}{300})^2 \text{ erg.}$$

Since the energy is constant, these expressions must be equal, so that

$$(x) = (300)^2 \times 10^7 = 9 \times 10^{11}.$$

$\therefore$ 1 farad $\equiv 9 \times 10^{11}$ e.s.u. of capacitance,

1 μF $\equiv 9 \times 10^5$ „ „ „

and 1 pF $\equiv 0{\cdot}9$ „ „ „

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* The joule is the practical unit of energy.

THE THEORY OF ISOTROPIC DIELECTRICS

Experimental basis for the theory.—Let us suppose that we have two condensers geometrically alike; one is an air condenser but the space between the plates of the other is filled with an insulating material such as sulphur. If both are charged by being connected to the same high potential battery, as suggested in Fig. 2·02, and then discharged in turn through a ballistic galvanometer, the throws of the galvanometer are not equal, but their ratio is constant however the potential difference common to each condenser is varied. The ratio of these throws is equal to the ***relative permittivity*** of the dielectric. It is also equal to the ratio of the capacitance of the condenser with the dielectric to that of the condenser with air (strictly a vacuum). [It is assumed that the plates of the condensers are large and close together in order to render edge effects negligible.]

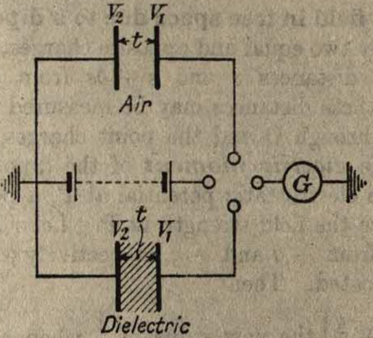

FIG. 2·02.—Geometrically identical condensers with and without a dielectric.

If t is the distance apart of the plates in each instance, V_1 and V_2 the potentials of the plates of the condensers, then

$$\frac{V_1 - V_2}{t}$$

is the same for each. For the air condenser this expression measures the field strength between the plates. What does it measure for the other condenser? Before this can be answered the following digression is necessary.

Electric doublets or electric dipoles.—If two electric charges, equal in magnitude but opposite in sign, coincide at any point the electric field strength due to these charges in the space around them vanishes. If, however, the charges suffer a small relative displacement any small electric charge introduced into the region round them experiences a force, i.e. there is an electric field of sensible magnitude in the neighbourhood of the two charges. Such a combination of electric charges is termed an ***electric dipole.***

Later on it will be found that molecules of such substances as ammonia NH_3, water H_2O, chloroform $CHCl_3$, etc., are permanent dipoles; other molecules such as those of methane CH_4, argon A, and oxygen O_2, are not permanent dipoles; they are said to be ***non-polar.***

The electric field in free space due to a dipole.—Consider the dipole formed by two equal and opposite charges, q and $-q$, placed respectively at distances s and $s + \delta s$ from a fixed point O, Fig. 2·03 (a); these distances may be measured along any smooth curve passing through O and the point charges. Then $q(\delta s) = \mu$ is known as the ***electric moment*** of the dipole. It is required to calculate the electrostatic potential at P, a point in free space, and then deduce the field strength at P. Let r and $r + \delta r$ be the distances of P from $-q$ and $+q$, respectively; θ and $\theta + \delta\theta$ are the angles indicated. Then

$$\begin{aligned} V &= -\frac{q}{\kappa_0}\left(\frac{1}{r}\right) + \frac{q}{\kappa_0}\left[\text{the new value of } \frac{1}{r} \text{ when } s \text{ becomes } s + \delta s\right] \\ &= -\frac{q}{\kappa_0}\left(\frac{1}{r}\right) + \frac{q}{\kappa_0}\left[\frac{1}{r} + \frac{\partial}{\partial s}\left(\frac{1}{r}\right)\delta s\right] \\ &= \frac{q\,\delta s}{\kappa_0}\frac{\partial}{\partial s}\left(\frac{1}{r}\right) = \frac{\mu}{\kappa_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial s} + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)\frac{\partial \theta}{\partial s}\right] \\ &= \frac{\mu}{\kappa_0}\left(-\frac{1}{r^2}\right)\cos(\pi - \theta) + 0 = \frac{\mu\cos\theta}{\kappa_0 r^2}. \end{aligned}$$

This may be written

$$V = \frac{\vec{\mu}\cdot\hat{r}}{\kappa_0 r^2} = \frac{\vec{\mu}\cdot\vec{r}}{\kappa_0 r^3}.$$

For the field components we have, cf. Fig. 2·03 (b),

$$E_r = -\frac{\partial V}{\partial r} = \frac{2\mu\cos\theta}{\kappa_0 r^3},$$

and

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{\mu\sin\theta}{\kappa_0 r^3}.$$

$$\therefore\ \vec{E} = \hat{r}\frac{2\mu\cos\theta}{\kappa_0 r^3} + \hat{\theta}\frac{\mu\sin\theta}{\kappa_0 r^3}.$$

Hence E, the magnitude of the resultant field at P, is given by

$$E = \frac{\mu}{\kappa_0 r^3}[1 + 3\cos^2\theta]^{\frac{1}{2}},$$

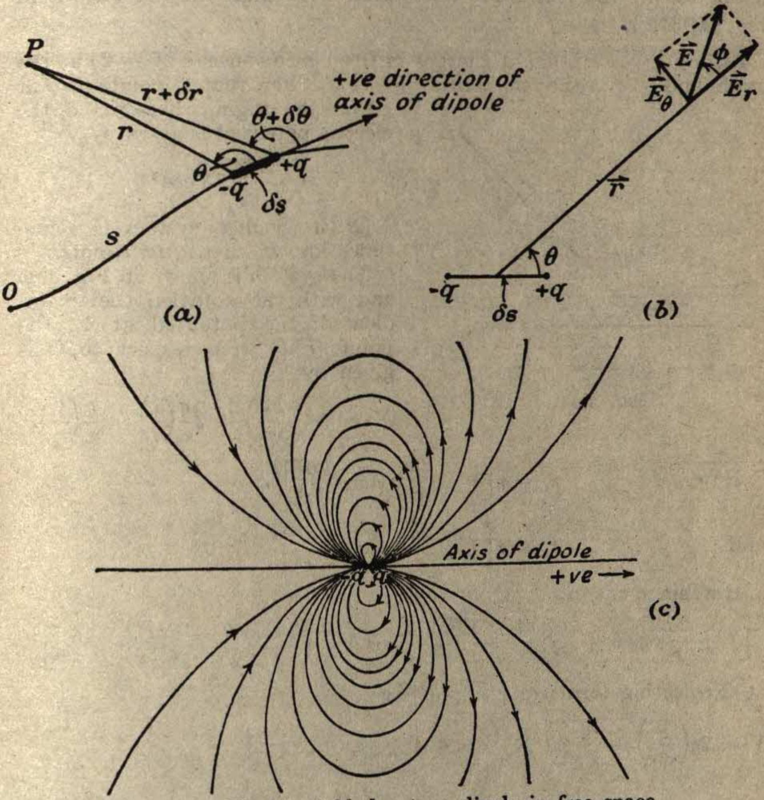

FIG. 2·03.—Electric field due to a dipole in free space.

and if E makes an angle ϕ with E_r,

$$\tan\phi = \frac{E_\theta}{E_r} = \tfrac{1}{2}\tan\theta.$$

To find the equation to the line of force through P, we have to use the well-known fact that $\tan\phi = r\dfrac{d\theta}{dr}$, for ϕ is the angle which the tangent at P makes with the radius vector. Since $\tan\phi = \frac{1}{2}\tan\theta$, we have

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}d\theta,$$

which, on integration, gives

$$\ln r = 2\ln\sin\theta + \ln A,$$

where A is a constant. This equation may be written

$$r = A\sin^2\theta.$$

Some of the lines of force associated with a dipole are shown in Fig. 2·03 (*c*).

Example.—A system of charges in free space consists of $+ 2q$ at the origin and $- q$ at the two points $(\pm a, 0)$. Show that at points distant r from the origin, where $r \gg a$, the electric potential is given by

$$V = -\frac{qa^2}{\kappa_0 r^3}(3\cos^2\theta - 1).$$

[N.B. Such a system is sometimes known as a ***quadrupole.***]

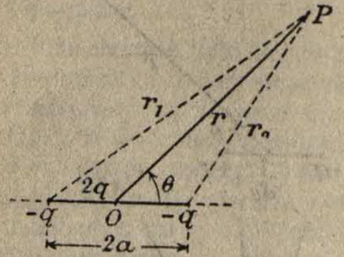

FIG. 2·04.

The system is shown in Fig. 2·04 and with the notation therein indicated, the potential at P, the point (r, θ) with respect to O is given by

$$V = -\frac{q}{\kappa_0}\left(\frac{1}{r_1}\right) + 2\frac{q}{\kappa_0}\left(\frac{1}{r}\right) - \frac{q}{\kappa_0}\left(\frac{1}{r_2}\right).$$

Now
$$r_1 = r\left[1 + \frac{2a}{r}\cos\theta + \frac{a^2}{r^2}\right]^{\frac{1}{2}},$$

and
$$r_2 = r\left[1 - \frac{2a}{r}\cos\theta + \frac{a^2}{r^2}\right]^{\frac{1}{2}}.$$

But, since $r \gg a$,

$$\left[1 \pm \frac{2a}{r}\cos\theta + \frac{a^2}{r^2}\right]^{-\frac{1}{2}} = \left[1 \mp \frac{a}{r}\cos\theta + \left(\frac{3\cos^2\theta - 1}{2}\right)\frac{a^2}{r^2} \mp \ldots\right]$$

∴ Neglecting terms in r^{-4}, we have

$$\begin{aligned} V = 2q\left(\frac{1}{\kappa_0 r}\right) - \frac{q}{\kappa_0 r}\Big[1 - \frac{a}{r}\cos\theta + \tfrac{3}{2}(\cos^2\theta - 1)\frac{a^2}{r^2} \\ + 1 + \frac{a}{r}\cos\theta + \tfrac{3}{2}(\cos^2\theta - 1)\frac{a^2}{r^2}\Big] \\ = -\frac{qa^2}{\kappa_0 r^3}(3\cos^2\theta - 1). \end{aligned}$$

This result may be obtained by regarding the quadrupole as two dipoles of moment μ and $- \mu$ respectively, with centres $\delta s = a$ apart. Then if P is the point (r, θ) with respect to the centre of the first dipole

$$\begin{aligned} V_P &= \frac{\mu\cos\theta}{\kappa_0 r^2} + \frac{(-\mu)}{\kappa_0}\left[\frac{\cos\theta}{r^2} + \frac{\partial}{\partial s}\frac{\cos\theta}{r^2}\delta s\right] \\ &= -\frac{\mu}{\kappa_0}\left[\frac{\partial}{\partial r}\frac{\cos\theta}{r^2}\frac{\partial r}{\partial s}\delta s + \frac{\partial}{\partial\theta}\frac{\cos\theta}{r^2}\frac{\partial\theta}{\partial s}\delta s\right] \\ &= -\frac{\mu}{\kappa_0}\left[\frac{(-2)}{r^3}\cos\theta(-\cos\theta)\delta s - \frac{\sin\theta}{r^2}\frac{\sin\theta}{r}\delta s\right] \\ &= -\frac{qa^2}{\kappa_0 r^3}[3\cos^2\theta - 1], \end{aligned}$$

if we write $\mu = qa$, and $\delta s = a$.

Dielectric polarization.—According to modern electrical theory an atom in its normal state is a configuration of electrons (negative charges) surrounding a relatively massive nucleus carrying a positive charge numerically equal to the total charge of the electrons. Hence, as a whole, the atom is uncharged. For the present we shall adopt the 'static' atom, i.e. we shall consider the electrons to be stationary relative to the nucleus. Since the total charge of a normal atom is zero, we may pair off each electron with an equal positive charge on the nucleus and regard each normal atom as an assemblage of electric doublets. On the whole the total electric moment is zero. It is known that a carbon atom has six orbital electrons and a corresponding positive charge on the nucleus. If we imagine the electrons to be coplanar, then the electric doublets formed in the manner described above are as indicated in Fig. 2·05(*a*).

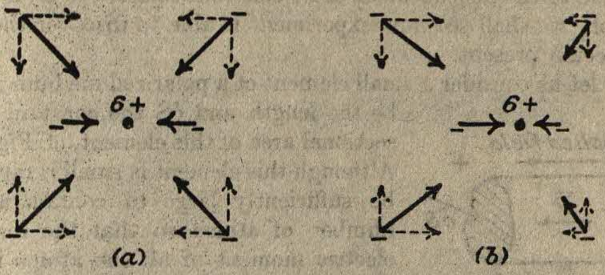

FIG. 2·05.—Electric moments of atoms.

When the above configuration of charges is subjected to an electric field, the negative charges are displaced, relative to the nucleus, as in Fig. 2·05(*b*). Now an electric moment is a vector represented by the straight line joining the charges. Hence, like any other vector quantity, it may be resolved into components. The resolved components in the present instance in the direction of the field and perpendicular to it are indicated by the dotted lines. These show that as a whole the atom has acquired an electric moment in the direction of the field. When the above occurs with each of the constituent atoms of a dielectric, the medium of the dielectric is said to be ***polarized.***

If we take a microscopic view of a dielectric and imagine a being sufficiently small to wander in and out among the atoms and to be provided with suitable measuring instruments, then such a being would be able to detect variations in the field strength in the space between the different electric charges. Actual instruments are only able to detect the combined effect of an exceedingly large number of atoms and their charges, and cannot reveal the changes which do occur locally in the medium. Such instruments only indicate mean values. In an unpolarized medium, if we assume a chaotic arrangement of atoms, the resultant electric moment per unit volume must be zero. When a dielectric is subjected to an electric field there is a tendency for each atom to acquire a resultant electric moment in the direction of the field, i.e. there will be a finite electric moment per unit volume. This is termed the ***dielectric polarization*** of the medium and is denoted by the letter P.

So far, reference has only been made to atoms or molecules which become polarized when acted upon by an external electric field. There are substances, however, whose molecules are permanently polarized—the case is analogous to that of paramagnetism—and the effect of an electric field is to cause the molecules of such substances to align themselves with their axes parallel to the direction of the field, in addition to a change in the electric moment of each atom due to the displacement of its electrons relative to its nucleus. Later on, we shall see how experiment is able to discover the type of molecule present.

Now let us consider a small element of a polarized medium ; let δl be the length and δS the constant cross-sectional area of this element, cf. Fig. 2·06. Although this element is small it must still be sufficiently large to contain a large number of atoms so that the resultant electric moment of all the atoms in this element is parallel to the length of the element which is supposed to be in the direction of the applied electric field. The electric moment of this element is by definition $P(\delta S . \delta l)$. Now this moment may be considered to arise from charges numerically equal to δq on the ends of the element. Its moment is then $\delta q . \delta l$. Equating these two expressions for the electric moment of the element, we have

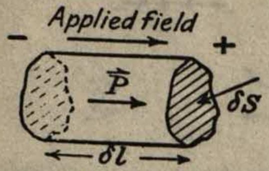

FIG. 2·06.—Dielectric polarization.

$$P . \delta S = \delta q,$$

or

$$P = \lim \frac{\delta q}{\delta S} = \frac{dq}{dS}.$$

Hence* P, *the dielectric polarization of the medium, is equal to the surface density of the electrification on the positive end of an element such as that we have considered. Here it must be remembered that when equivalent surface charges are being considered, the dielectric must be supposed non-existent.

Electric induction or electric displacement.—Let us consider unit cross-sectional area of a parallel plate condenser, (*a*) when the space between the plates is free from matter and (*b*) when the medium between them has a permittivity κ. If V_1 is the potential of the positive plate, V_2 that of the negative plate, t the distance apart of the two plates, and σ_1 the surface density of the electrification on the positive plate in the first instance, then

$$\frac{4\pi\sigma_1}{\kappa_0} = E = \frac{V_1 - V_2}{t} = \frac{V}{t}, \quad . \quad . \quad . \quad \text{(i)}$$

where $V = V_1 - V_2$, the drop in potential across the condenser.

Now insert the medium of permittivity κ between the condenser plates, the potential difference across the condenser being maintained equal to V. Then in theory and in practice (even with gases) an extremely narrow gap is left between the surface of the plates of the condenser and the boundaries of the medium. Owing to the polarization of the medium we get charges of surface density $\pm$ P at its boundaries, the charge due to dielectric polarization at the boundary near the positive plate of the condenser being negative, and vice versa, as indicated in Fig. 2·07. The charge $-$ P induces an opposite charge $+$ P on the positive plate of the condenser, so that the surface density of the electricity on the positive plate is now given by

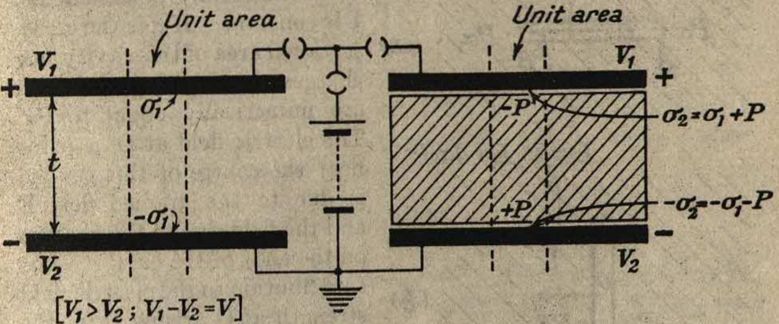
Unit area
Unit area
V_1
+
σ_1
t
$-\sigma_1$
−
V_2
$[V_1 > V_2;\ V_1 - V_2 = V]$
V_1
+
$-P$
$\sigma_2 = \sigma_1 + P$
$+P$
$-\sigma_2 = -\sigma_1 - P$
−
V_2

FIG. 2·07.—Electric induction.

$$\sigma_2 = \sigma_1 + P.$$

But experimentally $\sigma_2 = (\kappa)\sigma_1$. Hence

$$(\kappa)\sigma_1 = \sigma_1 + P \quad . \quad . \quad . \quad \text{(ii)}$$

Eliminating σ_1 from (i) and (ii) we have

$$(\kappa)\kappa_0\frac{V}{4\pi t} = \frac{\kappa_0 V}{4\pi t} + P,$$

or, since $\kappa = (\kappa)\kappa_0$, $\quad \kappa = \kappa_0 + 4\pi\dfrac{P}{\left(\dfrac{V}{t}\right)} = \kappa_0 + 4\pi\left[\dfrac{\text{Polarization}}{\text{Applied field}}\right]$

$$= \kappa_0 + 4\pi\chi, \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

where $\chi = \frac{P}{E}$, the ***electric susceptibility*** of the medium.

molecule of a gaseous dielectric has, on the average, an electric moment m, and that

$$m = \alpha E,$$

where E is the applied field and α a quantity whose nature is to be discovered. If there are n molecules per unit volume, then

$$nm = P = n\alpha E.$$

But $D = \kappa_0 E + 4\pi P$.

$$\therefore\ \kappa = \frac{D}{E} = \kappa_0 + 4\pi\frac{P}{E} = \kappa_0 + 4\pi n\alpha.$$

Hence, when κ is known, α may be determined if κ_0 is taken to be unity. Quite naturally, therefore, we ask ourselves what does this quantity represent ? **Faraday** imagined that α was a measure of the electric response of a molecule to an electric field, for according to him this field caused a displacement of the two differently charged portions of a molecule so that a dipole was formed. The dipole ceased to exist when the field was removed.

Debye (1927) conceived the idea that a molecule may possess a definite electric moment even before the electric field is applied, i.e. the distribution of charge in such a molecule is asymmetrical. Debye termed them ***polar molecules***, and the action of an electric field on them is twofold :

(*a*) it tends to align the molecules so that their electric axes are in the direction of the field ;

(*b*) it causes a displacement of the electrons in each atom relative to the nucleus in that atom.

Debye therefore wrote

$$\alpha = \alpha_1 + \alpha_2,$$

where α_1 is the contribution caused by the application of an external field, and α_2 is the contribution due to the permanent electric moments of the molecules when a random orientation of their axes no longer exists. He argued that α_2 will depend on the temperature of the dielectric ; as the temperature rises the molecules will be less liable to become orientated with their axes along the field. He showed that the permittivity of a gas whose molecules are permanent dipoles could be expressed by the formula

$$\kappa = a + \frac{b}{T},$$

where T is the absolute temperature and a and b are constants. For non-polar compounds κ is constant. If therefore κ is plotted against $\frac{1}{T}$, a straight line having a definite slope is obtained for gases whose molecules have a permanent electric moment; for non-polar compounds the straight line, cf. Fig. 2·09 (*a*), is parallel to the $\frac{1}{T}$ axis.

Ammonia NH_3, and methane CH_4, are examples of polar and non-polar molecules respectively. In the case of ammonia, Debye has shown that the hydrogen atoms (really ions) are arranged at the base of a regular tetrahedron, the nitrogen ion occupying the apex; this arrangement is shown in Fig. 2·09 (*b*). The angle HNH is 110°.

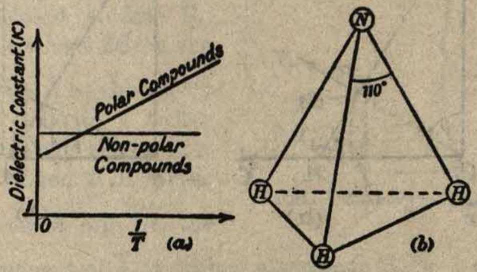

FIG. 2·09.—(*a*) The permittivities of gases and their variation with temperature, (*b*) an NH_3-molecule (polar).

Note on projected areas.—Let δs, Fig. 2·10 (*a*), be an element of length in the plane of the paper; let P be a point on δs. Take O_1, a point on the normal at P to δs and through O_1 draw two straight lines O_1X and O_1Y parallel to rectangular axes Ox, Oy and let δs produced intersect these lines in A and B respectively. If δx is the projection of δs on O_1X, which is also its projection on Ox,

$$\begin{aligned}\delta x &= \delta s \cos \gamma \\ &= n\,\delta s,\end{aligned}$$

where γ is the angle O_1P makes with Oy (or O_1Y), and $\cos \gamma = n$.

In order to obtain an area suitable for projection let AD and BC, Fig. 2·10 (*b*), be straight lines parallel to the plane of projection and normal to the straight line AB just considered. Further, let AD = BC. Then if the area ABCD is projected on to the plane O_1AD, the projected area is

$$\begin{aligned}O_1C_1DA &= O_1A \times AD \\ &= AB \cos \gamma \times AD \\ &= n \times \text{area ABCD}\end{aligned}$$

Now suppose S, Fig. 2·10 (c), is the plane area whose projection on the plane xOy is required. At any convenient point O_1, on the normal to S at P, draw three straight lines, O_1X, O_1Y, O_1Z parallel to the axes of reference Ox, Oy, Oz. Let the plane containing S intersect the axes O_1X, etc., in A, B and C respectively.

Through P draw a straight line QPR normal to PC to cut the periphery of S in Q and R. Let P_1, Q_1 and R_1 be the projections of P, Q and R, respectively on the plane XO_1Y. Then since PQ is normal to PC and also to O_1P, it is normal to every straight line in the plane O_1PC, i.e. QP is normal to PP_1. Hence the projection of PQ on XO_1Y is P_1Q_1 where P_1Q_1 = PQ. Thus QR = Q_1R_1.

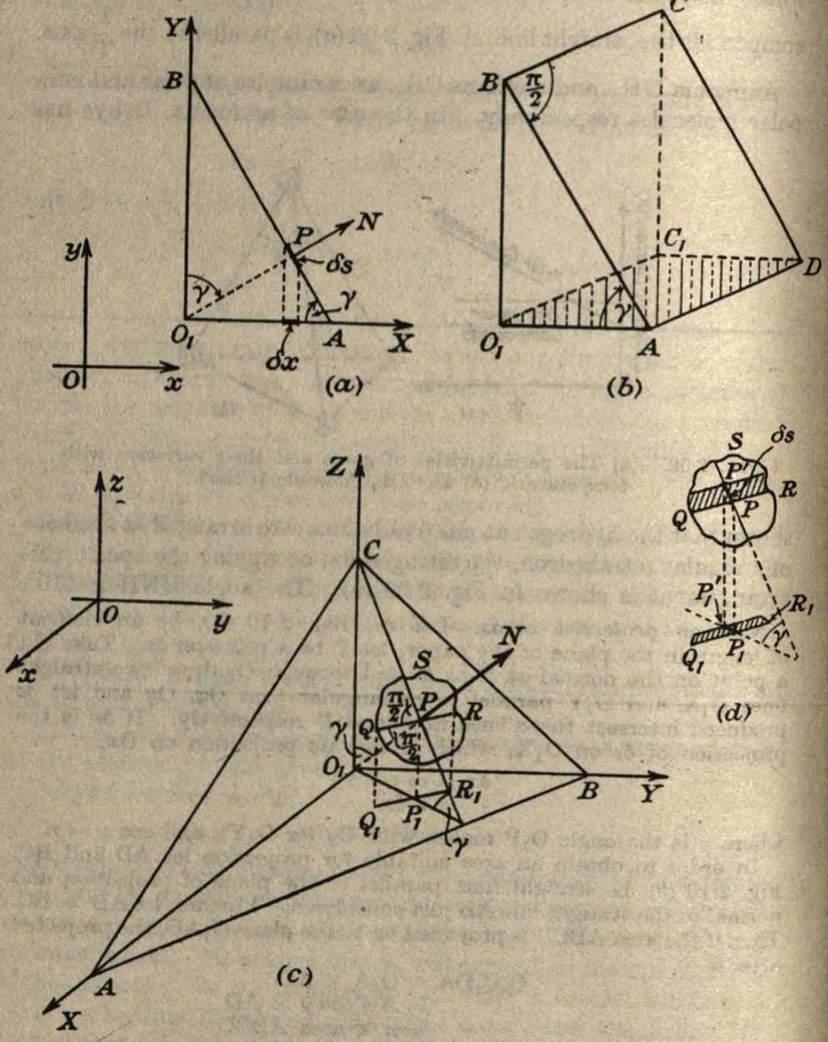

FIG. 2·10.—Projection of an area.

If, therefore, we form an element of area δS by drawing a line parallel to QR and at a distance δs, as measured along PC, from it, cf. Fig. 2·10 (d), the projection of this element on XO_1Y will be

$$Q_1R_1 \times n\ \delta s = n \times \delta S,$$

where (l, m, n) are the direction cosines for the normal at P. If $(\delta S)_{xy}$ is the projection of δS on xOy, we have

$$(\delta S)_{xy} = n\ \delta S.$$

Similarly, $(\delta S)_{yz} = l\ \delta S$, and $(\delta S)_{zx} = m\ \delta S$.

The flux of a vector quantity across a surface.—The flux of a vector quantity across a surface is defined in the following way. Let S, Fig. 2·11, be a surface with a unique normal at every point and let Γ be its periphery. Then S may be considered as part of a closed surface S_0. Consider an element δS on the surface and let $\vec{A}$ be the vector at δS; let $\hat{n}$ be unit vector along the outward drawn normal at δS. Then if θ is the angle between $\hat{n}$ and $\vec{A}$, the normal component of $\vec{A}$ at δS has magnitude $A_n = A\cos\theta = \vec{A}\cdot\hat{n}$. The scalar quantity $\vec{A}\cdot\hat{n}\,\delta S$ is called the flux of $\vec{A}$ across the surface δS.

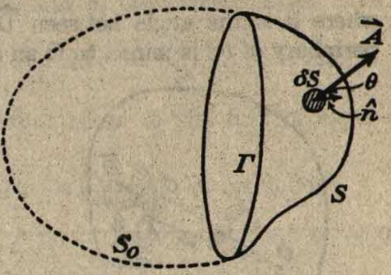

FIG. 2·11.—Flux of a vector.

By summing up the contributions to the flux from each element of the surface we determine the flux across the whole surface S; thus

$$\text{Flux of } \vec{A} \text{ across } S = \int \vec{A}\cdot\hat{n}\,dS,$$

where the integral is evaluated over the surface S.

In particular, if $\vec{D}$ is the electric induction at an element δS of a surface, $\int \vec{D}\cdot\hat{n}\,dS$ is the flux of electric induction across the surface S.

Gauss' theorem in electrostatics.—This theorem states that ***the flux of electric induction across any closed surface having a unique outward drawn normal at every point on it is 4π times the sum of the electrostatic charges enclosed by that surface.*** Thus

$$\int D_n\,dS = 4\pi\Sigma(q) = 4\pi Q,$$

or

$$\int \vec{D}\cdot\hat{n}\,dS = 4\pi\Sigma(q) = 4\pi Q,$$

where the integrations extend over the whole of the closed surface, and $Q = \Sigma(q)$ is the total charge enclosed.

Proof of Gauss' theorem in electrostatics.—Let q be a point charge of electricity situated at O, a point inside a closed surface S, Fig. 2·12 (*a*). This surface must have a unique outward drawn normal at every point on it. Let AB be a small element, δS, of the surface. Then the flux of electric induction across δS is $D \cos\theta\, \delta S$, where θ is the angle between $\vec{D}$ and $\hat{n}$. If every point on the periphery of δS is joined to O an elementary cone of solid angle $\delta\omega$ is obtained. A section of this cone is shown in Fig. 2·12 (*b*). Now

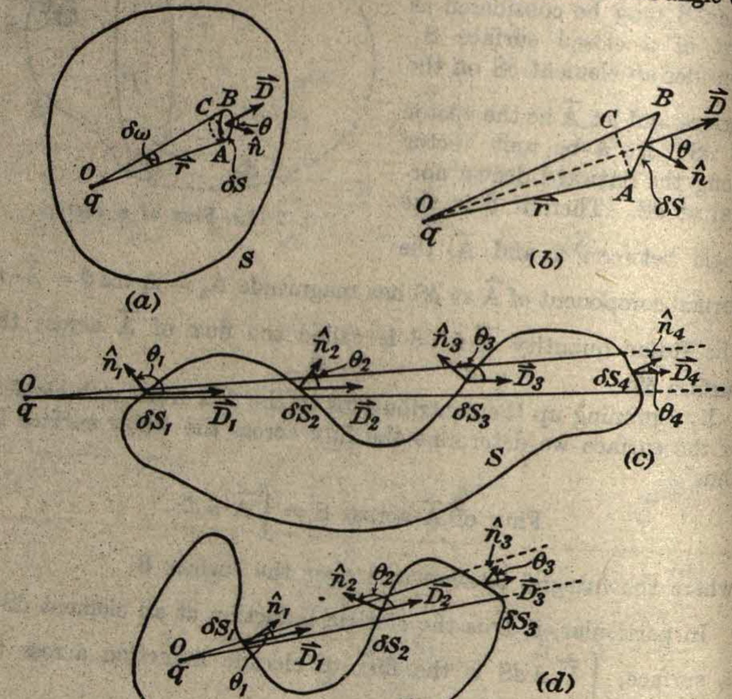

FIG. 2·12.—Proof of Gauss' theorem in electrostatics.

$\cos\theta\, \delta S$ = area of which AC is the trace $= r^2\, \delta\omega$. [$\overrightarrow{AC}$ is $\perp \overrightarrow{D}$]

$$\therefore \text{ Flux across } \delta S = D r^2\, \delta\omega.$$

But $D = \dfrac{q}{r^2}$. Therefore the flux across $\delta S = q\, \delta\omega$.

$$\therefore \text{ Flux across the closed surface} = \int q\, d\omega = q \int d\omega = 4\pi q.$$

When the point charge lies outside the surface S, every elementary cone from O which cuts the surface S must do so an even number of times—let us assume that this number is four and consider Fig. 2·12 (*c*). Let δS_1, δS_2 . . . be the areas of the closed surface intercepted by this elementary cone; the corresponding unit vectors along the outward drawn normals are shown. The angles θ_1 and θ_3 are obtuse; θ_2 and θ_4 are acute. Thus

$$\delta\omega = \frac{\delta S_1 \cos(\pi - \theta_1)}{r_1^2} = -\frac{\delta S_1}{r_1^2}\cos\theta_1 = \frac{\delta S_2}{r_2^2}\cos\theta_2 = \text{etc.}$$

The contribution from these four areas to the flux of electric induction is

$$D_1\cos\theta_1\,\delta S_1 + D_2\cos\theta_2\,\delta S_2 + D_3\cos\theta_3\,\delta S_3 + D_4\cos\theta_4\,\delta S_4$$
$$= -\frac{q}{r_1^2}r_1^2\,\delta\omega + \frac{q}{r_2^2}r_2^2\,\delta\omega - \frac{q}{r_3^2}r_3^2\,\delta\omega + \frac{q}{r_4^2}r_4^2\,\delta\omega = 0.$$

$\therefore$ Flux of electric induction across the whole surface is zero.

If the closed surface has a shape similar to that shown in Fig. 2·12 (*d*) and the point charge lies within this surface, then an elementary cone having its apex at the charge must intersect the surface an odd number of times. Let us assume this number to be three. Then the contribution from the three surface elements indicated to the flux of electric induction is

$$D_1\cos\theta_1\,\delta S_1 + D_2\cos\theta_2\,\delta S_2 + D_3\cos\theta_3\,\delta S_3$$
$$= +\frac{q}{r_1^2}r_1^2\,\delta\omega - \frac{q}{r_2^2}r_2^2\,\delta\omega + \frac{q}{r_3^2}r_3^2\,\delta\omega$$
$$= q\,\delta\omega - q\,\delta\omega + q\,\delta\omega = q\,\delta\omega.$$

Therefore the flux of electric induction across the whole surface is

$$\int q\,d\omega = q\int d\omega = 4\pi q.$$

Although the theorem has been established for a point charge it applies to any distribution of charge for this may be considered as a collection of point charges. Hence if Q is the total charge, i.e. $\Sigma\,(q)$, enclosed by a surface with a unique outward drawn normal at every point on it, then we have proved that

$$\int D_n\,dS \text{ or } \int D\cos\theta\,dS = 4\pi Q,$$

which is Gauss' theorem in electrostatics. An analytical expression in vector form for this theorem is

$$\int \vec{D}\cdot\hat{n}\,dS = 4\pi Q.$$

Lines and tubes of electric force and of electric induction.—It is possible to draw continuous lines in an electric field in free space such that the tangent at any point to one of them indicates the direction of the field strength at that point. It is also possible to draw ***lines of force***, for such is the name given to the above lines, in a dielectric placed in an electrostatic field. The tangent to such a line in a dielectric then indicates the direction of the force per unit charge on a small charge near the centre of a small needle-shaped cavity having its length parallel to the applied field and its centre at the point in question. It is also possible to draw other continuous curves in a dielectric; if these are such that the tangent at any point indicates the direction of the force per unit charge on a small positive charge placed in a cavity having the shape of a pill box with its faces normal to the applied field, such lines are termed ***lines of induction.***

If any area is considered in an electric field and lines of force or lines of induction are drawn through every point on the contour of the above area, a tube of force or a tube of induction is obtained.

Tubes of Induction.—Let A, Fig. 2·13, be a point in an electric field, the electric induction or electric displacement at A being D_1. Consider a small area δS_1 drawn round A so that it is normal to the direction of D_1. Construct the tube of induction having a cross-section δS_1 at A. Let δS_2 be the cross-section of the above tube at B, a point in the field where the electric displacement is D_2. Let us apply Gauss' theorem to this portion of the tube. The contribution to the flux of induction by the ends of the tube is $D_2 . \delta S_2 - D_1 . \delta S_1$, since D_1 and D_2 are normal to the areas δS_1 and δS_2 respectively. The contribution from the curved sides of the tube is zero, since the normal component of the displacement is zero at all points on the curved portions of the tube. If there is no charge enclosed in the portion of the tube considered, then

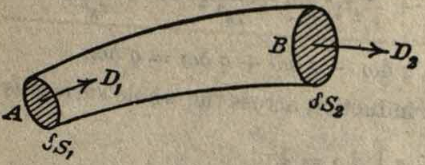

FIG. 2·13.—A tube of electric induction.

$$D_2 . \delta S_2 - D_1 . \delta S_1 = 0.$$

Thus the electric displacement at any point is inversely proportional to the area of cross-section of the tube at that point. In general, the product $D . \delta S$ is known as the ***strength of the tube.***

Definition.—*A unit tube of electric induction is one for which its strength*, viz. $D . \delta S$, *is unity.*

Conditions at the boundary between two isotropic dielectric media.—It is very important for us to determine how the electric induction, $\overrightarrow{D}$, and the field strength, $\overrightarrow{E}$, change as we pass across the interface between two different dielectrics. Consider first of all an element of area δS, Fig. 2·14 (*a*), of the interface, so small that its curvature may be neglected. Describe the prism generated by lines passing through the boundary of δS and at right angles to its plane. Let the prism be truncated by planes parallel to δS and let the height of the prism be a small quantity of the second order if δS is a quantity of the first order. Let $\vec{D}_1$ and $\vec{D}_2$ be the electric induction vectors in the media on either side of δS and let these vectors make angles θ_1 and θ_2 with the normal to δS.

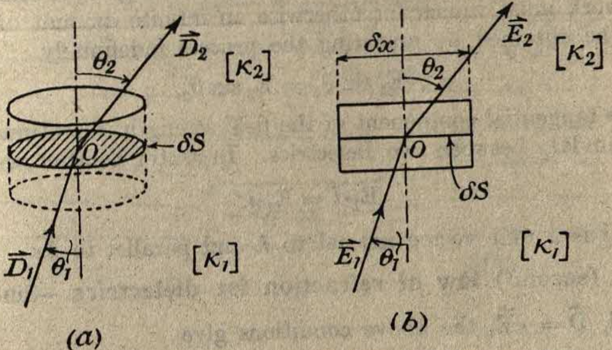

FIG. 2·14.—Conditions at the boundary between two isotropic dielectric media.

Apply Gauss' theorem to the prism. The electric induction is necessarily finite so that the flux of induction across the curved sides of the prism vanishes since its area is negligible compared with that of each plane end. The flux of electric induction is therefore

$$-\,D_1 \cos \theta_1 \, \delta S + D_2 \cos \theta_2 \, \delta S,$$

and this, by Gauss' theorem, is $4\pi\sigma\,\delta S$, if σ is the density of the electrification at the boundary.

$$\therefore\ D_2 \cos \theta_2 - D_1 \cos \theta_1 = 4\pi\sigma, \quad \text{i.e. } \vec{D}_2 \cdot \hat{n} - \vec{D}_1 \cdot \hat{n} = 4\pi\sigma.$$

In the majority of cases there will be no charge at the interface so that $D_2 \cos \theta_2 - D_1 \cos \theta_1 = 0$, i.e. the components of the electric induction normal to the interface will have the same value on either side of it—in other words, when there is no charge at the boundary between dielectrics the normal component of the electric induction is continuous from one medium to another. When this is not the case it is discontinuous by an amount equal to 4π times the charge per unit area at the interface.

A similar relation is satisfied by the tangential component of the field strength. Consider the rectangle, Fig. 2·14 (*b*), which forms the central section of the prism just contemplated. Let us calculate the work done in taking a small charge δq round the periphery of the elementary rectangle. The field strength is finite everywhere and so the contribution to the work done along the short sides vanishes in comparison with that done along the other sides. Since $E_1 \sin \theta_1$ and $E_2 \sin \theta_2$ are the tangential components of the electric field strengths at O, if δx is the length of the longer sides of the element, the work done is

$$\pm [(E_1 \sin \theta_1 - E_2 \sin \theta_2)\, \delta q]\, \delta x,$$

the sign depending on the sense in which the charge δq is conveyed. This work must vanish for otherwise an infinite amount of work could be obtained by repeating the process indefinitely.

$$\therefore\ E_1 \sin \theta_1 = E_2 \sin \theta_2,$$

i.e. the tangential component of the field strength is continuous at the boundary between two dielectrics. In vector notation we have

$$\vec{E}_1 \cdot \hat{t} = \vec{E}_2 \cdot \hat{t},$$

where $\hat{t}$ is a unit vector normal to $\hat{n}$ and parallel to $\vec{\delta x}$.

The (second) law of refraction for dielectrics.—Since, in general, $\vec{D} = \kappa \vec{E}$, the above conditions give

$$\kappa_1 \cot \theta_1 = \kappa_2 \cot \theta_2.$$

Thus $\kappa \cot \theta$ is an electrical invariant. This statement may be symbolized by the equation

$$\Delta[\kappa \cot \theta] = 0,$$

where $\Delta[x] \equiv x_1 - x_2$.

To illustrate the use which may be made of these laws let us investigate the field inside a spherical cavity within an isotropic dielectric; previously, cf. p. 26, we have been able only to obtain values for the field inside needle-shaped and disc-shaped cavities. The result we are about to obtain is not without importance in the theory of dielectrics.

The field inside a spherical cavity within an isotropic dielectric.—When such a dielectric completely fills the space previously occupied by a uniform electric field E, the field strength is unchanged although the medium is polarized. The lines of induction are parallel to the field and it has been shown that the force per unit charge in a needle-shaped cavity and in a flat cavity along and normal to a line of induction have values E and $E + \dfrac{4\pi P}{\kappa_0}$, respectively; the symbols are as defined before. When a spherical cavity of radius a exists in an isotropic dielectric of infinite extent the lines of induction are as shown in Fig. 2·15 (*a*). At a point Q on the surface of the cavity the second law of refraction gives

$$\frac{\kappa_0}{\tan\theta_1} = \frac{\kappa}{\tan\theta_2},$$

i.e. $\theta_2 > \theta_1$ since $\kappa > \kappa_0$.

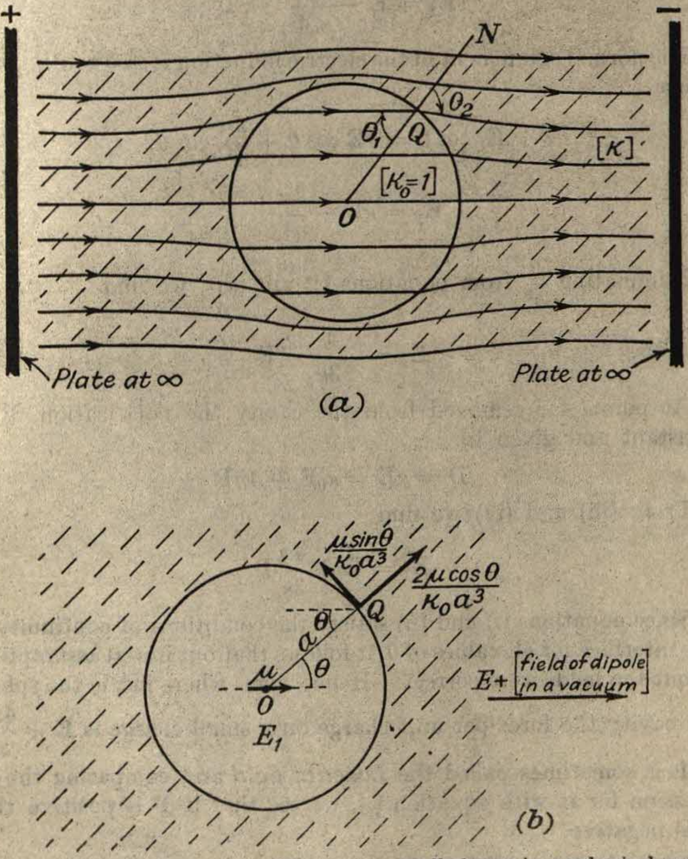

FIG. 2·15.—The force per unit charge on a small charge in a spherical cavity within an isotropic dielectric.

To determine the force per unit charge on a small charge placed within the spherical cavity let it be assumed that the field strength is E_1 everywhere; it is parallel to the original field E and this remains the field within the dielectric at points far removed from the cavity. At other points in the dielectric let the force per unit charge be assumed to be E plus that due to a dipole of moment μ

at the centre of the cavity, the whole of space, as far as this superimposed field is concerned, being free from material media, cf. Fig. 2·15 (*b*). Then at the point Q, viz. (a,θ), since the tangential component of the field strength is continuous,

$$E_1 \sin\theta = E\sin\theta - \frac{\mu}{\kappa_0 a^3}\sin\theta,$$

i.e.

$$E_1 = E - \frac{\mu}{\kappa_0 a^3} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The normal component of the electric induction is also continuous; hence

$$\kappa_0 E_1 \cos\theta = \kappa E\cos\theta + \frac{2\mu}{a^3}\cos\theta,$$

i.e.

$$\kappa_0 E_1 = \kappa E + \frac{2\mu}{a^3} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Eliminating $\frac{\mu}{a^3}$ from equations (i) and (ii), we find

$$E_1 = \frac{\kappa + 2\kappa_0}{3\kappa_0}E \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

At points far removed from the cavity the polarization, P, is constant and given by

$$D = \kappa E = \kappa_0 E + 4\pi P \quad . \quad . \quad . \quad \text{(iv)}$$

From (iii) and (iv) we find

$$E_1 = E + \frac{4\pi}{3\kappa_0}P.$$

Since equations (i) and (ii) satisfy the conditions of continuity at the interface for all values of θ, it follows that our initial assumptions about the fields were correct. Hence, everywhere inside the spherical cavity the force per unit charge on a small charge is $E + \frac{4\pi P}{3\kappa_0}$. This is sometimes called the ***Lorentz field*** and comparing the expression for it with equation (i), we see that if P is positive then μ is negative.

The electric field inside a hollow charged conductor of homogeneous material.—Let A, Fig. 2·16, be a hollow charged conductor of homogeneous material and let it be at a potential V. It is desired to show that if there is no electric charge enclosed by A, the field strength within A is zero. If it is not zero equipotential surfaces can be drawn within A and these cannot intersect A since this also is an equipotential. Let B be such a surface, the potential at all points on it being V_1. Suppose $V_1 > V$, so that at all points on B the electric field will have a finite component along the outward drawn normal. Since, in general, D = κE, where κ is the permittivity of the medium, it follows that if the field has a finite component along the normal considered, then D must also have a finite component in the same direction, i.e. $\vec{D}\cdot\hat{n}$ is always positive. Hence the flux of $\vec{D}$ across the surface B is always positive, i.e., by Gauss' theorem, there must be a positive charge within B and so within A.

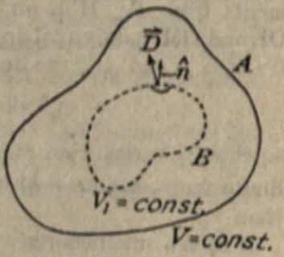

FIG. 2·16.—Electric field within a closed hollow conductor of homogeneous material.

Similarly if $V_1 < V$ the electric field will have a negative component along the outward drawn normal at every point on the surface of B. The flux of induction across B will therefore be negative, so that a negative charge must be present inside the hollow of the conductor.

Each conclusion thus deduced is contrary to our hypothesis that there is no charge within the hollow conductor. Thus the potential is constant throughout the hollow, i.e. the field therein is zero.

The field strength at points within a charged and isolated spherical conductor.—To examine the field strength at points within a charged and isolated hollow spherical conductor let O, Fig. 2·17 (*a*), be the centre of the spherical shell with its uniform distribution of electricity of density σ, and P a point within it. Let us assume a modified law of force and write r^{-n} instead of r^{-2}, where n is a positive quantity. Also let AB be the trace of a plane through P normal to OP and through P construct an elementary

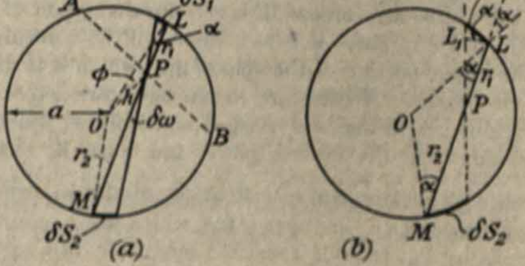

FIG. 2·17.—The inverse n^{th} power law in electrostatics; field inside a charged and isolated conductor in the form of a spherical shell.

cone, with solid angle $\delta\omega$, to cut the sphere in elements of area δS_1 and δS_2 respectively; let r_1 and r_2 be the distances of these elements from P. If L and M are the points indicated, then both OL and OM make equal angles, say α, with LM. From the diagram, if $OP = h$ and ϕ is the angle indicated,

$$r_1^2 = a^2 + h^2 - 2ah\cos\phi,$$

so that r_1 varies from $(a - h)$ to $+\sqrt{a^2 - h^2}$; the corresponding limits for r_2 are $(a + h)$ and $+\sqrt{a^2 - h^2}$, so that r_2 is never less than r_1.

If δq is a small charge at P, the resultant force, δF, on it and due to the elements considered, is directed along PM and given by

$$\delta F = \left[\frac{\sigma\delta S_1}{\kappa_0 r_1^{\,n}} - \frac{\sigma\delta S_2}{\kappa_0 r_2^{\,n}}\right]\delta q$$

$$= \frac{\sigma\delta\omega}{\kappa_0\cos\alpha}\left[\frac{1}{r_1^{\,n-2}} - \frac{1}{r_2^{\,n-2}}\right]\delta q$$

since, cf. Fig. 2·17 (*b*), the solid angle which δS_1 subtends at P is $\dfrac{\delta S_1\cos\alpha}{r_1^2}$; this is also $\dfrac{\delta S_2\cos\alpha}{r_2^2}$. Thus the field strength at P due to the elements considered, and in the direction PM, is given by

$$\delta E = \lim\frac{\delta F}{\delta q} = \frac{\sigma\delta\omega}{\kappa_0\cos\alpha}\left[\frac{1}{r_1^{\,n-2}} - \frac{1}{r_2^{\,n-2}}\right].$$

Since r_2 is never less than r_1, $\overrightarrow{\delta E}$ will be towards δS_2 if $n > 2$ and away from it if $n < 2$. Now the whole surface of the sphere may be divided in this way into pairs of elements and the resultant field at P will not be zero unless δE is zero for every pair of elements such as δS_1 and δS_2, since if $n > 2$ there will be a resultant field towards the larger portion of the sphere into which it is divided by AB, and vice versa. If therefore it can be shown experimentally that the field strength inside a charged and isolated sphere is zero, then the index n in the inverse power law must be exactly 2.

Maxwell's experimental verification of the inverse square law in electrostatics.—The apparatus, which was a modification of an earlier one by CAVENDISH, consisted essentially of a metal sphere surrounded by two metal hemispheres supported concentrically on an insulating stand; the inner sphere was held in position by means of an ebonite ring. In this way the insulating support for the inner sphere was never exposed to the action of any electric field and therefore never received any charge which might have been a disturbing factor when the inner sphere was tested for the presence of electricity on it. The outer sphere had a small hole in it so that by inserting a metal probe, connected to a quadrant electrometer, through it any charge on the inner sphere could be detected. Initially the two spheres were connected by a wire and raised to a high potential. They were then insulated from one another and the outer one earthed; finally the inner sphere was connected to the electrometer. 'Not the slightest effect on the electrometer could be observed,' wrote Maxwell.

Maxwell then calculated that if the repulsion had followed the law $r^{\beta-2}$, the potential of the inner sphere would have been $-0{\cdot}1478\beta V$ where V was the potential of the shell, and showed experimentally that

$$\beta < \pm \frac{1}{21600}.$$

It was in order to estimate the potential of the inner sphere if the value of β were not zero that both Cavendish and Maxwell used spherical condensers, but in any closed conductor with no charge within the electric field strength is zero.

A modern method for testing Coulomb's law of force between point charges.—At first sight it would appear that a considerable improvement in the accuracy of the test of this law by Maxwell's method should be possible to-day, for high voltages can now be produced and very small differences in potential detected. In all probability, however, Maxwell apparently reached the limit of accuracy attainable by his method for it is now known that the small charge which might remain on the inner sphere could be affected by spontaneous ionization even in a very short time; the most serious difficulty arises from contact potentials. In 1936 PLIMPTON and LAWTON devised the following method for testing the law of force between point charges and in it the effects of contact potentials were entirely eliminated. This was accomplished by placing the detector inside the inner globe and connecting it permanently to indicate any change in the potential of the inner globe relative to the outer one. It was shown that the presence of a conducting body inside the inner globe has no effect on the results and this made it possible for the dimensions of the detector to be chosen so that the detector and its shield was approximately equivalent to the lower half of the inner globe; this was removed from the system.

Fig. 2·18 shows the principal features of the apparatus used in this investigation. The globe A is made of iron and is five feet in diameter. The detector D is a five-stage resistance-capacitance coupled amplifier designed for a frequency of 2 cycle. sec.$^{-1}$; the actual detector being a vibration galvanometer G tuned to this frequency, the complete detector is known as a ***resonance electrometer.***

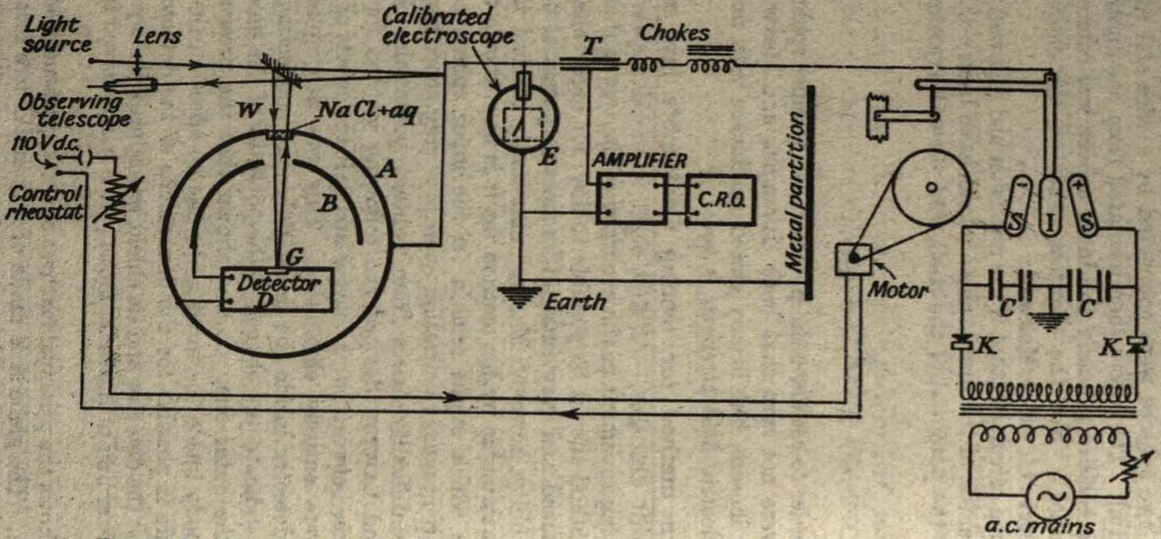

FIG. 2·18.—Apparatus for testing the inverse square law of force between electric charges. [After Plimpton and Lawton.]

The inner hemispherical dome B is also made from iron and is four feet in diameter. It is mounted on Pyrex glass insulators.

The galvanometer G, inside D, was illuminated and after the upper half of A was in situ it could be viewed through holes at the top of A and B. In order that there should be no break in the conducting surface of the outer globe A, a glass-bottomed vessel was threaded into the hole in A and filled with an aqueous solution of common salt; the liquid surface was flush with the external surface of A and a fine wire gauze, soldered to the walls of the containing vessel, was submerged in this solution to cover the glass.

In some preliminary experiments wherein the outer globe was charged and simply discharged by opening and closing circuits, the galvanometer gave deflexions which were due to mutual inductance, M, between the charging circuit and the inner conductor; by Faraday's law, cf. p. 224, this acquired a potential numerically equal to $M\frac{di}{dt}\left(\text{or } M\frac{d^2Q}{dt^2}\right)$, where Q is the charge on A at time t; i is the charging current. Such an effect must never be allowed to be large enough to be detectable, i.e. $\frac{di}{dt}$ has to be kept small. For this purpose the use of a resistor alone was not satisfactory since $\frac{di}{dt}$ is very large at the instant a circuit is made or broken; a quasi-static method was therefore adopted. In it a high voltage was applied to the outer sphere as a sinusoidal wave of low frequency, so as to build up a resonant vibration of the galvanometer. Apart from the great sensitivity of such a detector it also possessed another great advantage which may be explained as follows. The sensitivity of the amplifier is limited by the effect on the galvanometer caused by Brownian motion in the surrounding air. Such motions have frequencies of all values and, according to the equipartition law, the energy associated with any one frequency is the same for all frequencies. Now a vibration galvanometer responds to those Brownian motions which have a frequency approximately equal to its own, and this response is only negligible when the disturbance, or signal to be measured, is energetic and has a frequency equal to that of the galvanometer.

To obtain a high voltage with a very low frequency, a ***condenser-generator*** was developed. This consisted in part of three large metal combs; in the diagram S, S represent two of these, known as ***stators***; they were always stationary and had their teeth pointing upwards, while the third, the ***inductor*** I, was held with its teeth pointing downwards and caused to swing harmonically in and out of mesh alternatively with each stator. The stators were charged by means of a transformer and rectifiers K, K; the capacitance to earth of the stator was increased by adding large mica condensers C, C.

The generator was connected to A through graduated chokes to eliminate high frequency currents and the applied voltage read directly on a calibrated electroscope, E, for any position of the inductor I. To observe the wave-form of this voltage an oscilloscope was coupled to the line through a cylindrical air-condenser T. Since the impedance of such a condenser is high at the low frequency used, an amplifier was inserted before the oscilloscope. This was provided with a 60-cycle sweep, which became a very slender ellipse owing to pick-up from the surroundings. This ellipse moved up and down with the applied voltage and retained its smooth shape, a fact indicating the absence of high frequency components.

The connexions for the calibration of the oscilloscope are shown in Fig. 2·19. The outer globe A was earthed and a small known fraction, $R_1 \div (R_1 + R_2)$, of the voltage from the condenser-generator applied to B by means of the high resistance potential divider $R_1 + R_2$. The amplitude of the voltage from the generator was read directly on the oscilloscope, without sweep; since the deflexion is independent of frequency, the oscilloscope could be calibrated with the aid of the battery shown.

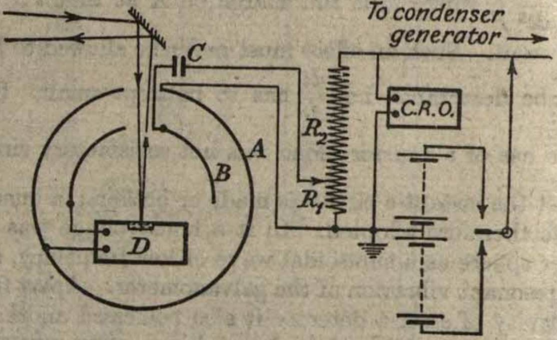

FIG. 2·19.—Calibration of the C.R.O. used in connexion with the verification of the inverse square law in electrostatics.

With the apparatus in working order and $\hat{V} > 3000$ volts no change in the small heat motion of the galvanometer was ever detected when the generator was started or stopped at random. Since the minimum voltage, $\hat{v}$, observable above the usual heat fluctuation was 1 μV., Plimpton and Lawton finally concluded that the exponent in the inverse law of force between point charges is 2 ± 10^{-9}.

In view of the accuracy thus attained it is necessary to consider any possible effect of gravity on the otherwise uniform distribution of charge on an isolated spherical conductor. If electrons with inertial mass m and charge e, where $e = -4{\cdot}803 \times 10^{-10}$ e.s.u., have weight mg, the potential, V, at a point on the surface of the sphere must be such that

$$e\frac{dV}{dz} = mg,$$

where z is the height of the point above datum level. Thus $\dfrac{dV}{dz}$ is constant, i.e. $V = az + b$, where a and b are constants. Calculation showed that the maximum potential difference over A did not exceed 10^{-10} volt; the effect of gravity does not therefore affect the conclusion already stated concerning the law of force in electrostatics.

Coulomb's theorem in electrostatics.—This theorem states that if D is the electric induction at a point just outside a conductor, then $\vec{D} = 4\pi\sigma\,\hat{n}$, where σ is the density of the electricity on the conductor and $\hat{n}$ is the unit vector along the outward drawn normal at the point considered. Thus $D = 4\pi\sigma$.

Since the surface of the conductor is an equipotential the vector $\vec{D}$ is always in a direction normal to the surface. Let δS, Fig. 2·20, be an element of the surface, S, of a conductor surrounded by a dielectric of permittivity κ. At every point on the boundary of δS let us draw perpendiculars to the surface; let the cylindrical surface thus formed be truncated by two neighbouring planes parallel to δS, one of the ends of the frustum so formed being inside the conductor and the other in the dielectric. These ends each have an area δS. Let us apply Gauss' theorem to the small closed surface so formed. Now within the conductor $\vec{D}$ is zero for the conductor is an equipotential. Hence for the surface considered $\vec{D}$ is everywhere zero except over the end δS lying within the medium.

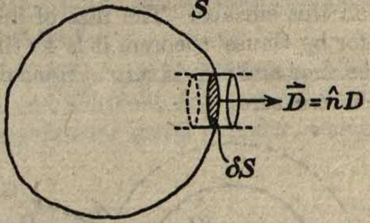

FIG. 2·20.—Coulomb's theorem.

∴ Flux of electric induction over the surfaces of the prismatic element = D δS, and this, by Gauss' theorem, is $4\pi.\sigma\delta S$. Hence

$$D = 4\pi\sigma,$$

or

$$\vec{D} = 4\pi\sigma\,\hat{n}.$$

Electric induction due to a uniformly charged spherical conductor.—(*a*) ***At points outside the sphere.*** Let A, Fig. 2·21, be a point situated outside a sphere of radius a and carrying a charge Q. Since the charge on the sphere is uniformly distributed, the electric displacement will be radial and equal at all points equidistant from O the centre of the sphere. Through A draw a spherical surface of radius OA, the centre being O. The electric induction at all points on this surface is D and is everywhere normal to the surface at the point considered. Hence the flux of induction across this spherical surface is

$$D \times (\text{area of surface of sphere of radius } r) = 4\pi r^2 D.$$

But by Gauss' theorem this is $4\pi Q$. Hence

$$D = \frac{Q}{r^2}.$$

The electric field strength is therefore $\frac{Q}{\kappa r^2}$, where κ is the permittivity of the surrounding medium; it is assumed to be isotropic. At points outside the sphere the electric induction and field are the same as if the charge were concentrated at the centre of the sphere.

(*b*) ***At points inside the sphere.*** If B is such a point and a spherical surface, with centre O and radius OB, is constructed, the electric induction, unless it is zero, must again be radial at all points on this surface. The flux of induction across this surface is zero, for by Gauss' theorem it is 4π times the charge inside the spherical surface and this is zero. Hence, inside the sphere, D, and therefore E, are zero.

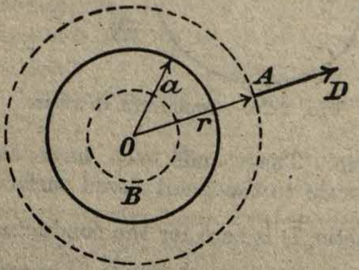

FIG. 2·21.—Electric induction due to a charged sphere.

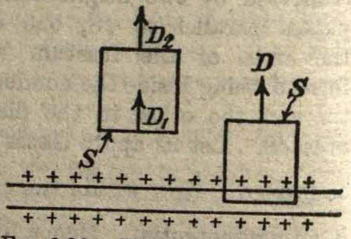

FIG. 2·22.—Electric induction due to a charged infinite plate.

Uniformly electrified infinite flat metal plate.—Let σ be the density of the electricity on ***each*** side of the plate. By symmetry, the electric displacement must everywhere be normal to the plate and have a constant value at all points in a plane parallel to it.

Under these conditions the area of cross-section of any tube of induction remains constant. Consider the portion of such a tube shown in Fig. 2·22. Let D_1 and D_2 be the values of the electric induction at the lower and upper ends of this cylinder. Let the areas of these ends be S. It is not necessary for these areas to be small since, by symmetry, D_1 and D_2 are constant at all points in the respective planes parallel to the given plate. The flux of induction across the surface of this cylinder is

$$D_2S - D_1S,$$

the contribution from the curved surfaces being zero. The above is zero by Gauss' theorem. Hence

$$D_1 = D_2,$$

or the electric induction is constant, and equal to D, say.

To determine the value of the induction consider the portion of a tube of induction indicated on the right of the diagram. This tube originates on a charge σS. Let the above cylinder be produced backwards so that it may be truncated inside the conductor. There can be no tubes of induction inside the charged plate, but that does not prevent us from drawing a portion of a closed surface inside the conductor. The only contribution to the flux of induction arises from the end of the tube in the dielectric, and amounts to DS. By Gauss' theorem this is $4\pi\sigma S$, so that $D = 4\pi\sigma$. In using this formula it must be remembered that σ is the surface density of the electricity on one side of the plate only.

A uniformly electrified infinite cylindrical conductor.—Let us consider the electric induction and the field due to a uniform distribution of electricity on a right circular metal cylinder, which

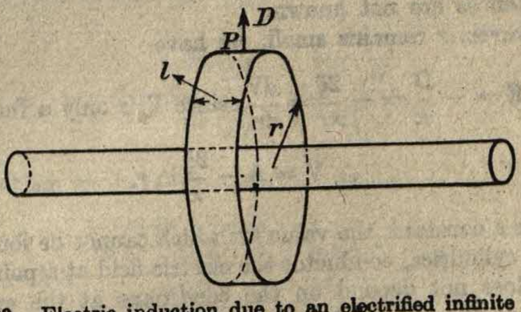

FIG. 2·23.—Electric induction due to an electrified infinite cylinder.

either extends to infinity in each direction or else has its ends so far removed from the region under consideration that their influence may be neglected. Let P, Fig. 2·23, be a point at distance r from the axis of the infinite cylinder, the charge of electricity per unit length being $\bar{q}$. By symmetry D will be radial and have the same value at all points equidistant from the axis of the cylinder. Through P describe a cylindrical surface coaxial with the charged cylinder, and construct two planes at distance l apart and normal to the axis of the cylinder to form a closed surface. The charge enclosed by this surface is $\bar{q}l$. The plane ends contribute nothing to the flux of induction across the closed surface considered. The flux across the curved surface of the cylinder is $D.2\pi rl$. By Gauss' theorem this is $4\pi\bar{q}l$. Hence

$$D = \frac{2\bar{q}}{r},$$

and

$$E = \frac{2\bar{q}}{\kappa r},$$

where κ is the permittivity of the medium surrounding the charged cylinder.

Each of these expressions is independent of the radius of cross-section of the cylinder, so that the field is the same as if the radius of cross-section were very small, i.e. if the conductor were a straight charged wire.

In the above argument it has tacitly been assumed that the distance of P from the axis of the cylinder is small, so that the influence of the charges at the ends of the cylinder is negligible in comparison with that of the nearer points of the cylinder. When r is very large the electric induction is not zero over the plane surfaces of the Gaussian surface used, so that the investigation cannot hold for large values of r. Hence we cannot find the electric potential at P by integration of the work done against the field in bringing up a small charge from infinity to P, because the field strengths at large distances are not known.

If, however, r remains small, we have

$$\frac{\partial V}{\partial r} = -E = -\frac{D}{\kappa} = -\frac{2\bar{q}}{\kappa r} = \frac{dV}{dr}, \text{ since V is only a function of } r.$$

$$\therefore V = A - \frac{2\bar{q}}{\kappa} \ln r,$$

where A is a constant, the value for which cannot be found. Thus for a long cylindrical conductor the electric field at a point near the cylinder does not depend on the conditions at the ends of the cylinder but the potential and capacitance do.

Example.—The filament in a certain diode may be regarded as a cylinder of radius a, the anode being a coaxial cylinder of radius b. The filament is earthed and the anode maintained at a potential V_0.

Show that at a point distance r, $[a < r < b]$, the potential V is given by

$$\mathrm{V} = \mathrm{V}_0 \frac{\log\left(\frac{r}{a}\right)}{\log\left(\frac{b}{a}\right)}.$$

If $a = 0{\cdot}04$ cm., $b = 0{\cdot}8$ cm. and $\mathrm{V}_0 = 300$ volts, calculate a value for the velocity of an electron, which starts from rest at the cathode, when it is 0·32 cm. from the surface of the cathode. It may be assumed that for an electron $\frac{e}{m} = -\,5{\cdot}28 \times 10^{17}$ e.s.u. gm.$^{-1}$.

Let P be a point in the region between the electrodes and at a distance r from the axis of the system. Since the inner electrode is the cathode let the charge on it per unit length be $-\,\bar{q}$; then $\bar{q}$ will be the charge per unit length on the inner surface of the anode. The corresponding charge on its outer surface will be $\bar{q}_0$, say, but it does not concern us here. Through P construct a Gaussian surface, which in this instance is a cylinder of radius r and unit length. If E is the field strength at P and κ_0 the permittivity of a vacuum, we have, by Gauss' theorem,

$$\kappa_0 \mathrm{E} \times 2\pi r = 4\pi(-\,\bar{q}), \quad \text{or} \quad \mathrm{E} = -\,\frac{2\bar{q}}{\kappa_0 r} = -\,\frac{d\mathrm{V}}{dr}.$$

$$\therefore\ \mathrm{V} = 2\frac{\bar{q}}{\kappa_0}\ln r + \mathrm{A},$$

where A is an integration constant. At $r = a$, $\mathrm{V} = 0$, so that $2\frac{\bar{q}}{\kappa_0}\ln a + \mathrm{A} = 0$. Hence

$$\mathrm{V} = 2\frac{\bar{q}}{\kappa_0}\ln\left(\frac{r}{a}\right).$$

But

$$\mathrm{V}_0 = 2\frac{\bar{q}}{\kappa_0}\ln\left(\frac{b}{a}\right).$$

Hence

$$\mathrm{V} = \mathrm{V}_0\frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)} = \mathrm{V}_0\frac{\log\left(\frac{r}{a}\right)}{\log\left(\frac{b}{a}\right)}.$$

In the problem we have $\mathrm{V}_0 = 1$ e.s.u. Hence

$$\mathrm{V} = 1 \times \frac{\log\left(\frac{0{\cdot}32 + 0{\cdot}04}{0{\cdot}04}\right)}{\log\left(\frac{0{\cdot}8}{0{\cdot}04}\right)} = \frac{\log 9}{\log 20} = 0{\cdot}73 \text{ e.s.u.}$$

If u is the velocity of the electron, m its mass and e its charge, since the total energy of the electron remains constant, we have

$$e(0) + \tfrac{1}{2}m(0^2) = e\mathrm{V} + \tfrac{1}{2}mu^2.$$

Hence $u = \sqrt{2 \times 5{\cdot}27 \times 10^{17} \times 0{\cdot}73} = 8{\cdot}8 \times 10^{8}$ cm.sec.$^{-1}$.

[In this calculation the variation of mass with velocity, cf. Vol. VI, is neglected.]

The stress at the surface of a charged conductor.—Let A and B, Fig. 2·24, be two points close to the surface of a charged conductor, one outside and the other inside. Let κ be the permittivity of the medium outside the conductor. The problem before us is to investigate the force on the charge residing on unit area of the surface of the conductor. Let E_1 and E_2 be the contributions to the field strength E at A made respectively by the electricity on (*a*) a surface element δS near to A and (*b*) the rest of the conductor. Then

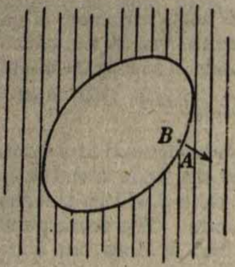

FIG. 2·24.—Stress at the surface of a charged conductor.

$$E = E_1 + E_2.$$

At B the electric field strength is zero. The contribution to this is $-E_1$ from the first part of the distribution, and E_2 from the remainder. Hence, $-E_1 + E_2 = 0$, or $E_1 = E_2 = \frac{1}{2}E$.

If σ is the surface density of the electrification at A, the charge on the element δS is $\sigma . \delta S$ and the force δF acting on this charge is caused by the electric field strength at the point considered due to the rest of the electricity on the conductor, i.e. the force is

$$\delta F = \sigma . \delta S . E_2 = \tfrac{1}{2}\sigma . \delta S . E.$$

Hence p, the surface stress, which is the force per unit area at the surface of the conductor, is given by

$$p = \lim \frac{\delta F}{\delta S} = \tfrac{1}{2}\sigma E.$$

But $D = 4\pi\sigma$, so that the expression for the surface stress becomes

$$p = \frac{DE}{8\pi} = \frac{\kappa E^2}{8\pi} = \frac{2\pi\sigma^2}{\kappa}.$$

The sign of this stress is independent of that of the charge and it is always directed outwards.

To prove that the stress is $\dfrac{2\pi\sigma^2}{\kappa}$ without assuming that the stress is determined by the field strength $\vec{E}$ we may proceed as follows. The proof is not a general one but it lends support to our argument.

Consider a parallel plate condenser, area A and thickness x.

Then if edge effects are neglected, we have, with the usual notation,

$$W = \frac{1}{2}\frac{Q^2}{C}, \quad \text{and} \quad C = \frac{\kappa A}{4\pi x}.$$

$$\therefore\ W = \frac{2\pi Q^2 x}{\kappa A}.$$

If F is the force on either plate tending to increase x, the work done by the force when x becomes $x + \delta x$ is equal to the decrease in the potential energy of the system, i.e. $F\,\delta x = -\,\delta W$.

$$\therefore\ F = -\frac{\partial W}{\partial x} = -\frac{2\pi Q^2}{\kappa A}.$$

$$\therefore\ \text{Force tending to decrease } x = \frac{2\pi Q^2}{\kappa A}.$$

$$\therefore\ \text{Surface stress} = p = \frac{2\pi Q^2}{\kappa A} \div A = \frac{2\pi\sigma^2}{\kappa}.$$

Experiment.—The existence of the above stress may be shown by producing a soap bubble at the end of an insulated metal tube. When the tube and therefore the soap bubble is connected to a source of high potential the bubble expands until the reduction in pressure inside the bubble compensates for the surface stress arising from the charge on the bubble.

Energy density in an electric field.—To find an expression for the energy density in an electric field let us consider the unit tube shown in Fig. 2·25, and make use of the fact that the flux of electric induction across any normal section of a unit tube of induction is unity, i.e. $D\,\delta S = 1$. The unit tube considered begins on the surface of a positively charged conductor A and ends on the negatively charged conductor B. If δS_A and δS_B are the 'terminal' cross-sections of the tube

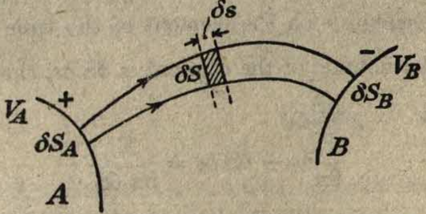

FIG. 2·25.—Energy density in an electric field.

$$D_A\,\delta S_A = 1 = D_B\,\delta S_B.$$

Now since $D_A = 4\pi\sigma_A$, where σ_A is the density of the electrification on δS_A, we have

$$\therefore\ 4\pi\sigma_A\,\delta S_A = 1,$$

or $\sigma_A \, \delta S_A =$ total charge (numerically) at either end of the unit tube

$$= \frac{1}{4\pi}.$$

Now the total energy of the charges at the ends of the tube considered is

$$\frac{1}{2}\left(\frac{1}{4\pi}\right)V_A + \frac{1}{2}\left(-\frac{1}{4\pi}\right)V_B = \frac{1}{8\pi}(V_A - V_B),$$

where V_A and V_B are the electrostatic potentials at A and B respectively. The location of this energy is somewhat indeterminate but if the tube lies in an insulating medium which will be polarized, then at least some of the above energy will be distributed throughout that part of the medium through which the tube passes. It is convenient to regard all the energy as being distributed in the portion of the medium (or space) through which the tube passes.

If $\vec{E}$ is the electric field strength at any point on one of the generators of the tube

$$\int_A^B \vec{E} \cdot d\vec{s} = \int_A^B E \, ds = \int_A^B -\frac{\partial V}{\partial s} ds = -\int_A^B dV = V_A - V_B.$$

$$\therefore \text{ Energy in the unit tube} = \int_A^B \frac{E}{8\pi} ds.$$

This energy may be distributed as we please along the tube; the simplest way is to distribute it at a rate $\frac{E}{8\pi}$ per unit length. Then the energy associated with the element of the tube shown will be $\frac{E}{8\pi}\delta s$. Since the volume of the element is $\delta S \, \delta s$, the density of the energy in it is

$$\frac{E}{8\pi}\delta s \div \delta S \, \delta s = \frac{E}{8\pi \, \delta S}.$$

But $D \, \delta S = 1$, so that the above expression for the energy density becomes

$$\frac{DE}{8\pi} = \frac{\kappa E^2}{8\pi}.$$

Thus by assigning an energy density $\frac{\kappa E^2}{8\pi}$ $\left(\text{or } \frac{DE}{8\pi}\right)$ to the medium, the total energy of the field is accounted for completely.

Stresses in dielectric media.—Coulomb's inverse square law has been the basis so far upon which the theory of electrostatics has been developed. It has been assumed that one electric charge exerts a certain definite force upon another electrical charge separated from it, but the mechanism whereby the action occurs has not been discussed or revealed. Faraday founded a school of thought which definitely refused to admit 'action at a distance' as an ultimate explanation of electric phenomena and maintained that some intervening medium capable of acting as a transmitting agent was essential. Both Faraday and Maxwell used the concept of tubes of induction and imagined that each such tube is in a state of tension so that the force of attraction between two unlike charges, for example, was transmitted along the tubes of induction. In order that such tubes should not contract it was found necessary to stipulate that there was a force of repulsion between adjacent tubes. This system of forces gave rise to stresses normal to the surface of the tube and directed outwards. The longitudinal tensile stress and the pressure due to the normal stresses are each equal to $\frac{\mathrm{DE}}{8\pi}$. [For a proof consult earlier editions of this book.]

The capacitance of a concentric spherical condenser.—Let O, Fig. 2·26 (*a*), be the centre of two concentric metal spheres with radii *a* and *b* respectively ($a < b$). If Q is the charge on the inner sphere then when the outer sphere is earthed the charge on its inner surface will be $-$ Q. If κ is the permittivity of the dielectric between the spheres then the field strength* at P, a point on the Gaussian surface of radius r and centre O, is given by

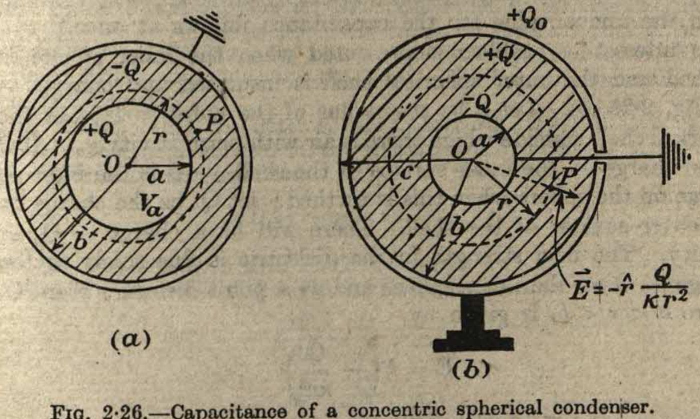

FIG. 2·26.—Capacitance of a concentric spherical condenser.

$$\mathrm{E} = \frac{\mathrm{Q}}{\kappa r^2}, \quad \text{or} \quad \vec{\mathrm{E}} = \hat{r}\frac{\mathrm{Q}}{\kappa r^2}.$$

---

* The expressions given are valid only because the charges are uniformly distributed on spherical surfaces.

Let V_a be the potential of the inner sphere; it may be evaluated as follows. If P_1 is a point on OP produced and such that $OP_1 = r + \delta r$, then the work done against the field per unit charge in taking a small charge from P_1 to P is

$$\frac{Q}{\kappa r^2}(P_1P) = \frac{Q}{\kappa r^2}(-\delta r). \quad [\because PP_1 = \delta r]$$

Since V_a is equal to the work done against the field per unit charge in taking a small charge from the outer to the inner sphere, we have

$$V_a = \int_b^a \frac{Q}{\kappa r^2}(-dr) = \frac{Q}{\kappa}\left[\frac{1}{a} - \frac{1}{b}\right].$$

The capacitance is therefore $\dfrac{\kappa ab}{b-a}$.

[The same result is obtained by writing $E = \dfrac{Q}{\kappa r^2} = -\dfrac{dV}{dr}$. Hence $V = \dfrac{Q}{\kappa}\dfrac{1}{r} + \alpha$, the constant of integration being determined by the boundary condition $r = b$, $V = 0$. Thus $\alpha = -\dfrac{Q}{\kappa}\left(\dfrac{1}{b}\right)$. The potential of the inner sphere is found by putting $r = a$, $V = V_a$, when the above value for the capacitance follows at once.]

An interesting problem is presented when the inner sphere is earthed and the outer spherical shell is insulated and charged—cf. Fig. 2·26 (*b*). Let c be the radius of the outer surface of the spherical shell, outside which there is air with permittivity κ_a. If Q is the charge on the inner surface of the shell, $-$ Q is the induced charge on the sphere when this is earthed; let Q_0 be the charge on the outer surface of the shell. There will be a charge $-Q_0$ at infinity. The field strength in the dielectric is due solely to the charge $-$ Q on the inner sphere and at a point distant r from O, where $a < r < b$, is given by

$$\vec{E} = \hat{r}\left(-\frac{Q}{\kappa r^2}\right).$$

The potential V of the shell is constant throughout and is given by $V - 0 =$ work done per unit charge and against the field in taking a small positive charge from the surface $r = a$ to the surface $r = b$, i.e.

$$V = \int_a^b - E dr = \int_a^b \frac{Q}{\kappa r^2} dr = \frac{Q}{\kappa}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Since charges on spherical conducting surfaces act at external points and points on their own surfaces as if they were concentrated at the centre of the sphere, the potential at a point just outside the outer shell, and therefore of the shell itself, is also given by

$$\frac{Q_0}{\kappa_a c} + \frac{Q}{\kappa_a c} - \frac{Q}{\kappa_a c} = \frac{Q_0}{\kappa_a c}.$$

$$\therefore \ \frac{Q}{\kappa}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q_0}{\kappa_a c} = V.$$

Since the total charge on the shell is $Q_0 + Q$, the capacitance of the system is given by

$$C = \frac{Q_0 + Q}{V} = \frac{Q\frac{\kappa_a c}{\kappa}\left[\frac{(b-a)}{ab}\right] + Q}{V} = \frac{Q\left[\kappa_a \frac{c(b-a)}{\kappa ab} + 1\right]}{Q\frac{(b-a)}{\kappa ab}}$$

$$= \kappa_a c + \frac{\kappa ab}{b-a}.$$

This result can be written down at once if we regard the condenser as consisting of two parts, of capacitances $\frac{\kappa ab}{b-a}$ and $\kappa_a c$, arranged in parallel.

Example.—Obtain a value for the capacitance of a spherical condenser for which $a = 4{\cdot}0$ cm., $b = 6{\cdot}0$ cm. and a spherical shell of permittivity 4·0 and thickness 1·0 cm. surrounds the inner sphere. The remainder of the space between the spheres is filled with an insulator of permittivity 6·0.

It is assumed that the outer sphere is earthed. Then if Q is the charge on the inner sphere, in the dielectric the potential V is, in general, given by the equation

$$D = -\kappa\frac{dV}{dr} = \frac{Q}{r^2},$$

where the symbols have their usual meanings.

Let V_1 be the potential at any point in the region $4 < r < 5$. Then

$$-4\frac{dV_1}{dr} = \frac{Q}{r^2},$$

i.e.

$$4V_1 = \frac{Q}{r} + A_1,$$

where A_1 is an integration constant. At $r = 5$ cm., we have

$$4\Big[V_1\Big]_{r=5} = \frac{Q}{5} + A_1.$$

In the region $5 < r < 6$, we have in a similar way

$$6V_2 = \frac{Q}{r} + A_2.$$

At $r = 6$, $V_2 = 0$, so that $A_2 = -\frac{Q}{6}$.

At $r = 5$, the potential is continuous. Hence

$$\Big[V_2\Big]_{r=5} = \frac{Q}{30} - \frac{Q}{36} = \Big[V_1\Big]_{r=5} = \frac{Q}{20} + \frac{A_1}{4}.$$

$$\therefore\ A_1 = -\frac{8}{45}Q.$$

If V_a is the potential of the inner sphere, then

$$V_a = \Big[V_1\Big]_{r=4} = \frac{Q}{16} - \frac{2Q}{45} = \frac{13}{16 \times 45}Q.$$

$$C = \frac{Q}{V_a} = 55{\cdot}4 \text{ e.s.u.}$$

Example.—It may be assumed that the charge density, ρ, in a spherical distribution of electricity is proportional to $\exp\left(-\frac{r}{r_0}\right)$ at a radial distance r from the centre of the distribution and r_0 is a characteristic length. If the total charge is $Z\varepsilon$, find expressions for the field strength and electric potential at a point distant R from the centre of the system.

[N.B. The above may be regarded as the distribution of charge within an atomic nucleus, where $\varepsilon = -e$, the protonic charge; the charge density falls off continuously and there is no way of getting outside the distribution. There is no discontinuity at $r = r_0$, although this could be taken to be the radius of the nucleus. The expression for the charge density cannot give the true distribution of electricity within the nucleus since the expression for ρ has a cusp at the origin.]

If $Q = Q(R)$ is the charge enclosed by a sphere of radius R and we write $\rho = A \exp\left(-\frac{r}{r_0}\right)$, where A is a constant, then

$$Q(R) = \int_0^R 4\pi\rho r^2\, dr = 4\pi A\int_0^R r^2 \exp\left(-\frac{r}{r_0}\right) dr$$
$$= 4\pi AI \text{ (say).}$$

To carry out the necessary integration, let $r = xr_0$. Then $dr = r_0\, dx$ and we have

$$I = \int_0^{\frac{R}{r_0}} r_0^3 x^2 \exp(-x)dx$$

$$= r_0^3\left[x^2(-1)\exp(-x) + 2\int x \exp(-x)\, dx\right]_0^{\frac{R}{r_0}}$$

$$= r_0^3\left[-x^2\exp(-x) + 2\{x(-1)\exp(-x) - \exp(-x)\}\right]_0^{\frac{R}{r_0}}$$

$$\therefore Q(R) = 4\pi Ar_0^3\left[-\left(\frac{R}{r_0}\right)^2 \exp\left(-\frac{R}{r_0}\right) - 2\frac{R}{r_0}\exp\left(-\frac{R}{r_0}\right) - 2\exp\left(-\frac{R}{r_0}\right) + 2\right]$$

$$= 8\pi Ar_0^3\left[1 - \exp\left(-\frac{R}{r_0}\right)\left\{1 + \frac{R}{r_0} + \frac{1}{2}\left(\frac{R}{r_0}\right)^2\right\}\right].$$

When $R \rightarrow \infty$, $Q(R) \rightarrow Z\varepsilon$, and we have

$$Z\varepsilon = 8\pi A r_0{}^3, \quad \text{i.e. } A = \frac{Z\varepsilon}{8\pi r_0{}^3}.$$

Now at $r = R$, the field is due entirely to the charge within the sphere of radius R. Hence, if κ is the permittivity of the medium in which the charges are situated,

$$E_{r=R} = \frac{Q(R)}{\kappa R^2} = \frac{Z\varepsilon}{\kappa R^2}\left[1 - \exp\left(-\frac{R}{r_0}\right)\left\{1 + \frac{R}{r_0} + \frac{1}{2}\left(\frac{R}{r_0}\right)^2\right\}\right].$$

Since, in general, $E = -\dfrac{dV}{dr}$, we have

$$V_{r=R} = \frac{Q(R)}{\kappa R} + B,$$

where B is an integration constant. But when $R \rightarrow \infty$, $V = 0$, so that $B = 0$. Hence

$$V_{r=R} = \frac{Z\varepsilon}{\kappa R}\left[1 - \exp\left(-\frac{R}{r_0}\right)\left\{1 + \frac{R}{r_0} + \frac{1}{2}\left(\frac{R}{r_0}\right)^2\right\}\right].$$

The capacitance per unit length of a long coaxial cylindrical condenser.—Let Fig. 2·26 (*a*), p. 53, represent the cross-section of a long coaxial cylindrical condenser, the outer shell being earthed while the inner cylinder is insulated and carries a charge $\bar{q}$ per unit length.

The field strength at P is $\dfrac{2\bar{q}}{\kappa r}$,

i.e.

$$-\frac{dV}{dr} = \frac{2}{\kappa}\frac{\bar{q}}{r}.$$

$$\therefore V = -\frac{2\bar{q}}{\kappa}\ln r + A,$$

where A is an integration constant.

Now at $r = b$, $V = 0$.

$$\therefore A = \frac{2\bar{q}}{\kappa}\ln b,$$

and at $r = a$, $V = V_a$, so that

$$V_a = -\frac{2\bar{q}}{\kappa}\ln a + \frac{2\bar{q}}{\kappa}\ln b.$$

$$\therefore \text{Capacitance per unit length} = \frac{\kappa}{2\ln\left(\dfrac{b}{a}\right)} = \bar{C} \text{ (say).}$$

The field strength within a charged cylindrical condenser.—Let V_a and V_b be the potentials of the conductors forming the condenser. Then, with the usual notation and assuming that the condenser is very long,

$$E = \frac{2\bar{q}}{\kappa r} = -\frac{dV}{dr}, \quad \text{or} \quad V = -\frac{2\bar{q}}{\kappa}\ln r + A,$$

where A is an integration constant. Hence

$$V_a - V_b = \frac{2\bar{q}}{\kappa}\ln\left(\frac{b}{a}\right).$$

Eliminating $\frac{2\bar{q}}{\kappa}$ from the expression for V, we obtain

$$V = -\frac{V_a - V_b}{\ln\left(\frac{b}{a}\right)}\ln r + A,$$

so that

$$E = -\frac{dV}{dr} = \frac{V_a - V_b}{\ln\left(\frac{b}{a}\right)}\cdot\frac{1}{r}.$$

This expression gives the field strength in terms of the potential difference across the condenser and known distances ; it is independent of the permittivity of the intervening insulator and, of course, of any charge on the outer surface of the larger cylinder.

A variable cylindrical air condenser.—To the ends of a wooden base there are attached two ebonite uprights carrying a long ebonite rod AB, Fig. 2·27 (*a*). This rod supports a brass tube M, the outer radius of which is *a*. N is a coaxial brass cylinder carried on two metal supports to one of which is fixed a spring S making contact with an earthed metal rail R. The inside radius of the tube N is *b*. The tube M may be raised to any desired potential difference by connecting the source of potential to the terminal T_2 which is in metallic connexion with M. A cross-section of the apparatus at N is shown in Fig. 2·27 (*b*).

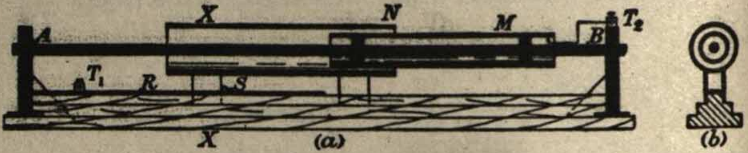

FIG. 2·27.—A variable cylindrical air condenser.

The absolute capacitance of such a condenser is unknown but by moving N a distance *l* to the right or left the capacitance may be respectively increased or decreased by an amount

$$\frac{\kappa l}{2\ln\left(\frac{b}{a}\right)},$$

where, for air, the numerical value of the permittivity κ may be taken as unity.

The capacitance of two parallel wires in air and carrying equal but opposite charges.—It will be assumed that the diameter, viz. $2a$, of each wire is small compared with their distance apart $2h$. Let $\bar{q}$ and $-\bar{q}$ be the charges per unit length on the wires. Consider the positively charged wire first. At a point distant r from its axis, and provided r is not too large, cf. p. 48, the field strength is $\frac{2\bar{q}}{\kappa_a r}$, where κ_a is the permittivity of air, i.e.

$$\frac{dV}{dr} = -\frac{2\bar{q}}{\kappa_a r},$$

where V is the potential at the point considered.

$$\therefore \kappa_a V = -2\bar{q} \ln r + A,$$

where A is a constant which cannot be determined.

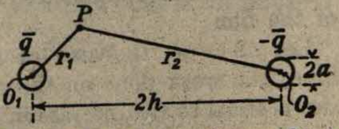

FIG. 2·28.—The capacitance of two parallel wires in air with equal but opposite charges per unit length.

Now consider the potential at a point P, cf. Fig. 2·28, in the field of the two wires. If O_1 and O_2 are the points in which the axes of the wires cut the plane of the diagram, then writing $O_1P = r_1$ and $O_2P = r_2$, we have

$$\kappa_a V = -2\bar{q} \ln r_1 + A_1 + 2\bar{q} \ln r_2 + A_2,$$

where A_1 and A_2 are the relevant constants. Now at the point where $r_1 = r_2 = h$, the potential is zero, for such a point is equidistant from the two wires, i.e. $A_1 + A_2 = 0$.

$$\therefore V = \frac{2\bar{q}}{\kappa_a} \ln \left(\frac{r_2}{r_1}\right).$$

When P is close to the first wire, $r_1 \rightarrow a$ and $r_2 \rightarrow 2h$.

$$\therefore V_{r=a} = \frac{2\bar{q}}{\kappa_a} \ln \left(\frac{2h}{a}\right).$$

$\therefore$ Capacitance per unit length of the system is given by

$$\bar{C} = \frac{\bar{q}}{V_{r=a}} = \frac{\kappa_a}{2 \ln \left(\frac{2h}{a}\right)}.$$

[N.B. The above formula is true for a telegraph wire lying at a height h above the surface of the earth (considered as flat).]

An electrified soap bubble.—Let r be the radius of a soap bubble, Fig. 2·29, carrying a surface distribution of electricity of density σ. This electricity will be on the outer surface only of the bubble since the solution is a conductor. Let O be the centre of the bubble. Consider a small element of the surface formed by neighbouring lines intersecting orthogonally on it. Let opposite pairs of sides of this element have lengths a and b respectively. Let P be the pressure outside the bubble; $P + p$ is the pressure inside, where p is to be determined. Let γ be the surface tension of the solution and let the bubble expand so that its radius becomes

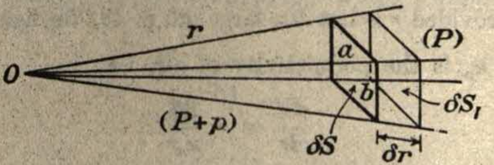

FIG. 2·29.—An electrified soap bubble.

$r + \delta r$, heat being supplied so that there is no change in temperature of the film.

Let $\delta S = ab$, the area of one side of the element considered. Since the work done on the system while the bubble increases in size is equal to the increase in free surface energy, cf. Vol. I, p. 455,

$$(P + p)\delta S\, \delta r + \frac{2\pi\sigma^2}{\kappa}\delta S\, \delta r - P\, \delta S\, \delta r = 2\gamma(\delta S_1 - \delta S),$$

where κ is the permittivity of the surrounding gaseous medium, δS_1 is the value of δS when r becomes $r + \delta r$, and the factor 2 is introduced because there are two surfaces to a film.

$$\therefore\ ab\, p\, \delta r + \frac{2\pi\sigma^2}{\kappa} ab\, \delta r = 2\gamma\left[a\left(1 + \frac{\delta r}{r}\right)b\left(1 + \frac{\delta r}{r}\right) - ab\right]$$

$$= 2\gamma\, ab\left(\frac{\delta r}{r} + \frac{\delta r}{r}\right).$$

$$\therefore\ p + \frac{2\pi\sigma^2}{\kappa} = \frac{4\gamma}{r}.$$

Poisson's and Laplace's equations for an electric field which is everywhere parallel to a fixed direction.—It will be assumed that the lines of electric induction in a certain field are everywhere parallel to the x-axis. Consider a small element of volume in this field, δx being the length of the element and δS its constant cross-sectional area—cf. Fig. 2·30.

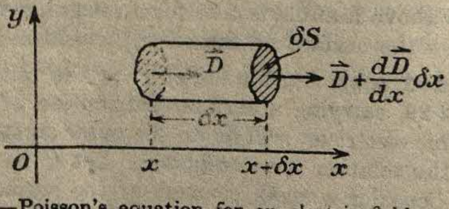

FIG. 2·30.—Poisson's equation for an electric field parallel to a fixed direction.

Let $\vec{D}$ be the electric induction parallel to the x-axis at all points in a plane at distance x from the plane through the origin of co-ordinates. At points in the plane defined by $x + \delta x$ the value of the electric induction will be $\vec{D} + \frac{d\vec{D}}{dx}\delta x$.

Consider the flux of electric induction across the faces of the element of volume selected. Since $\vec{D}$ is always parallel to Ox, the only contributions to the flux arise from the flux across the plane surfaces of the element. For the x-face the flux is

$$\vec{D}\cdot\hat{n}\,\delta S = D\,(\cos\pi)\,\delta S = -\,D\,\delta S,$$

since the outward drawn normal at this face makes an angle π with $\vec{D}$. Across the $(x + \delta x)$ face the flux is

$$\left(\vec{D} + \frac{d\vec{D}}{dx}\delta x\right)\cdot\hat{n}\,\delta S = \left(D + \frac{dD}{dx}\delta x\right)\delta S.$$

$\therefore$ Flux across the faces of the element is

$$-\,D\,\delta S + \left(D + \frac{dD}{dx}\delta x\right)\delta S = \frac{dD}{dx}\,\delta x\,\delta S,$$

and this by Gauss' theorem is $4\pi\rho\,\delta x\,\delta S$, where ρ is the volume density of the electricity in the element considered. Hence

$$\frac{dD}{dx} = 4\pi\rho,$$

i.e. $$\frac{d}{dx}(\kappa E) = 4\pi\rho. \quad [\kappa = \text{permittivity.}]$$

Since $E = -\frac{dV}{dx}$, the above equation may be written

$$\frac{d}{dx}\left(\kappa\frac{dV}{dx}\right) = -\,4\pi\rho.$$

This is ***Poisson's equation*** for the particular field considered. If $\rho = 0$, we have

$$\frac{d}{dx}\left(\kappa\frac{dV}{dx}\right) = 0,$$

which is ***Laplace's equation*** for the field considered. Its usefulness will appear in the sequel especially if the medium is such that κ is constant, for then Laplace's equation becomes

$$\frac{d^2V}{dx^2} = 0.$$

The capacitance of a parallel plate condenser.—We shall consider unit area of an infinite parallel plate condenser in order that the lines of induction shall be everywhere perpendicular to either plate, i.e. all irregularities caused by the edges of an actual condenser are ignored.

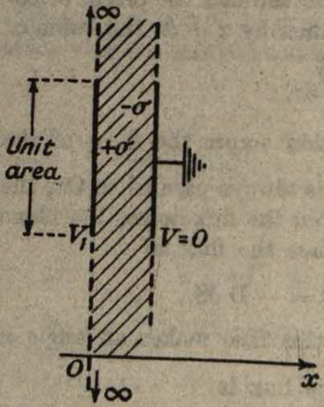

FIG. 2·31.—The capacitance of a parallel plate condenser.

Let the plane of the plate which is insulated and at potential V_1 contain O, the origin of a system of rectangular coordinates; the other plate is earthed as indicated in Fig. 2·31. Then, with the usual notation,

$$\vec{D} = \hat{\imath}D + \hat{\jmath}0 + \hat{k}0.$$

Suppose that the medium between the plates is isotropic and has a constant permittivity κ. Then since no free charges are present and the field is parallel to the x-axis, $\dfrac{d^2V}{dx^2} = 0$.

$$\therefore \frac{dV}{dx} = M, \quad \text{and} \quad V = Mx + N,$$

where M and N are integration constants to be determined by the boundary conditions.

Now at $x = 0$, $V = V_1$.

$$\therefore N = V_1.$$

And at $x = t$, $V = 0$.

$$\therefore 0 = Mt + V_1, \text{ i.e. } M = -\frac{V_1}{t}.$$

$$\therefore V = V_1 - \frac{V_1}{t}x.$$

At $x = 0$ we also have, by Coulomb's theorem,

$$\Big[E\Big]_{x=0} = \Big[-\frac{dV}{dx}\Big]_{x=0} = \frac{4\pi\sigma}{\kappa}.$$

$$\therefore \frac{V_1}{t} = \frac{4\pi\sigma}{\kappa}, \quad \text{or} \quad \frac{\sigma}{V_1} = \frac{\kappa}{4\pi t},$$

which is the capacitance per unit area of the condenser considered.

If each plate has an area A, the capacitance C is given by

$$C = \frac{\kappa A}{4\pi t},$$

in so far as 'fringing effects' are neglected.

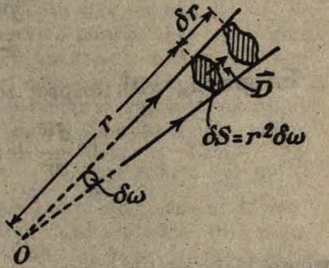

FIG. 2·32.—Poisson's equation for a radial field.

Poisson's and Laplace's equations for an electric field whose lines of induction are all straight and pass through a common point.—This is a so-called radial field. Consider a small tube of induction emanating from the common point O, Fig. 2·32; let $\delta\omega$ be the solid angle of this tube and consider the portion of the tube cut off by spheres having O as common centre and radii r and $r + \delta r$. If $\vec{D}$ is the electric induction at points on the sphere of radius r, then the flux of electric induction across the surfaces of the element considered is

$$-\, Dr^2\delta\omega + \left[Dr^2 + \frac{d}{dr}(Dr^2)\delta r\right]\delta\omega = \left[\frac{d}{dr}(Dr^2)\delta r\right]\delta\omega,$$

and this, by Gauss' theorem, is $4\pi\rho(r^2\,\delta r\,\delta\omega)$, where ρ is the density of the space distribution of electricity and $(r^2\delta\omega)\delta r$ is the volume of the element. Thus

$$\frac{1}{r^2}\frac{d}{dr}(Dr^2) = \left[\frac{dD}{dr} + \frac{2}{r}D\right] = 4\pi\rho,$$

and since

$$D = \kappa E = -\,\kappa\frac{dV}{dr} \qquad \left[\text{or}\quad \vec{D} = -\,\hat{r}\,\kappa\frac{dV}{dr}\right],$$

$$\therefore\ \frac{d}{dr}\left(\kappa\frac{dV}{dr}\right) + \frac{2}{r}\kappa\frac{dV}{dr} = -\,4\pi\rho.$$

This is *Poisson's equation* for this field. If $\rho = 0$, we have

$$\frac{d}{dr}\left(\kappa\frac{dV}{dr}\right) + \frac{2}{r}\kappa\frac{dV}{dr} = 0.$$

This is *Laplace's equation* for this particular field.

If, in addition, the permittivity is constant, the two equations become respectively

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = -\,\frac{4\pi\rho}{\kappa}, \quad \text{and} \quad \frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

The capacitance of a concentric spherical condenser with its outer shell earthed.—Since the induction $\vec{D}$ in the region between the charged surfaces of such a condenser satisfies the conditions given above, we have

$$\frac{\dfrac{d^2V}{dr^2}}{\dfrac{dV}{dr}} + \frac{2}{r} = 0.$$

Integrating with respect to r we get

$$\ln \frac{dV}{dr} + 2 \ln r = \text{constant}.$$

$$\therefore \ \frac{dV}{dr} = \frac{M}{r^2},$$

where M is an integration constant. Integrating again we get

$$V = -\frac{M}{r} + N,$$

where N is another integration constant.

For the condenser shown in Fig. 2·26 (*a*), p. 53, we have at $r = a$, $V = V_a$ (say) and at $r = b$, $V = 0$. Thus

$$V_a = -\frac{M}{a} + N, \quad \text{and} \quad 0 = -\frac{M}{b} + N.$$

Hence

$$M = \frac{V_a ab}{a - b}, \quad \text{and} \quad N = \frac{V_a a}{a - b}.$$

$$\therefore \ V = \frac{V_a ab}{b - a}\left[\frac{1}{r} - \frac{1}{b}\right].$$

Let σ_a be the surface density of the electricity on the inner surface. Then

$$4\pi\sigma_a = \kappa\Big[E\Big]_{r=a} = \kappa\left[-\frac{dV}{dr}\right]_{r=a} = -\kappa\frac{M}{a^2} = \frac{\kappa V_a b}{a(b - a)}.$$

$\therefore$ Charge on inner sphere $= 4\pi a^2\sigma_a = \kappa V_a \dfrac{ab}{b - a} = Q$, say.

$$\text{Capacitance} = \frac{Q}{V_a} = \frac{\kappa ab}{b - a}.$$

EXAMPLES II

[Where necessary, assume the permittivity of free space and of air to be κ_0 and κ_a respectively; the numerical value of each may be taken to be unity.]

2·01. Two small pith balls each of mass m are suspended in a vacuum from a point by two strings each of length l. When each ball receives a charge q e.s.u. it is observed that each string makes an angle ϕ with the vertical. Show that $q = 2l \sin \phi\ (\kappa_0 mg \tan \phi)^{\frac{1}{2}}$.

2·02. A small particle of mass m is given a charge $+ q$ e.s.u. and then projected with an initial velocity $\hat{\imath}u$ cm.sec.$^{-1}$ into a uniform electric field $- \hat{\imath}\mathrm{E}$. Obtain the differential equation for its motion and use it to find (a) the time taken for the particle to come to rest and (b) the distance traversed in that time. [Assume that the motion takes place in a vacuum.] $\left[\frac{mu}{q\mathrm{E}}\text{ sec., } \frac{1}{2}\frac{mu^2}{q\mathrm{E}}\text{ cm.}\right]$

2·03. A condenser consists of two parallel circular plates of 12 cm. diameter, separated by an air-gap of 0·10 cm. Calculate the work done in separating the plates to a distance of 0·50 cm. when (a) the plates are connected to a 150-volt battery; (b) the plates are charged from the same battery and then insulated from it before the separation occurs. [9 erg., 45 erg.]

2·04. A condenser has a capacitance of 10 μF. and is charged to a potential of 150 V. Calculate the charge in coulombs and in c.g.s. e.s.u. A second condenser has a capacitance of 2·0 μF. and is charged to a potential of 300 V. If, after charging, the two condensers are connected in parallel by wires of negligible capacitance, how much energy is dissipated? [$1·5 \times 10^{-3}$ coul., $4·5 \times 10^{6}$ e.s.u., 0·184 joule.]

2·05. Obtain, from first principles, an expression for the energy of a charged condenser.

A spherical oil drop, radius 10^{-4} cm. has on it at a certain time a total charge of 40 electrons. Calculate the energy that would be required to place an additional electron on the drop.

[Assume 1 faraday = 96522 coulomb., Avogadro's constant, N, is given by

$$\mathrm{N} = 6·023 \times 10^{23} \text{ atoms.(gm.-atom.)}^{-1}$$

or

$$\mathrm{N} = 6·023 \times 10^{23} \text{ molecules.mole}^{-1}.]$$

[$9·4 \times 10^{-14}$ erg.]

2·06. Derive expressions for the stress at the surface of a charged conductor and for the energy density in an electric field.

Two condensers of capacitances 0·10 μF. and 0·010 μF. are connected in series across a potential difference of 22 volts. If they are then insulated and connected in parallel, what is the loss of energy in ergs? [14·7 erg.]

2·07. Define *electric field strength* and *potential*. Establish the relation between them.

Two plane electrodes, 0·50 cm. apart, within an electron tube are maintained at a difference of potential of 90 V. Estimate the strength of the electric field in the interelectrode space. How long will it take for an electron, starting from rest at one electrode, to reach the other? [Assume that the charge on an electron is $- 4·80 \times 10^{-10}$ e.s.u. and that its mass is $9·11 \times 10^{-28}$ gm.] [180 V. cm.$^{-1}$, $1·78 \times 10^{-9}$ sec.]

2·08. Two insulated metal spheres of radius 4·0 cm. and 10·0 cm. carry equal charges of 4·0 e.s.u. Neglecting any mutual effect between the spheres, prove that if they are connected by a fine wire the loss in energy would be sufficient to raise a mass of 0·5 mgm. through a height of about 1 cm.

2·09. Derive an expression for the capacitance per unit area of a large parallel plate condenser which has half the distance between its plates occupied by a slab of material of permittivity κ and the remaining half by air. $[\{2\pi t(\kappa^{-1} + \kappa_a{}^{-1})\}^{-1}.]$

2·10. Two large parallel plates at a distance t apart in air are maintained at potentials 0 and V e.s.u. respectively. Find the stress at the surface of the plates and examine the effect of interposing a slab of insulating material, of uniform thickness αt $(0 < \alpha < 1)$ and permittivity κ, between the plates. $\left[\frac{\kappa_a V^2}{8\pi t^2}\left\{\frac{\kappa}{\kappa(1-\alpha)+\alpha\kappa_a}\right\}^2\right]$

2·11. A parallel plate condenser is made up of fifteen metal plates each 6 cm. by 4 cm., separated by sheets of mica of permittivity 6 and constant thickness 0·15 mm. Calculate a value for its capacitance in microfarads. The condenser is charged to 240 V. and then connected in parallel with an uncharged condenser of capacitance 0·002 μF. How much charge flows into this latter condenser and what is the voltage drop across it ? [0·0119 μF., 0·41 μC, 205·3 V.]

2·12. An insulated conductor, initially free from charge, is charged by repeated contacts with a plate which, after each contact, is given a charge Q from an electrophorus. If q is the charge on the conductor after the first operation, prove that the maximum charge which can be given to the conductor in this way is $Qq(Q - q)^{-1}$.

2·13. Three thin conducting spherical shells of radii 10, 20, and 40 cm. respectively are mounted concentrically in a vacuum, the two inner ones being insulated and the outer one earthed. The innermost shell carries a charge of 60 e.s.u. and the middle one a charge of 100 e.s.u. Calculate (*a*) the charges on each surface in the system, (*b*) the potentials of the two inner conductors.

[(*a*) 60, − 60, 160, − 160, 0 (*b*) 11, 8 e.s.u.]

2·14. A condenser is to have a capacitance of 0·05 μF. It is made up of sheets of tin-foil separated by sheets of mica. There are 30 sheets of mica and the area of one side of each piece of tin foil is 10 cm.^2. If the relative permittivity of mica is 6·5, determine an approximate thickness for the mica which must be used. [0·033 cm.]

2·15. Experiments show that in air at normal pressure the maximum space rate of fall in potential is 100 volt.metre.^{-1}. Calculate values for the maximum charge and potential on the surface of the earth, if this can be regarded as an isolated conducting sphere surrounded by air at normal pressure. [Radius = $6·37 \times 10^8$ cm.]

[$4·5 \times 10^5$ coulomb., 7·1 kV.]

2·16. A spark passes in air when the potential gradient at the surface of a charged conductor is 30 kV.cm.^{-1}. What must be the radius of an isolated metal sphere which may be raised to a potential of 2×10^6 V. before sparking into the surrounding air takes place ? How much energy will be stored just before a spark passes ?

[67 cm., $0·33 \times 10^6$ erg.]

2·17. Obtain expressions for the capacitance of a spherical condenser (*a*) when the outer sphere is earthed, (*b*) when the inner sphere is earthed. Assuming that air becomes a conductor when the electric field in it exceeds 100 e.s.u., calculate the maximum quantity of electricity that may be stored on an isolated sphere of radius 20 cm.

[4×10^4 e.s.u.]

2·18. Three concentric hollow conducting spheres have radii a, b, c, and $a < b < c$. The outer sphere is earthed. The inner and outer spheres are connected together by a fine wire to form one 'plate' of a condenser; the other sphere is the second 'plate' of the condenser. Show that its capacitance is $\frac{\kappa_a b^2(c - a)}{(b - a)(c - b)}$, if a, b and c are all measured in cm. and the spheres are in air.

2·19. Three concentric thin spherical shells made of metal have radii, a, b, c where $a < b < c$. The 'middle' sphere is earthed by means of a fine wire passing through a small hole in the outer sphere; the inner and outer spheres are connected together by a fine wire passing through a small hole in the sphere of radius b. Show that the capacitance of the system, assumed to be in air, is

$$\kappa_a\left[\frac{ba}{b - a} + \frac{c^2}{c - b}\right].$$

If the above system is replaced by one consisting of long coaxial cylinders show that the capacitance per unit length cannot be found.

2·20. A, B and C, D are the plates of two condensers of capacitances C_1 and C_2 respectively. The condensers are arranged in series by connecting B and C together. The system is charged so that D is at a potential V with respect to earth. The condenser C_2 is isolated from C_1 and then connected in parallel with it. Show that there is a loss of energy equal to

$$\frac{1}{2}\frac{(C_1 - C_2)^2C_1C_2}{(C_1 + C_2)^3}V^2.$$

2·21. Find the capacitance per unit length of a cylindrical condenser of which the conductors have radii 2·5 cm. and 4·5 cm. respectively, and the dielectric consists of two layers whose cylinder of contact is 3·5 cm. in radius, the inner layer having a relative permittivity 4·0 and the outer layer a relative permittivity 6·0. [$4{\cdot}4 \times 10^{-6}$ μF.]

2·22. A condenser consists of two coaxial cylinders of radii 5·0 and 10·0 cm. separated by a medium of permittivity 3·0. Calculate the pull per unit area on each of the charged surfaces when the potential difference between the cylinders is 200 volt.

[$4{\cdot}42 \times 10^{-3}$ dyne.cm.$^{-2}$ (inner); $1{\cdot}10 \times 10^{-3}$ dyne.cm.$^{-2}$ (outer).]

2·23. State Gauss' theorem. Apply it to find (*a*) the electric field strength at a given point between two long coaxial cylindrical conductors in air which differ in potential by V e.s.u., and (*b*) the capacitance per unit length of the system.

Calculate the capacitance per kilometre of a concentric cable, the core of which has a diameter of 3·0 mm., the metallic sheath an inner diameter of 9·0 mm., and the separating insulator a relative permittivity of 4·5. [0·227 μF.]

2·24. A coaxial cable consists of a metallic core 1·0 cm. in diameter surrounded by an earthed metallic sheath 4·0 cm. in diameter, the space between them being filled with a medium of dielectric constant 2·0. Calculate (*a*) the capacitance per 100 metres of cable, (*b*) the electric field strength at the surface of the core when it is maintained at 5000 volts with respect to earth. (G)

[(*a*) $8{\cdot}02 \times 10^{-3}$ μF., 7·2 kV. cm.$^{-1}$.]

2·25. Use Gauss' theorem to find an expression for the flux of electric induction across a plane circular area of radius a when a point charge q

lies at a distance z from the plane and on an axis normal to the area. Verify your result by another method. $\left[2\pi q\left\{1 - \frac{z}{(a^2 + z^2)^{\frac{1}{2}}}\right\}.\right]$

2·26. An air-condenser consists of two coaxial cylinders fitted with guard rings. The inner and outer diameters of the outer and inner cylinders are 40 cm. and 30 cm. respectively. If the axial length is 1·2 metres calculate a value for the capacitance of the condenser. What is the energy of the condenser when the potential difference across it is 1000 V. ? [$2{\cdot}32 \times 10^{-4}$ μF., $1{\cdot}16 \times 10^{-4}$ joule.]

2·27. A soap bubble has a radius of 1·2 cm. when charged to a potential of 12,000 V. Calculate the excess pressure inside the bubble, the surface tension of the soap solution being 24 dyne.cm.$^{-1}$. [35·9 dyne.cm.$^{-2}$.]

2·28. What must be the total charge on the surface of a soap bubble of radius 2 cm. in order that when the air inside the bubble is put into direct communication with the atmosphere the size of the bubble shall remain constant ? Assume that the surface tension of soap solution is 8π dyne.cm.$^{-1}$. [142 e.s.u.]

2·29. A spherical soap bubble, surface tension γ, is blown at the end of a thin metal tube and given a charge Q. Show that when the pressure inside the bubble is equal to that of the surrounding atmosphere, its radius a is determined by the equation $Q^2 = 32\pi\kappa_a\gamma a^3$.

2·30. An isolated spherical soap bubble has a radius a and the pressure inside is atmospheric. If γ is the surface tension of the soap solution show that the bubble has a potential $4\left[\sqrt{\frac{2\pi\gamma a}{\kappa_a}}\right]$ e.s.u.

2·31. On the basis of the inverse square law of force between point charges, calculate (a) the potential at a point in the field of a system of point charges, (b) the magnitude and direction of the electric field at a point (r, θ) at a considerable distance from an electric dipole of moment μ, r being the distance of the point from the centre of the dipole and θ the angle which the radius vector makes with the axis of the dipole. Show also that the equation to the line of force through this point is $Ar = \sin^2\theta$, where A is a constant. (G)

2·32. Calculate the final charges which occur on the positive plates of the three condensers, shown in Fig. Ex. 2·32 below after the switch S is closed. It may be assumed that the condensers are initially without charge.

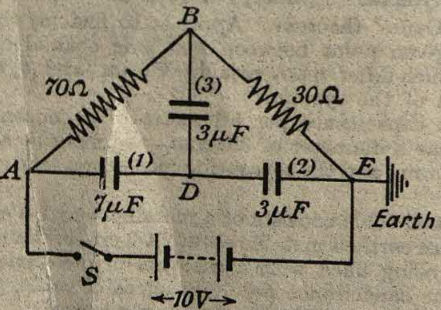

FIG. Ex. 2·32.

If the condensers (1) and (2) are interchanged what is the charge on (3) when S is closed?

[$Q_1 = 27{\cdot}5\ \mu$C., $Q_2 = 18{\cdot}2\ \mu$C., $Q_3 = 9{\cdot}3\mu$C.; zero.]

2·33. An electric layer of uniform density, $-\sigma$, is distributed over a spherical surface of radius a, while a second layer of density $+\sigma$ is distributed over a concentric spherical surface of radius $a + t$, where $\frac{t}{a} \longrightarrow 0$. Assuming that the system lies in free space, prove that the potential is zero at all external points and $4\pi\sigma t \div \varkappa_0$ at all internal points.

2·34. A condenser consists of a conducting sphere surrounded by a thin earthed metal shell. The centres are at a distance z apart. If the capacitance of the system is $\phi(z)$, where $\phi(z)$ is a known function, show that if the charge on the inner sphere is Q, the force tending to increase z is $\frac{1}{2}\frac{Q^2\phi'(z)}{[\phi(z)]^2}$. Also write down the energy of the system in terms of the difference of potential between the two components and again obtain a value for the force.

2·35. (*a*) A metal sphere of radius a is surrounded by a thin metal concentric shell of radius b and these have charges Q and 3Q respectively. Show that the energy of the system is $\frac{1}{2}Q^2\frac{(b + 15a)}{\kappa_a ab}$.

(*b*) If the charges in (*a*) are changed to mQ and nQ respectively what does the expression for the energy become?

(*c*) If the charges are unaltered but the radius of the outer surface of the shell is c, obtain an expression for the energy of the system.

$$\left[(b)\ \frac{1}{2}\frac{Q^2}{\kappa_a}\left\{\frac{m^2}{a} + \frac{n^2}{b} + \frac{2mn}{b}\right\};\ (c)\ \frac{1}{2}\frac{Q^2}{\kappa_a}\left\{\frac{(b - a)}{ab} + \frac{16}{c}\right\}\right].$$

2·36. A point charge q is placed at the origin in a medium for which the permittivity is $\kappa_0(1 + \alpha r^{-1})$, where κ_0 and α are constants. Show that the potential at a point distant r from the point charge q is

$$\frac{q}{\kappa_0\alpha}\ln(1 + \alpha r^{-1}).$$

2·37. C_1 and C_2 are two parallel plate condensers in air. Each plate has an area S. Initially the plates of C_1 are separated by a distance x and charged to a potential difference V; the plates of C_2 are uncharged and separated by a distance y. The plates of C_1 are now separated so that its capacitance is reduced to $\frac{1}{N}$th its initial value: C_1 and C_2 are then connected in parallel. Show that if edge effects are neglected the gain in energy of the system is

$$\frac{SV^2}{8\pi x}\left[\frac{(N - 1)y - Nx}{Nx + y}\right].$$

2·38. S_1 and S_2 are two metal cylinders in air arranged coaxially. The outer radius of S_1 is a, while the inner radius of S_2 is b. S_1 is maintained at a potential V while S_2 is earthed. Prove that when a length x of S_1 lies within S_2 the energy of that portion of the condenser is

$$\frac{\kappa_a xV^2}{4\ln\left(\frac{b}{a}\right)} + \alpha,$$

where α is an end correction. Hence show that an axial force of magnitude $\frac{1}{4}\frac{\kappa_a V^2}{\ln\left(\frac{b}{a}\right)}$ tends to draw S_1 into S_2.

[N.B.—The expression for the capacitance is in error since end-effects have been neglected; hence α cannot be found, but if the end-effect is independent of x, the expression for the force is correct.]

2·39. Two condensers A and B each consist of a pair of parallel circular plates of radius r. Initially the plates of A are separated by a distance α and charged to a potential difference V, while the plates of B are at a distance β apart and are uncharged. The plates of A are then moved to a distance $m\alpha$ and the two condensers then connected in parallel. Prove that the gain in potential energy of the system is

$$\frac{\kappa_a r^2 V^2[(m-1)\beta - m\alpha]}{8\alpha(m\alpha+\beta)}.$$

2·40. If r is the least distance of a point P from an infinite straight line along which electricity is distributed with constant density $\bar{q}$, show that the field strength at P is $\frac{2\bar{q}}{\kappa r}$, where κ is the constant permittivity of the medium surrounding the distribution. Show that the corresponding potential is $A - \frac{2\bar{q}}{\kappa}\ln r$, where A is an indeterminate constant.

2·41. Show that in an electrostatic problem in which the potential depends solely upon the radial distance r, the differential equation for V is

$$\frac{1}{r^2}\frac{d}{dr}\left(\kappa r^2\frac{dV}{dr}\right) = -4\pi\rho.$$

A charge q lies at the origin in a medium whose permittivity is $1 + \frac{\beta}{r}$, where β is independent of r. Show that the potential is

$$V = \frac{q}{\beta}\ln\left(1 + \frac{\beta}{r}\right).$$

2·42. By means of Gauss' theorem show that the electric potential at a point distant r from the axis of two long coaxial metal cylinders, the inner one of which is earthed while the outer one is maintained at a potential V_0, is given by the relation

$$V = V_0\left[\left\{\log\left(\frac{r}{a}\right)\right\} \div \left\{\log\left(\frac{b}{a}\right)\right\}\right],$$

where a and b are the radii of the cylinders and $b > r > a$. A vacuum diode consists of a cylindrical cathode 0·04 cm. in radius, mounted coaxially within a cylindrical anode 0·40 cm. in radius. The potential of the anode is 200 volts above that of the cathode. If an electron starts from rest at the surface of the cathode, obtain a value for its velocity when it has travelled 0·16 cm. towards the anode. Space charge effects may be assumed negligible and the specific charge $\left(\frac{e}{m}\right)$ for an electron may be taken as $-5{\cdot}27 \times 10^{17}$ e.s.u. gm.$^{-1}$.

[$7{\cdot}02 \times 10^8$ cm.sec.$^{-1}$.]

CHAPTER III

THE SUPERPOSITION OF ELECTRICAL EFFECTS. SOME ELECTROSTATIC INSTRUMENTS AND MEASUREMENTS

The coefficients of potential, capacitance and induction.—Suppose that we have two conductors fixed in position and initially uncharged. Let us call them (1) and (2). Suppose that the result of placing unit charge on conductor (1) and leaving the other uncharged is to raise the potentials of the conductors to p_{11} and p_{21}, respectively; these coefficients do not depend on the potentials or on the charges but simply upon the size, shape and position of the two conductors, and on the intervening medium.

Then on putting a charge Q_1 on (1) the potentials of the conductors become $p_{11}Q_1$ and $p_{21}Q_1$. Similarly, placing unit charge on conductor (2) gives potentials p_{12}, p_{22}. The p's are called ***coefficients of potential.***

If Q_1 and Q_2 are the charges given to the two conductors, the corresponding potentials V_1 and V_2 are given by

$$V_1 = p_{11}Q_1 + p_{12}Q_2 \quad . \quad . \quad . \quad \text{(i)}$$

and

$$V_2 = p_{21}Q_1 + p_{22}Q_2 \quad . \quad . \quad . \quad \text{(ii)}$$

These equations may be solved as shown on p. 72; we find

$$Q_1 = c_{11}V_1 + c_{12}V_2,$$

and

$$Q_2 = c_{21}V_1 + c_{22}V_2,$$

where

$$c_{11} = \frac{p_{22}}{p_{11}p_{22} - p_{12}p_{21}}, \quad c_{12} = \frac{-p_{12}}{p_{11}p_{22} - p_{12}p_{21}}, \text{ etc.}$$

The coefficients c_{11} and c_{22} are called ***coefficients of capacitance,*** while c_{12} and c_{21} are ***coefficients of induction.***

To prove $p_{12} = p_{21}$.—Let us suppose that a system consisting of two conductors is charged by bringing up the charge Q_1 from an infinite distance to the first conductor, the second conductor being uncharged; when this has been done let the charge Q_2 be brought up from infinity to the second conductor. Then the work done in the first part of the operation is equal to the energy of the system under these conditions, viz.

$$\tfrac{1}{2}Q_1V_1 + \tfrac{1}{2}0V_2 = \tfrac{1}{2}Q_1(p_{11}Q_1) = \tfrac{1}{2}p_{11}Q_1^2.$$

To find the work done in bringing up Q_2 let this charge be brought up in instalments so that at a certain time a charge θQ_2 has been brought up, where $0 < \theta < 1$. The potential of the second conductor at this stage will be

$$p_{21}Q_1 + p_{22}(\theta Q_2).$$

When the charge on the second conductor is increased by bringing up a charge $Q_2(\delta\theta)$ from infinity, the work done is

$$[p_{21}Q_1 + p_{22}(\theta Q_2)]Q_2\,\delta\theta.$$

The total work done in bringing up Q_2 is the integral of the above expression over the range $\theta = 0$ to $\theta = 1$, i.e.

$$p_{21}Q_1Q_2 + \tfrac{1}{2}p_{22}Q_2^2.$$

The total work done in bringing up Q_1 and then Q_2 is

$$W = \tfrac{1}{2}p_{11}Q_1^2 + p_{21}Q_1Q_2 + \tfrac{1}{2}p_{22}Q_2^2.$$

But if Q_2 had been brought up first and then Q_1 in instalments, we should have found that

$$W = \tfrac{1}{2}p_{22}Q_2^2 + p_{12}Q_2Q_1 + \tfrac{1}{2}p_{11}Q_1^2.$$

$$\therefore\ p_{12} = p_{21}.$$

To prove $c_{12} = c_{21}$.—Calling $p_{12} = p_{21} = p$, we have from our definition of the potential coefficients

$$p_{11}Q_1 + pQ_2 - V_1 = 0,$$
$$pQ_1 + p_{22}Q_2 - V_2 = 0.$$

Solving these equations for Q_1 and Q_2 we have

$$\frac{Q_1}{\begin{vmatrix} p, & -V_1 \\ p_{22}, & -V_2 \end{vmatrix}} = \frac{Q_2}{\begin{vmatrix} -V_1, & p_{11} \\ -V_2, & p \end{vmatrix}} = \frac{1}{\begin{vmatrix} p_{11}, & p \\ p, & p_{22} \end{vmatrix}}.$$

$$\therefore\ Q_1 = \frac{-pV_2 + p_{22}V_1}{p_{11}p_{22} - p^2} = c_{11}V_1 + c_{12}V_2,$$

and $$Q_2 = \frac{-pV_1 + p_{11}V_2}{p_{11}p_{22} - p^2} = c_{21}V_1 + c_{22}V_2.$$

Hence the c's are known and we have $c_{12} = c_{21}$.

Example.—Suppose that the first conductor is a sphere of radius a, Fig. 3·01, its centre being at O, and the second conductor is a very small sphere placed with its centre at a point P. Let κ be the permittivity of the surrounding medium.

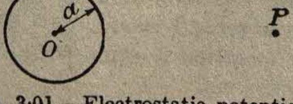

FIG. 3·01.—Electrostatic potential of a sphere due to a point charge.

(*a*) Suppose that P is *outside* the sphere of radius a. If unit charge is placed on this sphere

$$V_P = \frac{1}{\kappa OP},$$

since the charge on the sphere acts as if it were concentrated at its centre O. Thus $p_{21} = \frac{1}{\kappa OP}$. But since $p_{21} = p_{12}$, unit charge at P raises the potential of the sphere by an amount $\frac{1}{\kappa OP}$ if the sphere is insulated.

(*b*) If P is *inside* the sphere of radius a, then when unit charge is given to the sphere, the potential at P is raised by an amount $\frac{1}{\kappa a}$, i.e. $p_{21} = \frac{1}{\kappa a}$. Since $p_{21} = p_{12}$, it follows that if the sphere is insulated and unit charge is placed at an internal point P, the increase in potential of the sphere is $\frac{1}{\kappa a}$, and this is independent of the position of P so long as P remains inside the sphere.

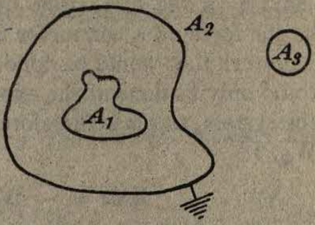

FIG. 3·02.—Electrical screening.

Electric screening.—As an example of the use of the coefficients of capacitance let us consider three conductors A_1, A_2 and A_3, arranged as in Fig. 3·02. Then A_2 is a closed surface with A_1 inside and A_3 outside it. The relations between the charges and the potentials are

$$Q_1 = c_{11}V_1 + c_{12}V_2 + c_{13}V_3 \quad . \quad . \quad . \quad \text{(i)}$$

$$Q_2 = c_{21}V_1 + c_{22}V_2 + c_{23}V_3 \quad . \quad . \quad . \quad \text{(ii)}$$

and

$$Q_3 = c_{31}V_1 + c_{32}V_2 + c_{33}V_3 \quad . \quad . \quad . \quad \text{(iii)}$$

To discern what we can about the c's let us consider the following special case. Let A_2 be earthed so that $V_2 = 0$ and let A_1 be without charge. Then since A_2 is free from inside charges its potential is zero at all points within. Under these conditions equation (i) becomes

$$0 = 0 + 0 + c_{13}V_3.$$

But V_3 may have any value whatsoever. Hence

$$c_{13} = 0 = c_{31}.$$

This proves that A_3 may be raised to any potential without affecting A_1, or vice versa. The conductor A_2 is said to screen A_1 from the field due to A_3; thus the interaction between A_1 and A_3 is completely nullified by interposing a closed conductor surrounding A_1 and maintaining it at zero potential.

The forces and couples on any member of a system of charged conductors.—The stress at the surface of a conductor has been shown [cf. p. 50] to be $\frac{2\pi\sigma^2}{\kappa}$. We are now in a position to determine the *resultant* force (or couple) on any conductor in a system in terms of the electrical coefficients appropriate to that system. As usual, we limit ourselves to a system of two conductors. Let the positions of these conductors be specified by the coordinates x_1, x_2, so that the p's and c's are functions of the x's. W, the energy of the system is consequently also a function of the x's. Hence, in general, a change in any one particular coordinate x will change some or all of the p's and c's and so alter the energy of the system.

If W_Q is the energy of the system expressed in terms of the charges on the conductors and the coefficients of potential, let X be the force on a conductor tending to produce a displacement δx. The work that would be done in such a displacement is $X\,\delta x$ and this could only be done at the expense of the store of potential energy in the system, and it is therefore equal to the corresponding decrease in W_Q, i.e.

$$X\,\delta x = -\,\delta W_Q, \quad \text{or} \quad X = -\frac{\partial W_Q}{\partial x}.$$

In this differentiation the Q's are constant and the differentiation is performed on the p's which alone are functions of x.

If W_V is the energy of the system in terms of the potentials of the conductors and the c's, then although $W_Q = W_V$ it is not open to us to substitute W_V for W_Q in the above equation because of our hypothesis that the charges remain constant during the displacement.

If we take as an alternative hypothesis that the potentials are kept constant while the conductors are displaced, this introduces another physical consideration, viz. that any alteration in the positions of the conductors alters the potentials and in such circumstances the potentials can only be kept constant by connecting each conductor to a battery, i.e. with a source of energy on which it can draw. In such instances it cannot be argued that the work done in a small displacement is the exact equivalent of the loss of energy, because we are assuming that there is a large store of energy available for doing work.

Now, in general, $W = \frac{1}{2}\Sigma(QV)$ Thus

$$W_Q = \tfrac{1}{2}Q_1V_1 + \tfrac{1}{2}Q_2V_2$$
$$= \tfrac{1}{2}p_{11}Q_1^2 + p_{12}Q_1Q_2 + \tfrac{1}{2}p_{22}Q_2^2, \quad [\because p_{12} = p_{21}].$$
$$\therefore \frac{\partial W_Q}{\partial Q_1} = p_{11}Q_1 + p_{12}Q_2 = V_1,$$

and
$$\frac{\partial W_Q}{\partial Q_2} = p_{12}Q_1 + p_{22}Q_2 = V_2.$$

Similarly, since

$$Q_1 = c_{11}V_1 + c_{12}V_2,$$

and
$$Q_2 = c_{12}V_1 + c_{22}V_2,$$

$$W_V = \tfrac{1}{2}Q_1V_1 + \tfrac{1}{2}Q_2V_2 = \tfrac{1}{2}c_{11}V_1^2 + c_{12}V_1V_2 + \tfrac{1}{2}c_{22}V_2^2.$$

$$\therefore \frac{\partial W_V}{\partial V_1} = c_{11}V_1 + c_{12}V_2 = Q_1,$$

and
$$\frac{\partial W_V}{\partial V_2} = c_{12}V_1 + c_{22}V_2 = Q_2.$$

Now
$$W_Q + W_V = Q_1V_1 + Q_2V_2.$$

Consider a small displacement δx without any restrictions on charges and potentials, i.e. suppose that charges, potentials and positions all undergo small changes.

Then
$$\delta W_Q + \delta W_V = \delta(Q_1V_1 + Q_2V_2).$$

Now W_Q is a function of Q_1, Q_2 and x, the distance between the two conductors; similarly for W_V.

$$\therefore \frac{\partial W_Q}{\partial Q_1}\delta Q_1 + \frac{\partial W_Q}{\partial Q_2}\delta Q_2 + \frac{\partial W_Q}{\partial x}\delta x + \frac{\partial W_V}{\partial V_1}\delta V_1 + \frac{\partial W_V}{\partial V_2}\delta V_2 + \frac{\partial W_V}{\partial x}\delta x$$
$$= Q_1\delta V_1 + V_1\delta Q_1 + Q_2\delta V_2 + V_2\delta Q_2.$$

$$\therefore \frac{\partial W_Q}{\partial x} + \frac{\partial W_V}{\partial x} = 0.$$

$$X = -\frac{\partial W_Q}{\partial x} = +\frac{\partial W_V}{\partial x}.$$

Theorem.—*When a system of conductors is maintained with each at a constant potential by connexion to a source of electricity, the total energy supplied by the sources in any displacement of the system is double the increase in the electrical energy of the system.*

Consider charges $Q_1, Q_2, \ldots$ at potentials $V_1, V_2, \ldots$ Then the total energy is

$$W = \tfrac{1}{2}Q_1V_1 + \tfrac{1}{2}Q_2V_2 + \cdots$$

Let a displacement be given to the system so that the charges become $(Q_1 + q_1)$, $(Q_2 + q_2)$, . . . the potentials of the individual conductors being kept constant. The energy of the system is then

$$W + w = \tfrac{1}{2}(Q_1 + q_1)V_1 + \tfrac{1}{2}(Q_2 + q_2)V_2 + \cdots$$

so that the increase in energy is

$$w = \tfrac{1}{2}q_1V_1 + \tfrac{1}{2}q_2V_2 + \cdots$$

Now the quantities $q_1, q_2, \ldots$ have been drawn from sources of energy at constant potentials $V_1, V_2, \ldots$ and therefore the energy given up by these sources is

$$q_1V_1 + q_2V_2 + \ldots = 2w,$$

which establishes the theorem.

Example.—Let us use a parallel plate air condenser to illustrate the above theory. The condenser is shown in Fig. 3·03. It consists of

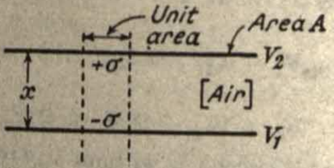

FIG. 3·03.

two parallel plates each of area A, the charge on unit area of the positive plate being σ. Let V_1 and V_2 be the respective potentials $(V_1 < V_2)$ and suppose that the plates are at a distance x apart.

Then, if κ_a is the permittivity of air and edge effects are neglected,

$$V_2 - V_1 = \frac{4\pi\sigma x}{\kappa_a}, \quad \text{and} \quad W = \tfrac{1}{2}A\sigma(V_2 - V_1).$$

$$\therefore W_Q = \frac{2\pi A\sigma^2 x}{\kappa_a}, \quad \text{and} \quad W^V = \frac{\kappa_a A(V_2 - V_1)^2}{8\pi x}.$$

The force tending to increase x is either

$$-\frac{\partial W_Q}{\partial x} = -\frac{2\pi A\sigma^2}{\kappa_a}, \quad \text{or} \quad +\frac{\partial W_V}{\partial x} = -\kappa_a\frac{A(V_2 - V_1)^2}{8\pi x^2} = -\frac{2\pi A\sigma^2}{\kappa_a}.$$

The numerical value of the stress is therefore $\dfrac{2\pi\sigma^2}{\kappa_a}$ everywhere and is such that the resulting forces tend to draw the plates closer together.

The attracted disc electrometer.—This electrometer consists essentially of a guard-ring condenser and a balance, the underlying principle being that the mechanical pull on the movable plate of a condenser is balanced against the gravitational pull on a known mass. A, Fig. 3·04, is the lower plate of the condenser. It is supported by an insulated screw and may be raised to any desired potential. The upper plate of the condenser consists of a central circular section B, surrounded by a wide concentric ring C. This is the so-called guard-ring. B is supported from one arm of a balance as shown. The clearance between B and C is sufficient for B to move freely and yet not sufficient to disturb the homogeneity of the field in the central region of the condenser. C and the support for the beam of the balance are earthed, i.e. B is permanently earthed also. The balance is first adjusted so that B lies flush in the plane containing C. The beam is then just in contact with the stop K. This adjustment may be effected by adding sand to the balance pan D.

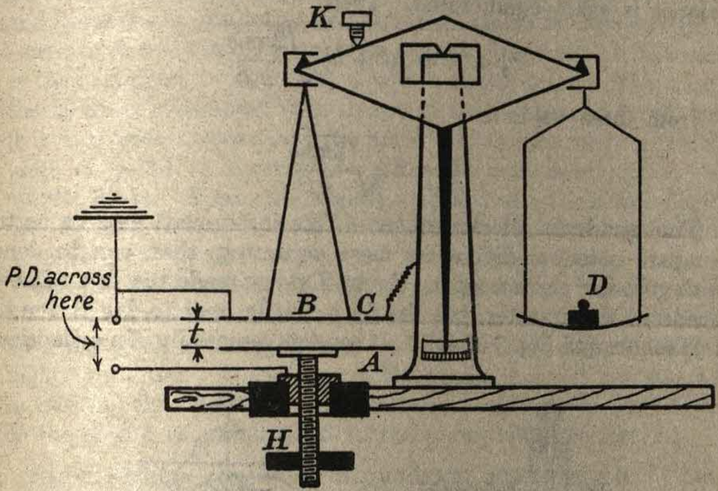

FIG. 3·04.—An attracted disc electrometer.

Let V be the potential of the lower plate at a distance t from the upper one and m be the additional mass required in D to restore equilibrium, i.e. to make B and C coplanar again. Let σ be the numerical value of the density of the charges on the central portions of the condenser plates. If S is the area of the plate B (strictly the mean area of the aperture in C and the plate B), the total pull, mg, on it is $\frac{2\pi\sigma^2 S}{\kappa_a}$, where κ_a is the permittivity of air. But

$$\frac{V}{t} = \text{electric field strength} = \frac{4\pi\sigma}{\kappa_a}.$$

$$\therefore\ V = 2t\sqrt{\frac{2\pi mg}{\kappa_a S}}.$$

Since, if κ_a is assumed to be unity, all the quantities on the right-hand side of the above equation are known, or measurable, V may be calculated.

In actual practice it is found difficult to measure t accurately, so that the following modified procedure is adopted. Let A be connected to a constant source of potential, V_0, the plates being at a distance t apart. Then

$$V_0 = 2t\sqrt{\frac{2\pi mg}{\kappa_a S}}.$$

Now let the potential to be measured, say V, be connected in series with V_0, the total potential being $V_0 + V$. Let A be lowered.

by means of the screw H, through a vertical distance h until the balance is again equilibrated. Then

$$\mathrm{V}_0 + \mathrm{V} = 2(t + h)\sqrt{\frac{2\pi mg}{\kappa_a \mathrm{S}}}.$$

From these we have

$$\mathrm{V} = 2h\sqrt{\frac{2\pi mg}{\kappa_a \mathrm{S}}}.$$

The quadrant electrometer.—This instrument enables us to compare potential differences more accurately than can be done with gold-leaf electroscopes. **Lord Kelvin** made the first reliable quadrant electrometer, but the form chiefly used to-day is due to **Dolezalek** [cf. Fig. 3·05 (*a*)]. It consists essentially of a cylindrical box divided into quadrants, one of which is shown in Fig. 3·05 (*b*). Diagonally opposite quadrants are connected by thin wires and an aluminium needle is suspended symmetrically in a horizontal plane between them. The needle is suspended by a fine phosphor bronze wire which is raised to about 100 volts by being connected to one terminal of a battery, the other terminal being earthed—cf. Fig. 3·05 (*c*). Each quadrant is supported on a quartz pillar which should never be touched by the hand if the insulation is to remain unimpaired. Communication to the quadrants is made by means of metal rods and springs passing through the brass case surrounding the instrument but insulated from it.

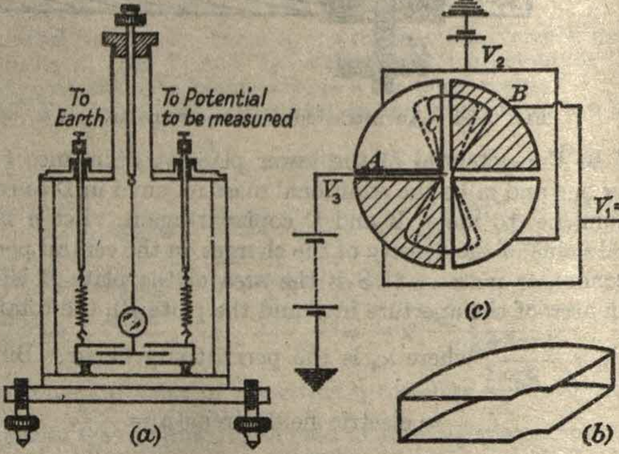

Fig. 3·05. (*a*) A Dolezalek quadrant electrometer.
(*b*) A quadrant.
(*c*) Diagrammatic representation of a quadrant electrometer and its connexions.

In some of these instruments the suspension consists of a quartz fibre which is chosen on account of its constant elastic properties and because the insulation resistance of quartz is so high that the needle does not lose its charge for a considerable time. The needle is then charged by touching it with a charged rod. The base of the instrument is fitted with screws so that it may be levelled and it is advisable to surround the entire instrument with an earthed piece of gauze to protect the quadrants and needle from stray electric fields. A mirror is rigidly attached to a thick wire fixed to the needle and coaxial with the suspension, so that small angular displacements of the needle may be measured.

The principle underlying this instrument is that when there is a difference of potential between the two pairs of quadrants, the needle, having a positive charge, moves away from the quadrants with the higher potential. The energy of the needle is spent in doing work in twisting the fibre. For small potential differences the deflexion of the needle is proportional to the potential difference.

Approximate theory of a quadrant electrometer.*—Let A be the area of *one face* of the needle and S be the area of *one face* of the needle under the V_1 quadrants; then A − S is the area of *one face* of the needle under the V_2 quadrants.

If h is the separation above and below the needle and a quadrant, we can regard the system as one of plate condensers if edge-effects are neglected. Then the energy of the electrical system is

$$\begin{aligned} W_V &= \Sigma\, \tfrac{1}{2}(CV^2) \\ &= 2 \times \frac{1}{2}\frac{S}{4\pi h}(V_3 - V_1)^2 + 2 \times \frac{1}{2}\frac{A - S}{4\pi h}(V_3 - V_2)^2. \end{aligned}$$

The couple tending to turn the needle in the direction of θ increasing is $+\dfrac{dW_V}{d\theta}$. Now S is the only term in the expression for the energy which is a function of θ.

$$\therefore \frac{dW_V}{d\theta} = \frac{(V_3 - V_1)^2 - (V_3 - V_2)^2}{4\pi h}\frac{dS}{d\theta}.$$

Now $r^2\,\delta\theta = \delta S$, if r is the constant radius of the needle.

$$\therefore \frac{dW_V}{d\theta} = \frac{(2V_3 - V_1 - V_2)(V_2 - V_1)}{4\pi h}r^2,$$

and in the position of equilibrium this is $\tau\theta$, where τ is the torsional constant for the suspension. Thus, α being a constant,

$$\theta = \alpha(V_2 - V_1)[V_3 - \tfrac{1}{2}(V_1 + V_2)].$$

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* Cf. earlier editions of this book for a more complete theory of this instrument.

Two methods of using a quadrant electrometer.—The deflexion of the needle of a quadrant electrometer has been shown to be given by the expression

$$\theta = \alpha(V_2 - V_1)[V_3 - \tfrac{1}{2}(V_1 + V_2)],$$

where α is a constant. If $V_3 \gg V_2$ and V_1 we may write for a fixed value of V_3,

$$\theta = k(V_2 - V_1),$$

where k is a constant. In such circumstances the instrument is said to be used ***heterostatically.***

If the insulated quadrants are at the same potential as the needle then we have, if $V_1 = 0$,

$$\theta = \alpha V_2[V_2 - \tfrac{1}{2}V_2] = \tfrac{1}{2}\alpha V_2^2 = \beta V^2. \quad [\beta = \tfrac{1}{2}\alpha = \text{constant and } V_2 = V \text{ (say).}]$$

In this instance the deflexion of the needle is directly proportional to the square of the potential on the needle and the instrument is said to be used ***idiostatically.*** It should be noted that V is the root-mean-square (r.m.s.) value if an alternating potential is applied—cf. p. 507.

The theory of an electrostatic voltmeter.—It has just been shown how a quadrant electrometer may be adapted to measure alternating potential differences. The formula involved may be derived by considering the energy stored in the system and the method applies to many commercial forms of electrostatic voltmeter. To proceed, let us therefore consider the arrangement shown in Fig. 3·06 (*a*); A and B are the fixed plates of the system while N is a moving component suitably suspended and capable of

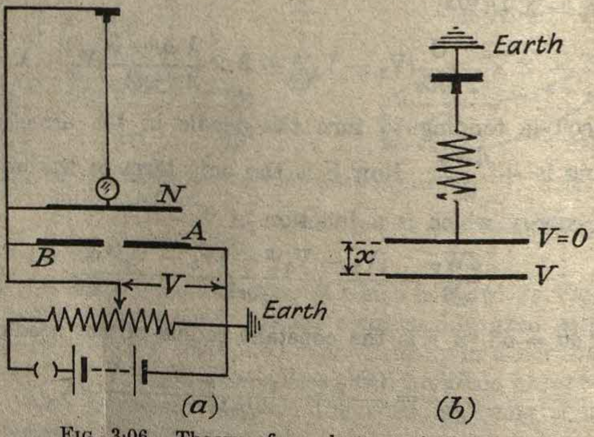

FIG. 3·06.—Theory of an electrostatic voltmeter.

rotation about a vertical axis. A is earthed and the potential difference V applied as shown. Let C be the capacitance of the system and suppose that N takes up an angular position defined by θ. Let the applied potential difference be increased to $V + \delta V$ so that θ becomes $\theta + \delta\theta$. In this process a charge $\delta Q = V\delta C + C\delta V$ flows on to the parts B and N.

The energy supplied by the battery is $V.\delta Q = V^2.\delta C + CV\delta V$, while the increase in the energy of the field is $\frac{1}{2}V^2.\delta C + CV\delta V$; the increase in the energy of the suspension is $\tau\theta.\delta\theta$, where τ is its torsional constant. Using the principle of the conservation of energy, we have

$$V^2.\delta C + CV\delta V = \tfrac{1}{2}V^2.\delta C + CV\delta V + \tau\theta\delta\theta,$$

i.e.

$$\tau\theta = \tfrac{1}{2}V^2\frac{dC}{d\theta}.$$

Hence

$$\theta = AV^2\frac{dC}{d\theta},$$

where A is an instrumental constant. In a well-designed instrument $\frac{dC}{d\theta}$ can be made constant for its working range, so that the deflexion is directly proportional to the square of the applied voltage.

The foregoing treatment may be applied to an instrument with a direct-attraction type of movement, cf. Fig. 3·06 (*b*), by writing x to define the separation of the plates; this variable replaces the θ of the earlier analysis. Then, if F is the deflexional force,

$$F = \tfrac{1}{2}V^2\frac{dC}{dx}.$$

[For a close-spaced parallel plate air-condenser, in which edge effects may be ignored, $C = \frac{\kappa_a A}{4\pi x}$, so that $\frac{dC}{dx} = \frac{\kappa_a A}{4\pi x^2}$, which is not a constant.]

Two commercial forms of electrostatic voltmeter.—The structure of a Kelvin multicellular voltmeter is shown in Fig. 3·07 (*a*); it is essentially a quadrant electrometer with a large number of moving vanes, rigidly connected together to form the needle, and a correspondingly large number of fixed quadrants. The large number of quadrantal cells is necessary, especially when the range of the instrument is low, to obtain a sufficiently high deflecting couple; the range of the instrument is usually from 100–1,000 volts, although instruments to cover the range below 40 volts have been constructed.

Since the moving system is suspended, all bearing friction is eliminated and the coach spring is added to protect the suspension from damage by shock; similarly, a clamp is used when the instrument is in transit. The pointer and scale, which must be calibrated, are of the 'edge-wise' type, and a torsion head, which may be rotated very slowly by a worm-wheel attachment, enables the pointer to be set at zero.

A simple form of electrostatic voltmeter, suitable for the range from 1–10 kV., is shown in Fig. 3·07 (*b*). The needle N, which is gravity controlled, is made from aluminium sheet and moves in a vertical plane between two fixed parallel vanes of which only one, A, is shown. The needle is earthed and the potential to be measured applied to the fixed vanes. A pointer P is attached to the needle and its end moves across a scale S, which is standardized by using known voltages; by adding small masses m_1 and m_2 to short arms attached to the lower part of the needle the controlling torque may be varied and this alters the range of the instrument. Since the instrument is gravity controlled the scale does not follow a square law. Its case has a glass front which is lightly aluminized so that electrostatic charges do not accumulate on the glass; if this is not done the indications of the instrument tend to be faulty.

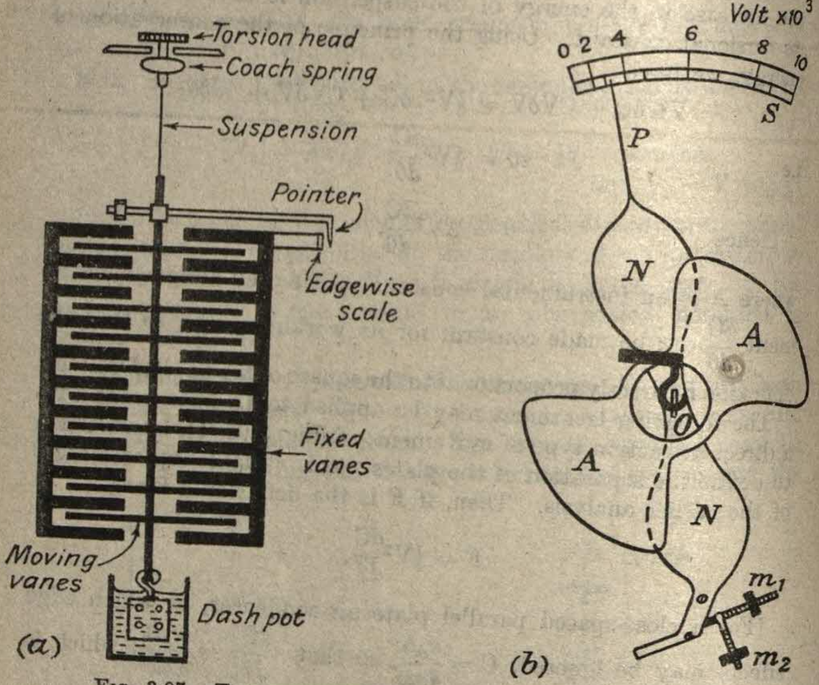

FIG. 3·07.—Two commercial forms of electrostatic voltmeter.

Example.—A horizontal circular insulated plate a, Fig. 3·08, is close to a similar earthed plate b. A light metal rod on an insulated pivot O carries a similar plate c. The centres of these circular plates lie on a vertical line and, when all the plates are at earth potential, c is midway between a and b; the rod is kept horizontal by a counterpoise M and a small rider of mass m. Show that when the potentials of a and c above earth are respectively v and V, equilibrium will be restored by moving m through a distance z given by $z = k(2\mathrm{V}v - v^2)$, where k is a constant.

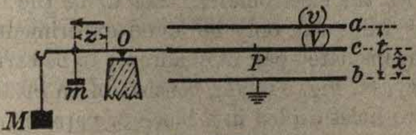

FIG. 3·08.

Find the voltage sensitivity, $\frac{dz}{dv}$, in cm. volt.$^{-1}$, if the area of each plate is 15 cm.^{2}, the distance between a and b is 0·20 cm., V is 100 volts, m is 0·010 gm., and the distance between O and the centre P of the plate c is 5·0 cm.

If A is the area of each plate, x and t the distances indicated, the potential energy of the system is given by

$$\mathrm{W} = \frac{1}{2}\frac{\kappa_a \mathrm{A}}{4\pi x}.\mathrm{V}^2 + \frac{1}{2}\frac{\kappa_a \mathrm{A}}{4\pi(t - x)}.(\mathrm{V} - v)^2,$$

where κ_a is the permittivity of air; its numerical value may be taken as unity.

$\therefore$ Force tending to increase x is

$$\frac{\partial \mathrm{W}}{\partial x} = -\frac{\kappa_a \mathrm{A}}{8\pi x^2}.\mathrm{V}^2 + \frac{\kappa_a \mathrm{A}}{8\pi}.\frac{1}{(t - x)^2}.(\mathrm{V} - v)^2.$$

If this force is F when $x = \frac{1}{2}t$, we have

$$\mathrm{F} = \left(\frac{\partial \mathrm{W}}{\partial x}\right)_{x=\frac{1}{2}t} = -\frac{\kappa_a \mathrm{A}}{2\pi t^2}(2\mathrm{V}v - v^2).$$

If z is the outwards displacement of m required to restore equilibrium, and OP $= l$,

$$mgz = |\mathrm{F}|\,l = k(2\mathrm{V}v - v^2),$$

where

$$k = \frac{\kappa_a \mathrm{A} l}{2\pi t^2}, \text{ a constant.}$$

$$\therefore \quad \frac{dz}{dv} = \frac{2k}{mg}(\mathrm{V} - v).$$

$$\therefore \quad \left(\frac{dz}{dv}\right)_{\text{e.s.u.}} = \frac{2 \times 15 \times 5}{2\pi \times (0{\cdot}20)^2 \times 0{\cdot}010^2 \times 980}.\frac{95}{300}.$$

$$\therefore \quad \left(\frac{dz}{dv}\right)_{\text{volt}} = \frac{150 \times 10^4}{6{\cdot}28 \times 4 \times 980}.\frac{95}{300}.\frac{1}{30}$$

$$= 0{\cdot}0064 \text{ cm. volt.}^{-1}.$$

To determine experimentally the capacitance of a quadrant electrometer and a small air condenser arranged in parallel with it.—[This is often referred to as a determination of the capacitance of an electrometer and its connexions.] Let E, Fig. 3·09, be the electrometer whose needle is maintained at a high potential (100 V.) by means of a battery A. It will be supposed that a fixed air condenser, C′, is permanently connected in parallel with the electrometer. Let C_0 be the capacitance of this system; its value can only be found experimentally. The key K_0 enables the insulated pair of quadrants to be earthed when necessary. This key, cf. Fig. 3·09 (*b*), consists of an insulated piece of wire bridging two holes drilled in a block of paraffin wax and containing a small quantity of calcium chloride. After a short exposure to the atmosphere the chloride becomes moist and conducting. By closing the key K_1 the insulated quadrants may be connected to one pole of a battery B, the other pole being earthed. C is a standard parallel plate air condenser; one plate is earthed while the other may be connected to the insulated quadrants of the electrometer by closing the key K_2. Let K_2 be open and K_1 closed, the deflexion of the electrometer needle being θ_1. If Q is the charge on the insulated quadrants and the connexions to them, C_0, their capacitance, is given by $Q = C_0V = \alpha C_0\theta_1$, where V is the potential difference applied and α a constant. When K_1 is opened the deflexion is unaltered if the electrometer is in working order, but on closing the key K_2 the deflexion is reduced to θ_2, the charge Q being shared between the capacitances C_0 and C. Then

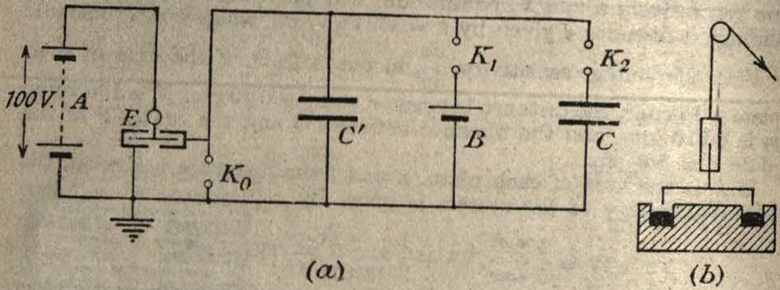

FIG. 3·09.—Capacitance of a quadrant electrometer and its connexions.

$$Q = \alpha(C_0 + C)\theta_2.$$

$$\therefore (C_0 + C)\theta_2 = C_0\theta_1,$$

or

$$C_0 = C\frac{\theta_2}{\theta_1 - \theta_2}.$$

Measurement of an ionization current by means of a quadrant electrometer.—***(a) The steady deflexion method.***—If two plates of an air condenser are maintained with a potential difference across them and the air between them is exposed to the action of X-rays or other ionizing agent a small current flows between the plates; it is usually too small to be detected by a galvanometer. One method of measuring such a current is due to **Bronson** (1906) and is as follows. A resistance R, Fig. 3·10 (*a*), of the order 10^{10} ohms, is joined in series with a battery B of from 30 to 300 volts or more and a condenser C. When the air between the plates of the condenser is exposed to X-rays a current flows in this circuit, thereby creating a potential difference between the ends of R. These are connected to the opposite pairs of quadrants of an electrometer and the steady deflexion of the needle is recorded. Let this be θ_1. The electrometer is calibrated by placing a Daniell cell [1·08 volts] across its diagonally opposite quadrants. Let the deflexion be θ_2. Then the potential difference corresponding to θ_1 is $1{\cdot}08 \times \left(\dfrac{\theta_1}{\theta_2}\right)$. The current in the circuit is therefore $\dfrac{1{\cdot}08}{R} \times \dfrac{\theta_1}{\theta_2}$ amperes, if R is measured in ohms.

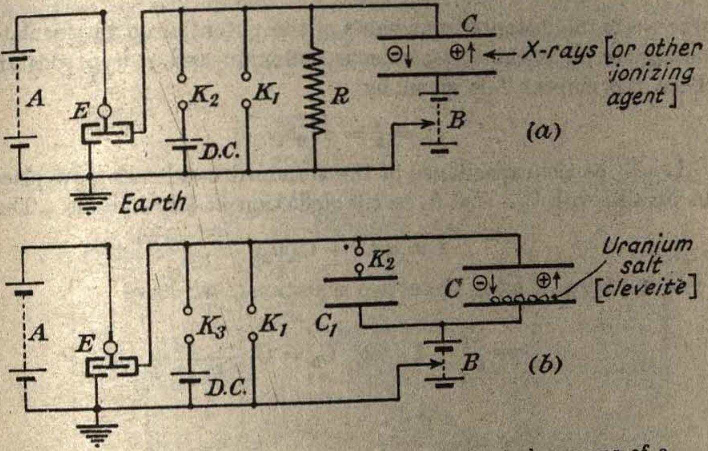

Fig. 3·10.—Measurement of ionization currents by means of a quadrant electrometer.

(b) The steady-rate method.—Suppose that a layer of some radioactive material, such as cleveite, is placed on the lower plate of a condenser C, Fig. 3·10 (*b*). This plate is connected to the positive electrode of a battery B, the other electrode (of the portion used) being at earth potential. E is a quadrant electrometer whose insulated quadrants are connected to the upper plate of the condenser C; they may be earthed by means of the key K_1. C_1 is a second air-condenser of known capacitance which may be placed in parallel with C, the key K_2 being used to carry out this operation.

All electrical quantities are at first expressed in practical units.

Let q be the charge at any instant on the insulated quadrants and the condenser plate to which they are connected. Let C_0 be the capacitance of the ionization chamber together with that of the electrometer and its connexions. This is the capacitance we have previously determined experimentally; here it is eliminated. Then

$$q = C_0 v = C_0(\alpha\theta_1),$$

where v is the instantaneous value of the potential on the insulated quadrants, θ_1 is the instantaneous deflexion and α is a constant. Then the current I is given by

$$I = \dot{q} = \alpha C_0 \dot{\theta}_1.$$

Let C_1 be the capacitance of the standard condenser now placed in parallel with C_0. Let θ_2 be the deflexion at any instant. Then

$$I = \alpha(C_1 + C_0)\dot{\theta}_2.$$

Eliminating α from these two equations, we have

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{C_1}{C_0} + 1, \quad \text{or} \quad C_0 = C_1 \cdot \frac{\dot{\theta}_2}{(\dot{\theta}_1 - \dot{\theta}_2)}.$$

$$\therefore\ I = \alpha C_1 \frac{\dot{\theta}_1\dot{\theta}_2}{\dot{\theta}_1 - \dot{\theta}_2}.$$

To standardize the electrometer the procedure is as follows. By means of a key K_3 a Daniell cell of e.m.f. 1·08 volts is placed across the quadrants; let the steady deflexion be ϕ. Then $1{\cdot}08 = \alpha\phi$.

$$I = \frac{1{\cdot}08}{\phi} C_1 \frac{\dot{\theta}_1\dot{\theta}_2}{\dot{\theta}_1 - \dot{\theta}_2} \text{ amp.}$$

$$= \frac{1{\cdot}08}{\phi} C_s \frac{\dot{\theta}_1\dot{\theta}_2}{\dot{\theta}_1 - \dot{\theta}_2} \times \frac{1}{9 \times 10^{11}} \text{ amp.},$$

where C_s is the capacitance in e.s.u. of the standard condenser C_1.

The Lindemann electrometer.—This instrument, which works on the same principle as a quadrant electrometer, was designed in 1924 for the measurement of small potential differences and very small currents. It is very compact, has a stable zero, a constant sensitivity and, unlike a quadrant electrometer, it does not require levelling, i.e. it is insensitive to changes of level and of tilt.

The working parts consist of four polished plates, insulated from each other but connected diagonally; these plates, cf. Fig. 3·11 (*a*), are fixed in position and are known as the quadrants. In addition, there is the moving system which is mounted mid-way between the two pairs of plates. It consists of a quartz 'needle' mounted on a gilded quartz fibre, 6μ thick and stretched on a U-shaped quartz frame as shown in Fig. 3·11 (*b*); the tension in this fibre is about one half the breaking load and is independent of temperature. Each upright limb of this frame has a saw-slot about 0·1 cm. deep and in these the ends of the torsion fibre are mounted. One end is fixed in a slot by means of Wood's metal and at the same time a thin copper wire is inserted to serve as a lead. A load of appropriate size is attached temporarily to the protruding end of the fibre to impart the necessary tension. Shellac is melted at the slot concerned and when it has re-solidified the protruding part of the fibre is removed. Across this torsion fibre the so-called needle is fixed. This lies vertically and symmetrically across the torsion fibre and is free to rotate in the space between the quadrants, the point where the two fibres are attached to each other being at the centre of the system. To facilitate balancing, the needle is composed of two equal parts placed parallel to one another; each is a gilded fibre about 25μ in diameter. Their ends are sealed together and the whole rigidly attached in the plane of the quartz frame to the torsion fibre by means of electrolytically deposited copper. By means of shellac a short length of quartz fibre P is fixed to the upper end of the needle; it is always parallel to the torsion fibre and serves as a pointer which enables the rotation of the needle to be determined by observing, through a microscope pointing downwards, the displacement of P.

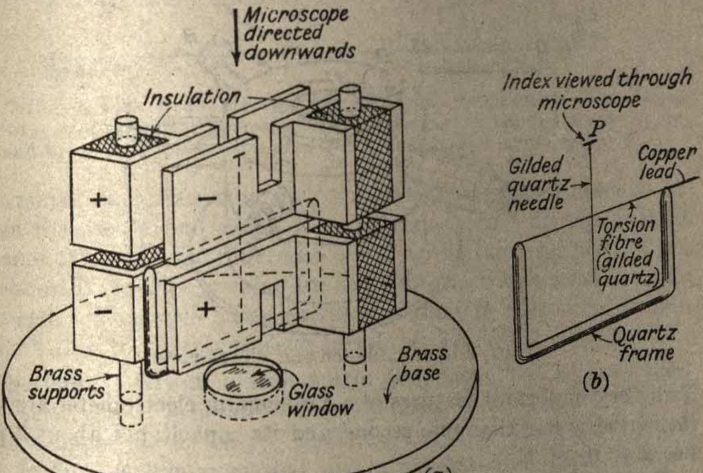

FIG. 3·11.—Essential features of a Lindemann electrometer.

The above arrangement is enclosed in a cylindrical brass box about 4 cm. in diameter and 3 cm. deep; Fig. 3·12 shows the external appearance of the instrument. A small glass window is provided in both the top and bottom of the box. This permits the microscope to be focused on the index P, when any motion imparted to the needle is shown by the displacement of the image of P on the graticule scale in the eye-piece of the microscope. A and B are terminals connected to the respective quadrants, while C is a third terminal connected to the needle. A small cylindrical chamber D contains silica gel to absorb moisture from the interior of the instrument and the terminal E permits the case to be earthed.

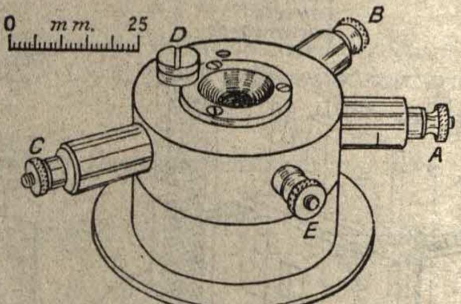

FIG. 3·12.—A Lindemann electrometer.

Two very important features of a Lindemann electrometer are that its period is less than one second and its capacitance about 2 pF., i.e. 2×10^{-12} F.

Now although the principle of working is the same as that of a quadrant electrometer, it must be noted that with a Lindemann electrometer the potential to be measured is applied to the moving system while the quadrants are kept at fixed potentials; with a quadrant electrometer these functions are reversed. The needle is deflected towards that pair of quadrants whose potential differs most from its own.

Fig. 3·13 illustrates a simple circuit showing how this electrometer may be used to measure a photoelectric current. To set up the instrument the terminals to the plates and the needle are all earthed and the 'mechanical zero' viewed on the eye-piece scale. The potentials of the plates are then adjusted to have equal and opposite values when the needle remains at the mechanical zero. Usually a H.T. battery with 3 V tappings is used for this purpose, a fine adjustment being made with the aid of a potential divider incorporated at the centre of the H.T. battery; the diagram is self explanatory.

In practice the plates are at ± 15 V. with respect to earth; higher values lead to instability but just before this stage is reached the instrument is very sensitive. The deflexion of the needle, as measured on the graticule, is a linear function of the potential difference applied to the needle and a sensitivity of 400 eye-piece divisions per volt is not uncommon. When the sensitivity has been determined experimentally, any unknown potential difference between the needle and earth is readily deduced. In the present instance this potential difference exists across the large and known resistance R, so that a value for the photoelectric current is easily obtained. With such small currents as these it is essential to add a guard-ring to the photoelectric cell, cf. Vol. VI.

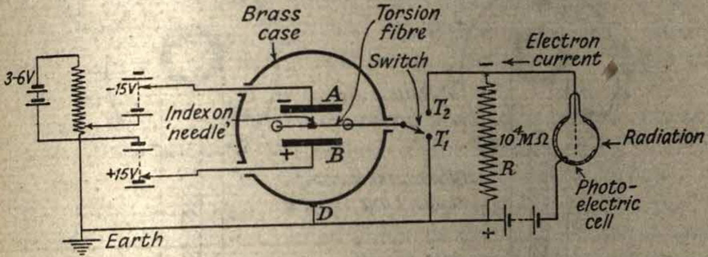

FIG. 3·13.—A Lindemann electrometer arranged to measure a photoelectric current. Any potential difference to be measured is applied across T_1, T_2 and the switch enables the needle to be earthed or connected to T_2.

The van de Graaff electrostatic generator.*—The basic idea of a belt type of generator probably dates from Kelvin's famous water dropper [John Gray, 'Electrical Influence Machines', 1890] and, in fact, Kelvin suggested such a generator in which charges would be carried to the electrode on a conveyor belt consisting of alternately insulated metal segments. The first really successful generator of this type is due to VAN DE GRAAFF.

The principle of operation of a van de Graaff electrostatic generator is shown schematically in Fig. 3·14. A and B are two metal spheres about 50 cm. in diameter and they serve respectively as the positive and negative high-tension terminals of the generator; belts made from multi-ply rubberized fabric are shown transporting to them positive and negative electricity from the charging outfit. The charge is 'sprayed' on to each belt as it passes between a metallic surface and one or more sharp points so adjusted as to maintain a brush-discharge between the surface and the neighbouring point. Thus at D, the small metallic sphere is charged positively from the generator and so the neighbouring metallic point receives a negative charge. This is discharged from the point and collects on the belt which carries the charge upwards. When this charge arrives inside the large sphere B it is collected by the point G, i.e. an induced positive charge appears on G while an induced negative charge appears on the surface of B. The charge

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* *Phys. Rev.*, **43**, 149, 1933.

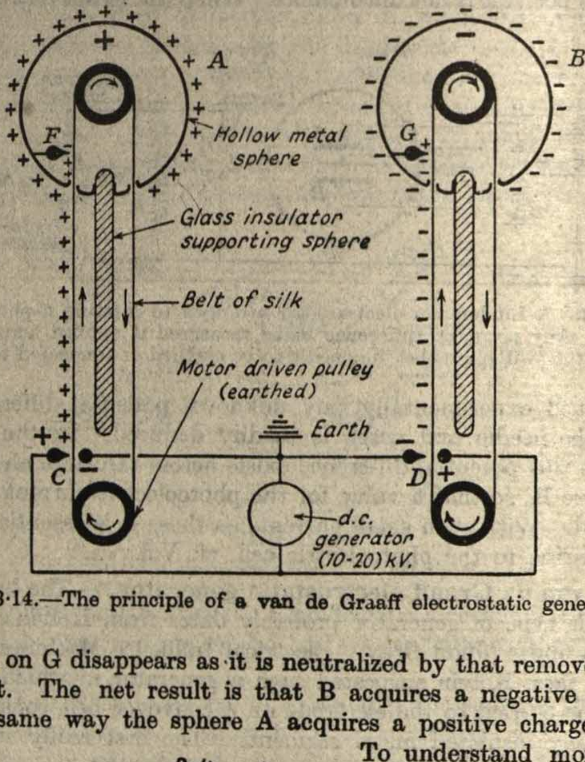

A
B
F
G
Hollow metal sphere
Glass insulator supporting sphere
Belt of silk
Motor driven pulley (earthed)
Earth
C
D
d.c. generator (10-20) kV.

FIG. 3·14.—The principle of a van de Graaff electrostatic generator.

induced on G disappears as it is neutralized by that removed from the belt. The net result is that B acquires a negative charge. In the same way the sphere A acquires a positive charge.

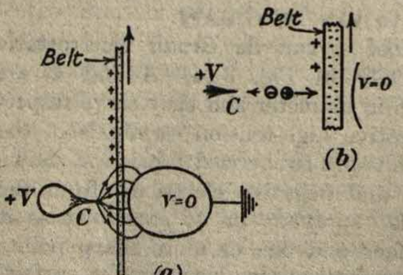
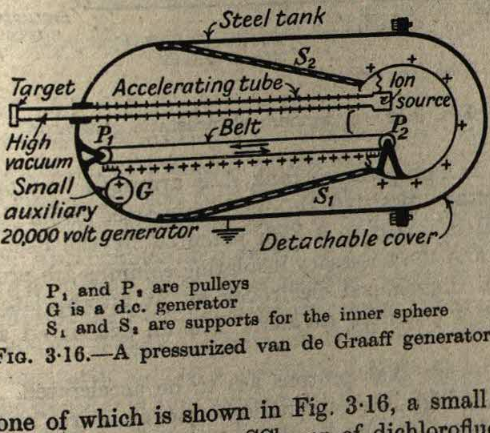


FIG. 3·15.—The charge-spraying process in a van de Graaff high-voltage generator.

To understand more fully how this 'spraying process' works, let us consider the state of affairs between the positively charged and pointed conductor C and the negatively charged small sphere close to it. In between, as shown in detail in Fig. 3·15 (*a*), is the belt of insulating material which travels upwards. Because of the large electrostatic field in the air near to the point C ionization occurs, i.e. both positive and negative ions are formed. The supply of ions is copious in the region between C and the belt, cf. Fig. 3·15 (*b*), and the positive ions, drifting with the field, strike the belt and adhere to it; these ions are carried upwards. On the other hand the negative ions move towards the pointed conductor C. Inside the sphere the charge is removed from the belt by the well-known and simple 'action of a point' process. It should also be noted that at the same time a relatively small number of negative ions may be sprayed on to the other side of the belt; these will not be removed until the belt passes over the lower pulley which is earthed.

The voltage which is attained by this device is limited only by the corona breakdown at the surface of the spheres, and this depends upon their size and the degree of smoothness of their surfaces. The current output is equal to the rate at which the charge is carried into the spheres and is therefore equal to the product of the surface density of the electricity on the belt and the areal velocity with which it enters the sphere concerned. To provide as large a current as possible the belts are designed to operate at the greatest possible speed and multiple belts can be used.

Pressurized van de Graaff generator.—Higher voltages can be obtained from a van de Graaff generator if the entire electrode system is enclosed in a high pressure tank and the conveyor belt made from rubberized fabric. Air is not used in the tank since the presence of oxygen increases the risk of fire should a spark occur. The mainly cylindrical vessel, which is about six metres long and somewhat less than two metres in diameter, is therefore filled with nitrogen at a pressure of about seven atmospheres and this reduces corona losses. In still more recent forms of this type of high voltage generator, one of which is shown in Fig. 3·16, a small percentage of carbon tetrachloride vapour, CCl_4, or of dichlorofluoromethane vapour, CCl_2F_2 and known commercially as 'freon', is added to the nitrogen; this reduces still further the corona losses. Freon



P_1 and P_2 are pulleys
G is a d.c. generator
S_1 and S_2 are supports for the inner sphere

FIG. 3·16.—A pressurized van de Graaff generator.

boils at − 29·8° C., and at 20° C. its equilibrium vapour pressure is about six atmospheres. Its dielectric strength is about equal to that of air at a pressure three times as high but its use with electrostatic generators is complicated by its slow decomposition under conditions of ionization and sparking. Since 1946 some workers have replaced freon by sulphur hexafluoride, SF_6, with a sublimation point at −63·8° C. It is much superior to freon on account of its greater chemical stability and an equilibrium vapour pressure of twenty-four atmospheres at room temperature.

The voltages attained with these generators are usually in the range 1–6 MV and by using suitable electronic controls may be stabilized to within ±0·1 per cent. Such an instrument is indispensable for precision measurements concerning nuclear reactions with energies not exceeding 10 MeV.

Tandem electrostatic generators.—Since 1960 a technique has been developed for utilizing the terminal potential of a van de Graaff generator several times in succession, i.e. the energies of accelerated particles are increased several times above that acquired by a single acceleration. Here only a two-stage tandem electrostatic generator will be discussed; Fig. 3·17 indicates the mode of operation. A high-pressure tank A is arranged horizontally with the high-voltage positive terminal supported from both ends; this terminal is raised to a high potential by the usual charging-belt system and the necessary insulating columns are provided with intermediate ring-shaped electrodes set at appropriate potentials by means of a potential divider. T is an evacuated accelerating tube.

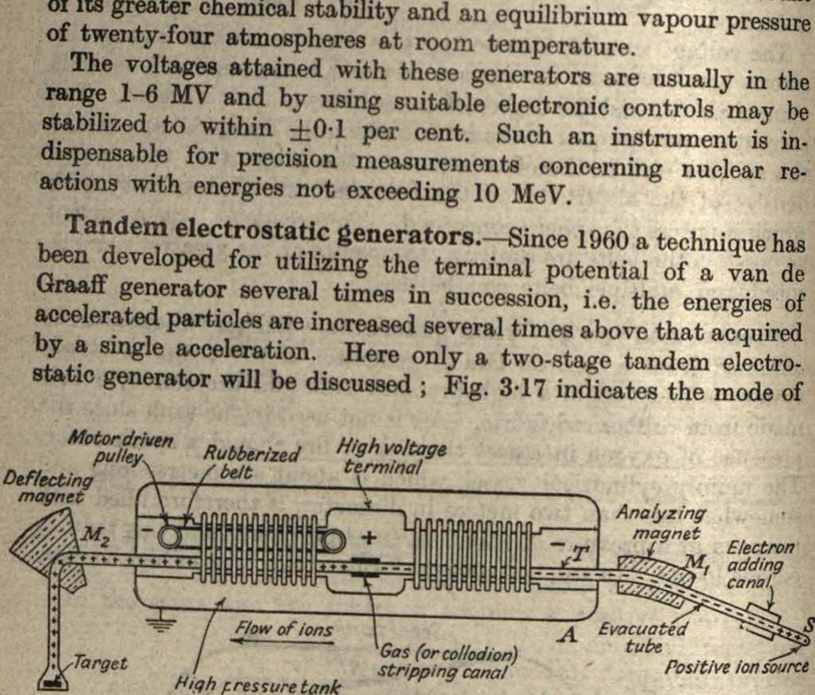


FIG. 3·17.—A tandem electrostatic generator.

Let us suppose that protons are to be accelerated. These are produced in a high-intensity radio-frequency source at S and then pass through an 'electron adding' canal. This contains gas at such a pressure that about one per cent of the protons emerge as *negatively* charged atomic hydrogen ions. The negative-ion beam is deflected by the analysing magnet M_1, so that monoenergetic ions enter the accelerating tube and move towards the high-voltage terminal. Inside this the swiftly moving negative ions pass into a 'stripping canal', where a very thin sheet of collodion, or gas at a suitable pressure, causes two electrons to be removed from each ion, i.e. protons appear; the energy loss in this process is only a few hundred electron-volts, i.e. it is negligible. These positive ions are then accelerated to earth potential so that their final energy is thereby doubled. A second magnet M_2 deflects the protons through 90° before they bombard the target.

The measurement of high d.c. (or z.f.) potential differences.—Fig. 3·18 (*a*) shows diagrammatically the essential features of an attracted disc electrometer for the measurement of potential differences up to 0·25 MV. A hemispherical metal dome S encloses a sensitive balance B from one arm of which hangs a movable metal disc D. This is flush with the lower surface of a metal plate R which serves as a guard ring; the clearance between D and R is about 0·01 cm. The diameter of the disc is within the range 10 cm. to 20 cm., while that of R is 100 cm. Light reflected from a plane mirror M attached to the beam of the balance increases its sensitivity and also enables an observer at a safe distance from the instrument to ascertain when D is restored to its zero position. This is brought about by adding standard masses to the appropriate pan of the balance. A bank of condensers P between a high potential electrode V and earth acts as a potential divider and the electric field across the gap AD is kept uniform by providing equally spaced 'guard-hoops' such as HH; each hoop is connected to the appropriate point in the potential divider P.

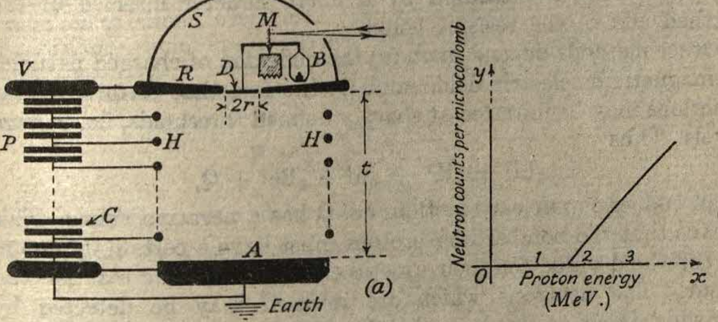


FIG. 3·18.—The measurement of high d.c. potential differences.

If $2r$ is the effective diameter of the disc we have, with the usual notation,

$$mg = \pi r^2 . \frac{2\pi\sigma^2}{\kappa_a},$$

where $$4\pi\sigma = \kappa_a E = \kappa_a \frac{V}{t}.$$

$$\therefore\ V = \sqrt{\frac{8mg}{\kappa_a}} . \frac{t}{r} \text{ e.s.u.}$$

An alternative method for ascertaining the terminal potential of an electrostatic generator is to measure the current through a stack of high resistors in parallel with the machine. By carefully shielding these resistors they may be made free from corona losses; then the method is an accurate one. To effect this the resistors are connected between successive equipotential rings and the total resistance of this bleeder column should be of the order of 10^4 megohms per million volts so that the current through it is about 100 μA. This is measured by a microammeter inserted at the earthed end of the resistor column.

Other methods depend upon (*a*) the deflexion of charged particles in magnetic or electric fields, and (*b*) the fact that certain nuclear reactions may be initiated at sharply defined thresholds, i.e. voltage levels. Thus

$${}_3\mathrm{Li}^7 + {}_1\mathrm{H}^1 \rightarrow {}_0\mathrm{n}^1 + {}_4\mathrm{Be}^7 + \mathrm{Q},$$

is an endoergic nuclear reaction, i.e. Q has a negative value. This means that the bombarding protons must have a certain minimum or threshold value for their kinetic energy before the reaction can occur. The neutrons which are released may be detected by allowing them to enter a chamber filled with boron trifluoride. The reaction is represented by the equation

$${}_0\mathrm{n}^1 + {}_5\mathrm{B}^{10} \rightarrow ({}_5\mathrm{B}^{11}) \rightarrow {}_3\mathrm{Li}^7 + {}_2\mathrm{He}^4,$$

and it is the strongly ionizing property of the α-particles which permits them to be detected. As a result of these experiments a graph similar to that shown in Fig. 3·18 (*b*) may be drawn; it shows the threshold value of the energy of the bombarding protons to be (1·882 ± 0·001) MeV.

When a series of such values for several nuclear reactions is known, an accurate means for standardizing high voltages becomes available.

EXAMPLES III

3·01. Each time the plate of an electrophorus is charged it acquires a charge Q and when it is brought into contact with an uncharged insulated conductor this acquires a charge αQ, where $0 < \alpha < 1$.

Prove that the maximum charge which may be given to the conductor after repeated contacts is $\alpha Q(1 - \alpha)^{-1}$.

3·02. Describe, in detail, how the force between the plates of a condenser is utilized in an absolute electrometer.

Two condensers of capacitances 0·1 μF. and 0·01 μF. are connected in series across a potential difference of 22 volt. If they are then insulated and connected in parallel, what is the loss of energy in ergs ? [14·7 erg.]

3·03. Two parallel plate condensers of capacitance 0·1 μF. and 100 pF. are arranged in parallel with one side of the system earthed and the other insulated. The potential of the insulated plates is raised to 12 V. and then the system is isolated. Calculate a value for the potential difference across the smaller condenser when the earthed plate of the other is removed to infinity. [$1{\cdot}2 \times 10^4$ V.]

3·04. An attracted disc electrometer is immersed in an insulating oil whose permittivity is 4. How must the distance between the plates be changed so that the suspended disc may remain in the plane of the guard-ring (*a*) if the charges on the plates are kept constant, (*b*) if the potentials of the plates are kept constant ?

[(*a*) impossible, (*b*) $t_k = 2t$.]

3·05. The upper plate of a parallel plate condenser is earthed and suspended from a closely coiled helical spring, for which a force f dynes is required to extend it by 1 cm. The plates are horizontal and initially at a distance a cm. apart. When the potential of the lower plate is raised to a value V e.s.u. and the upper plate is at a distance x cm. below its initial position, show that the resultant downward force acting upon it is given by

$$F = \frac{V^2A}{8\pi(a - x)^2} - fx,$$

where A is the area of a plate. Find $\dfrac{dF}{dx}$ and show that if F = 0 when $x = x_0$, then for a small displacement δx from this position

$$\delta F = f\left(\frac{3x_0 - a}{a - x_0}\right)\delta x.$$

Discuss the equilibrium of the system.

3·06. Three parallel plates, A, B and C, each 16π cm.^{2} in area, are maintained at potentials, 40, 90 and 0 volts respectively. B is between A and C, A and B being separated by 1 mm. and B and C by 3 mm. What force will prevent B from moving ? [3·56 dyne from B to C.]

3·07. The plates of an air condenser are connected to an electrometer and the system is charged. The deflexion of the needle is recorded and then the space between the plates is completely filled with a substance of permittivity 5, the condenser plates remaining insulated during this process. The deflexion is again observed and it is found to be less in the ratio 500 : 101. If the capacitance of the air condenser is 0·01 μF., find that of the electrometer and its connexions. [$1{\cdot}25 \times 10^{-4}$ μF.]

3·08. Obtain an expression for the energy of a system of conductors in terms of their charges and potentials.

Show that if the plates of a parallel plate condenser are maintained at a constant potential difference by a battery of cells, and one plate is permitted to move slightly, the work done by electric forces is half the energy taken from the battery during the displacement.

3·09. A quadrant electrometer and its connexions have a capacitance of 60 cm. The instrument is used heterostatically and the insulated quadrants are charged. They are then connected to a parallel-plate air condenser without an initial charge and the deflexion of the needle is reduced to one-third of its initial value. The distance between the condenser plates is then reduced by 0·4 cm., and the deflexion is halved. Obtain values for the initial capacitance of the condenser and the separation of its plates. [120 e.s.u., 0·667 cm.]

3·10. Describe a quadrant electrometer and explain how it may be used (*a*) to compare two resistances, and (*b*) to measure an alternating potential whose r.m.s. value is about 50 V. Comment on (*a*).

3·11. An electroscope has a capacitance of 20 pF. and its leaves diverge 30 scale divisions when they are raised to a potential of 450 V. If in this position the divergence of the leaves decreases at a rate of one division in 2 minutes owing to defective insulation, find the leakage current and the resistance of the insulation. Indicate any assumptions you make. [$2{\cdot}50 \times 10^{-12}$ amp., $1{\cdot}77 \times 10^{14}$ ohm.]

3·12. The case of a gold-leaf electroscope and one plate of a condenser which forms part of an ionization chamber are earthed. The leaf is connected to the upper plate which is raised to a high potential. When X-rays ionize the air between the plates, the potential of the leaf falls at the rate of 90 V.min.$^{-1}$. If the capacitance of the electroscope is 4 cm. and the plates of the condenser each have an area of 60π cm.^{2} and are 2·5 cm. apart, obtain a value for the ionization current in amperes. [5×10^{-9} amp.]

3·13. A high-voltage generator for the production of high speed electrons consists of an inner spherical electrode of radius a, surrounded by an earthed concentric spherical conductor of internal radius b. The inner electrode is supported on a uniform solid conical insulator whose material has a permittivity κ. The generators of the surface of this insulator are normal to the surfaces of the electrodes and each makes an angle θ with the axis of the cone. Find the capacitance of the system when a high vacuum is maintained therein.

The inner electrode is raised to a high potential and from it an electron, charge $-e$, mass m, and initially at rest, is released. If the electron arrives with velocity u at the outer electrode, prove that the charge on the electrode is

$$-\frac{ab}{4(b-a)}[(\kappa+\kappa_0)-(\kappa-\kappa_0)\cos\theta]\frac{m}{e}u^2.$$

How will the above expression be modified if the generator is in the form of a cylindrical condenser whose cross-section is identical with that of the sphere? End-effects are to be neglected and it may be assumed that the support runs along the whole length, λ, of the generator.

$$\left[\tfrac{1}{2}[(\kappa+\kappa_0)+(\kappa-\kappa_0)\cos\theta]\frac{ab}{b-a};\ -\frac{\lambda}{4\ln\left(\frac{b}{a}\right)}\left[1+(\kappa-\kappa_0)\frac{\theta}{\pi}\right]\frac{mu^2}{e}.\right]$$

3·14. Prove that the capacitance per unit length of a coaxial cylindrical condenser is $\dfrac{\kappa}{2\ln\left(\frac{b}{a}\right)}$, where the symbols have their usual meanings.

Suppose that the inner conductor of such a condenser is supported by triplets of sectorial spacers arranged in planes normal to the axis of the condenser. Six spacers are made by dividing a circular ring of insulating material into six equal parts ; the ring has inner and outer radii a and b, is of thickness t, and its material has a permittivity 3. If there are N triplets of spacers per unit length of the line, prove that the average capacitance per unit length of line is $\kappa_a\left[\dfrac{1 + Nt}{2 \ln\left(\dfrac{b}{a}\right)}\right]$.

3·15. An air condenser is formed by two equal rectangular metal plates, height h and breadth b. The planes containing the plates are inclined to one another at an angle θ, which is not large, and the distance from the nearer edge of each plate to the line of intersection of the planes is a. It may be assumed that the lines of force are circular and the field strength, E, constant in magnitude along any such line. If V is the potential difference between the plates, and r the least distance of a point from the above line of intersection, neglecting all edge-effects, prove that

$$E = \frac{V}{r\theta},$$

and if σ is the surface density of the electrification on the positive plate, $\sigma = \dfrac{\kappa_a V}{4\pi r\theta}$.

Write down an expression for the energy of the condenser and show that the couple tending to decrease θ is

$$\frac{\kappa_a V^2 h}{8\pi\theta^2} \ln\left(1 + \frac{b}{a}\right).$$

3·16. Obtain an expression for the strength of an electric field in terms of the potential gradient along the lines of force.

Suppose that two straight lines OA, OB, enclose an angle of $\dfrac{\pi}{8}$ and that along OA lengths OD and DA, equal to 15 cm. and 60 cm. respectively are marked off. Similarly, lengths OE and EB, respectively equal to 15 cm. and 60 cm., are marked off along OB. Now suppose that the lines DA and EB represent the edges of a variable air condenser consisting of two square metal plates carried on insulating arms which are hinged at O. If a difference of potential of 6000 V. is maintained across the plates, find the couple which the mutual attraction between the plates exerts about the hinge at O. [510 dyne.cm.]

CHAPTER IV

THE FUNDAMENTAL PROPERTIES OF MAGNETS

Natural and artificial magnets.—The peculiar properties of a mineral known as magnetite were known in classical times. This mineral consists mainly of an oxide of iron, Fe_3O_4, and it is sometimes known as lodestone. Its remarkable properties are (*a*) that a large piece of the mineral is capable of picking up smaller pieces of the same material or of iron, and (*b*) that when freely suspended it always comes to rest in one particular position at any one place. Moreover it was found that this somewhat peculiar substance was able to impart similar properties to pieces of iron when these were stroked from one end to the other with a piece of lodestone. The iron had become magnetized by this simple operation, i.e. it had acquired the remarkable properties of lodestone itself. Pieces of lodestone are known as natural magnets, a name serving to distinguish them from magnets which are produced artificially.

Little progress in the science of magnetism, except that about the twelfth-century navigators had learned to use a magnet to indicate direction at sea, was made until Dr. Gilbert, a physician to Queen Elizabeth I, inaugurated his experiments on magnetism and in 1600 published his famous treatise '*De Magnete*'. He discovered that there are two kinds of magnetic pole, the positive or north-seeking and the negative or south-seeking. They are characterized by the fact that like poles repel one another, whereas unlike poles attract one another.

The law of force between magnetic poles.—Coulomb, using his celebrated torsion balance, established the fact that in air the force between two magnetic poles varies inversely as the square of the distance apart. He also showed that this force is directly proportional to the 'strengths' of the poles. If F is this force, considered positive when the force is one of repulsion, between two poles of strength m_1 and m_2 at a distance r apart in air, then Coulomb found that

$$F = \alpha \frac{m_1 m_2}{r^2},$$

where α is a constant. Since the units of distance have already been fixed it is convenient to select a unit of magnetic pole strength or, more briefly, the unit magnetic pole, so that the above constant α is unity. Thus the unit magnetic pole is a pole of positive magnetism of such strength that when placed 1 cm. away from an equal pole in a vacuum, the repulsive force between them is one dyne.

Then $$\mathrm{F} = \frac{m_1 m_2}{r^2}.$$

Since, in practice, it is impossible to obtain a 'point' pole of unit strength, the above definition is not all that could be desired, for the question arises as to the precise points in the poles between which the distance is to be measured and, moreover, if poles of such strength are brought together there is a tendency for one to reduce the strength of the other. This latter difficulty is only apparent, however, for poles of $\frac{1}{n}$th the unit strength could be considered, where n is large, when the force would be n^{-2} dyne.

Moreover, strictly speaking, if μ_0 is the permeability, cf. pp. 109 and 363, of free space, the equation for the force of repulsion between point poles in free space should be written

$$\mathrm{F} = \frac{m_1 m_2}{\mu_0 r^2}.$$

This equation is correct dimensionally.

Strength of a magnetic field.—The field strength at a point in a magnetic field is given in magnitude and direction by the force per unit pole which a small north-seeking pole would experience if situated at the point in question, provided that the small pole is so feeble that its presence does not disturb the configuration of the original field.

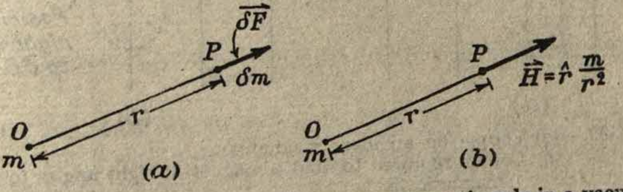

$\overrightarrow{\delta F}$
P
δm
r
O
m
(a)
P
$\vec{H} = \hat{r}\,\frac{m}{r^2}$
r
O
m
(b)

FIG. 4·01.—Magnetic field strength due to a magnetic pole in a vacuum.

The field strength, H, at a point P in a vacuum and at a distance r from a magnetic pole of strength m may be calculated as follows. Let O, Fig. 4·01, be the point at which the magnetic pole is situated and let δF be the force acting on a small pole δm at P. Then

$$\delta F = \frac{m\,\delta m}{r^2}, \quad \text{or} \quad \frac{\delta F}{\delta m} = \frac{m}{r^2}.$$

$$\therefore\ H = \lim \frac{\delta F}{\delta m} = \frac{m}{r^2}.$$

In vector notation this is written

$$\vec{H} = \hat{r}\frac{m}{r^2},$$

where $\hat{r}$ is a unit vector in the direction of r increasing.

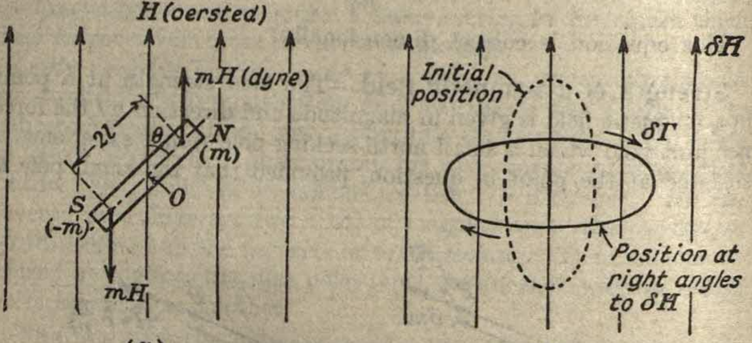
When the pole is surrounded by a medium, with permeability μ, the above equations become

$$H = \frac{m}{\mu r^2}, \quad \text{and} \quad \vec{H} = \hat{r}\frac{m}{\mu r^2}.$$

Hence, strictly speaking, in a vacuum, with permeability μ_0 (numerical value unity),

$$H = \frac{m}{\mu_0 r^2}, \quad \text{and} \quad \vec{H} = \hat{r}\frac{m}{\mu_0 r^2}.$$

Couple on an ideal magnet in a uniform field.—Let NS, Fig. 4·02 (*a*), be an ideal magnet, i.e. a magnet with point poles.



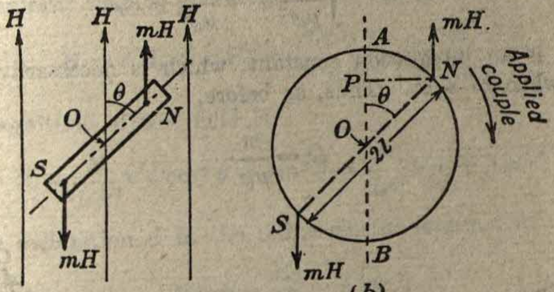
(*a*) (*b*)

Fig. 4·02.—(*a*) Couple on an ideal magnet.
(*b*) Couple required to hold a magnet at right angles to a small magnetic field.

Suppose that it is held at rest in a uniform magnetic field of strength H and its axis makes an angle θ with the field. Let $2l$ be the distance between the poles. The force on each pole is mH, but since the forces are in contrary senses they constitute a couple of moment $mH(2l)\sin\theta$. The quantity $2ml$ is the magnetic moment of the magnet; it is denoted by the symbol M. Thus the couple on the magnet, due to the field H, is MH sin θ and a couple equal and opposite to this must be applied to the magnet to hold it at rest and with its axis inclined at an angle θ to the field H.

Magnetic moment.—The definition of magnetic moment given in the preceding paragraph is not altogether satisfactory for it involves the idea of a magnetic pole and, moreover, the magnetic field in which the magnet is placed must affect the distribution of magnetism in the magnet. If, however, the magnet is placed in a uniform magnetic field of strength δH, a couple will be exerted on the magnet whenever its axis does not coincide with the direction of the field δH. Let $\delta \Gamma$ be the couple required to hold the magnet at right angles to the field δH, cf. Fig. 4·02 (*b*). Then $\lim \dfrac{\delta \Gamma}{\delta H} = M$ is called the magnetic moment of the magnet.

The work required to effect the rotation of a freely suspended magnet in a uniform horizontal magnetic field.—To calculate the work which must be done on a magnet in order to deflect it from its equilibrium position in a uniform horizontal magnetic field into the position it actually occupies, we may proceed as follows.



(*a*) (*b*)

FIG. 4·03.—The work required to effect the rotation of a magnet in a uniform magnetic field.

Consider Fig. 4·03 (*a*), in which the axis of a bar magnet NS makes an angle θ with a uniform magnetic field H; the magnet is assumed to be suspended by a torsionless fibre passing through O, the centre of the magnet. The forces on each pole are mH. The work done on the magnet in effecting its displacement from its equilibrium position AB, cf. Fig. 4·03 (*b*), is $2(mH) . AP$, where P is the projection of N on AB. If $2l$ is the magnetic length of the magnet, i.e. $l = OA$, the work done is

$$\begin{aligned} 2mH(OA - OP) &= 2mlH(1 - \cos\theta) \\ &= MH(1 - \cos\theta). \end{aligned}$$

It is convenient to refer to this work as the ***rotational potential energy*** of the magnet; this is necessary in order to avoid confusion with the potential energy, cf. p. 388, which a magnet possesses because it is magnetized.

Potential due to a magnetic pole in free space.—The potential at a point in a magnetic field is defined as the work done against the field per unit positive pole in bringing up a small pole from infinity to the point in question.

Thus, if the field is due to a single pole m in free space, the magnetic potential, Ω, at a point distant r from it is given by

$$\Omega = \lim \frac{\delta \mathrm{W}}{\delta m} = \lim \frac{\left[\int_{\infty}^{r} \frac{m}{\mu_0 r^2}(-\,dr)\right]\delta m}{\delta m} = \frac{m}{\mu_0 r}.$$

Alternatively, we may use the facts that $\mathrm{H} = -\frac{d\Omega}{dr}$ and, for a single pole in free space, $\mathrm{H} = \frac{m}{\mu_0 r^2}$. Hence

$$\Omega = -\int \frac{m}{\mu_0 r^2} dr = \frac{m}{\mu_0 r} + \mathrm{A},$$

where A is an integration constant, which is necessarily zero if $\Omega \rightarrow 0$ when $r \rightarrow \infty$. Thus, as before,

$$\Omega = \frac{m}{\mu_0 r}.$$

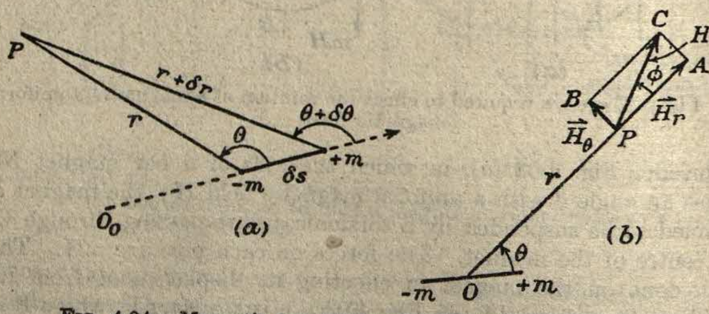


FIG. 4·04.—Magnetic potential and field strength due to a magnetic dipole.

Potential and field strength due to a dipole.—The magnetic potential at a point P in the field of the dipole shown in Fig. 4·04 (*a*), and assumed to be in a vacuum, is given by

$$\begin{aligned}\Omega_{\mathrm{P}} &= -\frac{m}{\mu_0}\left(\frac{1}{r}\right) + \frac{m}{\mu_0}\left[\frac{1}{r} + \frac{\partial}{\partial s}\left(\frac{1}{r}\right)\delta s\right] \\ &= \frac{m}{\mu_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial s} + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)\frac{\partial \theta}{\partial s}\right]\delta s \\ &= \frac{m}{\mu_0}\left(-\frac{1}{r^2}\right)\cos(\pi - \theta)\,\delta s + 0 = \frac{\mathrm{M}\cos\theta}{\mu_0 r^2},\end{aligned}$$

where M is the magnetic moment of the dipole. This expression for Ω_{P} may be written $\frac{\vec{\mathrm{M}}\cdot\hat{r}}{\mu_0 r^2}$ or $\frac{\vec{\mathrm{M}}\cdot\vec{r}}{\mu_0 r^3}$.

The components of the field strength at P, cf. Fig. 4·04 (*b*), are given in the usual way by

$$\mathrm{H}_r = -\frac{\partial \Omega}{\partial r} = \frac{2\mathrm{M}\cos\theta}{\mu_0 r^3},$$

and
$$\mathrm{H}_\theta = -\frac{1}{r}\frac{\partial \Omega}{\partial \theta} = \frac{\mathrm{M}\sin\theta}{\mu_0 r^3}.$$

The resultant field is given by

$$\vec{\mathrm{H}} = \hat{r}\,\mathrm{H}_r + \hat{\theta}\,\mathrm{H}_\theta = \hat{r}\frac{2\mathrm{M}\cos\theta}{\mu_0 r^3} + \hat{\theta}\frac{\mathrm{M}\sin\theta}{\mu_0 r^3}.$$

The magnitude of this field is

$$\mathrm{H} = \frac{\mathrm{M}}{\mu_0 r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{\mathrm{M}}{\mu_0 r^3}\sqrt{1 + 3\cos^2\theta},$$

and its inclination ϕ to the radius vector is given by

$$\tan\phi = \frac{\mathrm{H}_\theta}{\mathrm{H}_r} = \tfrac{1}{2}\tan\theta.$$

Magnetic field due to a dipole at a point (*a*) on its axis and (*b*) on a line passing through the centre of the magnet and normal to the axis.—***(a) Let the point be (r, 0).*** Assuming that the magnet lies in a vacuum, we have $\mathrm{H}_r = \frac{2\mathrm{M}}{\mu_0 r^3}$ and $\mathrm{H}_\theta = 0$, so that the magnetic field is directed along the axis of the dipole.

(b) Let the point be (r, ½π). Then $\mathrm{H}_r = 0$ and $\mathrm{H}_\theta = \frac{\mathrm{M}}{}$ so that the field is parallel to the axis of the dipole and its dire contrary to the positive direction of the above axis.

Gauss' verification of the inverse square law.—In developing the equations for the field strength due to a magnetic dipole the inverse square law has been assumed. If we assume that the force between poles varies inversely as the nth power of the distance, then, at a point distant r from a pole of strength m situated in an isotropic medium with permeability μ, the magnetic potential is given by

$$\Omega = \int_{\infty}^{r} \left(\frac{m}{\mu r^{n}}\right)(-\,dr) = \frac{m}{\mu(n-1)r^{n-1}}.$$

The potential due to a dipole of length δs is given by

$$\begin{aligned}\Omega &= -\frac{m}{\mu}\left[\frac{1}{(n-1)r^{n-1}}\right] + \frac{m}{\mu}\left[\frac{1}{n-1}\left\{\frac{1}{r^{n-1}} + \frac{\partial}{\partial s}\frac{1}{r^{n-1}}\delta s\right\}\right]\\ &= \frac{m}{\mu(n-1)}\left[\frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}}\right)\frac{\partial r}{\partial s}\delta s + \frac{\partial}{\partial \theta}\left(\frac{1}{r^{n-1}}\right)\frac{\partial \theta}{\partial s}\delta s\right]\\ &= \frac{\mathrm{M}}{\mu(n-1)}\cos\theta.\frac{n-1}{r^{n}} + 0 = \frac{\mathrm{M}\cos\theta}{\mu r^{n}}.\end{aligned}$$

$$\therefore\ \mathrm{H}_r = -\frac{\partial\Omega}{\partial r} = \frac{n\mathrm{M}\cos\theta}{\mu r^{n+1}}, \quad\text{and}\quad \mathrm{H}_\theta = -\frac{1}{r}\frac{\partial\Omega}{\partial\theta} = \frac{\mathrm{M}\sin\theta}{\mu r^{n+1}}.$$

It therefore follows that at the points $(r, 0)$ and $(r, \frac{1}{2}\pi)$, the tangent A and B positions of Gauss, respectively, the fields are given by

$$\mathrm{H}_{\mathrm{A}} = \frac{n\mathrm{M}}{\mu r^{n+1}} \quad\text{and}\quad \mathrm{H}_{\mathrm{B}} = \frac{\mathrm{M}}{\mu r^{n+1}};$$

each field is parallel to the axis of the magnet.

It is this last fact which enables us, when the surrounding medium is air, simply to compare by means of a deflexion magnetometer, H_{A} and H_{B} with H_0, the horizontal component of the earth's magnetic field and, hence, following Gauss to show that $n = 2$.

In 1832 GAUSS carried out a series of experiments on the above lines. The magnet which he used was about a foot long and had a mass of about 1 lb. The magnetometer was of the reflecting type and carried a plane mirror at the end of the suspended needle. A horizontal scale in mm. etc. and more than a metre long was fixed above a telescope five metres away from the magnetometer needle. The telescope was arranged so that the image of the horizontal scale above it, which was formed by reflexion in the plane mirror, could be observed. In this way small angular displacements of the magnetometer needle could be measured; a displacement of one scale division across the fiducial mark in the observing telescope corresponded to an angular rotation of 22 seconds. A rotation of one-tenth this amount could be estimated by eye. Now Gauss was fully appreciative of the fact that the chief uncertainty in these experiments was in estimating the distance between the poles of the magnet. He therefore decided to work with the magnet at distances between 1 and 4 metres from the magnetometer so that the fraction $\frac{l}{r}$ was always small and certainly negligible when r exceeded 2 metres.

Some of the results he obtained on 24–28th January 1832 are included in the following table.

Distance of magnet from magnetometer (r)	Deflexion of needle	
	tan A position	tan B position
1·1 metre	2° 13′ 51·2″	1° 10′ 19·3″
2·0 ,, ...	37′ 16·2″	19′ 1·6″
4·0 ,, ...	4′ 35·9″	2′ 22·2″

Since $\tan \theta_A \simeq 2 \tan \theta_B$, it was verified that n is equal to 2 within the limits of experimental error. Now it would be very remarkable if such a universal law should contain an index 2 ± a small fraction. It is therefore concluded that the value of n is exactly 2.

It is interesting to note that with the same apparatus Gauss made the first absolute determination of H_0, the horizontal component of the earth's magnetic field. Until then, values for H_0 at different places had only been compared by an oscillation method.

Potential and field strength due to a finite magnet at the points $(r, 0)$, $(r, \frac{1}{2}\pi)$.—(*a*) Let NS, Fig. 4·05 (*a*), be the finite magnet, of pole strength m and magnetic length $2l$. If P is the point $(r, 0)$ and it lies in an isotropic medium with permeability μ, we have

$$\Omega_P = \frac{1}{\mu}\left[\frac{m}{r-l} - \frac{m}{r+l}\right] = \frac{2ml}{\mu(r^2 - l^2)} = \frac{M}{\mu(r^2 - l^2)}. \qquad [M = 2ml.]$$

$$\therefore\ [H_r]_P = -\frac{\partial \Omega_P}{\partial r} = \frac{2Mr}{\mu(r^2 - l^2)^2},$$

and $$[H_\theta]_P = -\frac{1}{r}\frac{\partial \Omega_P}{\partial \theta} = 0.$$

Thus the magnetic field at P is directed along r in the direction of r increasing.

(*b*) If P is the point $(r, \frac{1}{2}\pi)$, Fig. 4·05 (*b*), the magnetic potential at P is, by symmetry, zero, and the field strength cannot therefore be evaluated by differentiation at once. Consider, however, the potential at Q, the point whose rectangular coordinates referred to

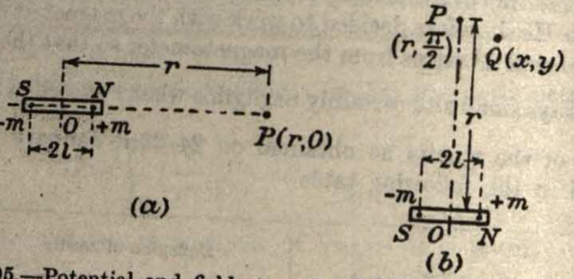


FIG. 4·05.—Potential and field strengths due to a finite magnet at the points $(r,\ 0)$ and $(r,\ \frac{1}{2}\pi)$.

O, the centre of the magnet, are $(x,\ y)$. Then

$$\Omega_Q = \frac{1}{\mu}\left[\frac{m}{\sqrt{(l-x)^2+y^2}} - \frac{m}{\sqrt{(l+x)^2+y^2}}\right].$$

Let the magnetic field strength at Q be $(\mathrm{H}_x,\ \mathrm{H}_y)$. Then

$$\mathrm{H}_x = -\frac{\partial\Omega_Q}{\partial x} = -\frac{m}{\mu}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{(l-x)^2+y^2}} - \frac{1}{\sqrt{(l+x)^2+y^2}}\right]$$

$$= -\frac{m}{\mu}\left[-\frac{(l-x)(-1)}{\{(l-x)^2+y^2\}^{\frac{3}{2}}} + \frac{(l+x)}{\{(l+x)^2+y^2\}^{\frac{3}{2}}}\right].$$

$$\therefore \left[\mathrm{H}_x\right]_{\substack{x=0\\y=r}} = \frac{m}{\mu}\left[\frac{-l}{(r^2+l^2)^{\frac{3}{2}}} - \frac{l}{(r^2+l^2)^{\frac{3}{2}}}\right] = -\frac{\mathrm{M}}{\mu(r^2+l^2)^{\frac{3}{2}}}.$$

Similarly

$$\left[\mathrm{H}_y\right]_{\substack{x=0\\y=r}} = 0,$$

so that the resultant field has magnitude $\frac{\mathrm{M}}{\mu(r^2+l^2)^{\frac{3}{2}}}$ and is directed in the negative direction and parallel to the x-direction.

The magnetic field strength due to a bar magnet at a point on its axis.—Let m be the pole-strength, $2l$ the magnetic length of the magnet NS, Fig. 4·06 (a), and r the distance of the point A, at which the field is required, from the centre of the magnet.

Let the magnet lie in a medium with permeability μ and suppose a small pole δm is placed at A. Then the force on the pole due to $+m$ is $\frac{m\,\delta m}{\mu(r-l)^2}$. This is considered positive. The force due to $-m$ is $-\frac{m\,\delta m}{\mu(r+l)^2}$. These two forces act along the direction of r

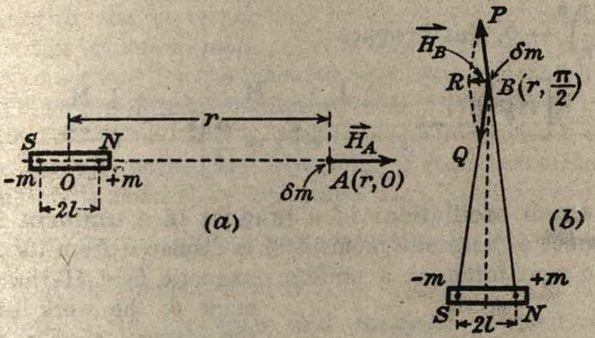


FIG. 4·06.—Field strengths due to a finite magnet at the points $(r, 0)$ $(r, \frac{1}{2}\pi)$; the tangent-A and the tangent-B positions of Gauss.

increasing so that their resultant δF_A is given by

$$\delta F_A = \frac{1}{\mu}\left[\frac{m}{(r-l)^2} - \frac{m}{(r+l)^2}\right]\delta m = \frac{2Mr\,\delta m}{\mu(r^2-l^2)^2},$$

where M is the magnetic moment of the magnet.

The field strength at A is the force per unit positive pole, viz. $\delta F_A \div \delta m = H_A$, say.

$$\therefore\ H_A = \frac{2Mr}{\mu(r^2-l^2)^2}, \quad \text{or} \quad \vec{H}_A = \hat{r}\,\frac{2Mr}{\mu(r^2-l^2)^2}.$$

If $\left(\frac{l}{r}\right)^2 \rightarrow 0$, this expression becomes

$$\left[\vec{H}_A\right]_{\left(\frac{l}{r}\right)^2\rightarrow 0} = \hat{r}\,\frac{2Mr}{\mu r^4\left[1-\frac{l^2}{r^2}\right]^2} = \hat{r}\,\frac{2M}{\mu r^3}.$$

The field strength due to a bar magnet at a point on the straight line bisecting its length at right angles.—Let B, Fig. 4·06 (*b*), be the point $(r, \frac{1}{2}\pi)$ and suppose that a pole δm is at B. The forces on it are $\dfrac{m\,\delta m}{\mu(r^2+l^2)}$ along BP and $\dfrac{m\,\delta m}{\mu(r^2+l^2)}$ along BQ.

The resultant of these forces is along BR, where BR is parallel to NS, and is given by

$$\frac{1}{\mu}\cdot\frac{2m\,\delta m}{(r^2+l^2)}\cdot\frac{l}{(r^2+l^2)^{\frac{1}{2}}} = \delta F_B, \text{ (say)}.$$

$$\therefore\ H_B = \lim\frac{\delta F_B}{\delta m} = \frac{1}{\mu}\,\frac{M}{(r^2+l^2)^{\frac{3}{2}}}, \quad \text{or} \quad \vec{H}_B = \hat{\theta}\,\frac{1}{\mu}\,\frac{M}{(r^2+l^2)^{\frac{3}{2}}},$$

where $\hat{\theta}$ is a unit vector perpendicular to $\hat{r}$ and parallel to NS.

When $\left(\frac{l}{r}\right)^2 \to 0$, this becomes

$$\left[\vec{\mathrm{H}}_{\mathrm{B}}\right]_{\left(\frac{l}{r}\right)^2\to 0} = \hat{\theta}\cdot\frac{1}{\mu}\cdot\frac{\mathrm{M}}{r^3\left[1+\frac{l^2}{r^2}\right]^{\frac{3}{2}}} = \hat{\theta}\cdot\frac{1}{\mu}\cdot\frac{\mathrm{M}}{r^3}.$$

The small oscillations of a magnet in a uniform field.—If a magnet of magnetic moment M is displaced from its position of static equilibrium in a uniform magnetic field H through an angle θ, the work done on the magnet, cf. Fig. 4·07, is

$$\mathrm{MH}(1-\cos\theta).$$

This measures the rotational potential energy of the magnet in its displaced position; in the position of static equilibrium the rotational potential energy of the magnet is taken to be zero.

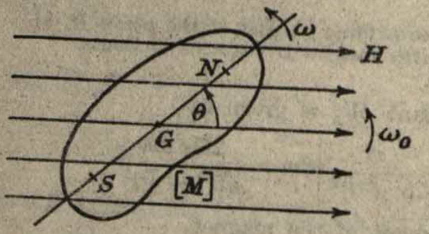


FIG. 4·07.—The small oscillations of a magnet in a uniform magnetic field.

Now suppose that a magnet oscillates in a horizontal plane and about a vertical axis through its centre of gravity G. Let ω_0 be the angular velocity with which the magnet crosses its position of static equilibrium. If ω is its angular velocity when its axis makes an angle θ with the above position we have, by the principle of the conservation of energy,

$$\tfrac{1}{2}\mathrm{I}\omega_0^2 + 0 = \tfrac{1}{2}\mathrm{I}\omega^2 + \mathrm{MH}(1-\cos\theta)$$
$$= \tfrac{1}{2}\mathrm{I}\dot{\theta}^2 + \mathrm{MH}(1-\cos\theta).$$

Differentiating with respect to time, we get

$$0 = \mathrm{I}\dot{\theta}.\ddot{\theta} + \mathrm{MH}\sin\theta.\dot{\theta}.$$

If $\dot{\theta}$ is not zero, and $\theta \to 0$, we have

$$\ddot{\theta} + \frac{\mathrm{MH}}{\mathrm{I}}\theta = 0.$$

Thus the motion is simple harmonic with period $\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{\mathrm{MH}}}$.

Magnetic media.—In making our investigations concerning magnets we began by assuming them to be situated in a vacuum. When an isotropic medium is present the necessary changes have been represented by the introduction of a factor μ, the permeability of the medium. The justification for this procedure comes from analogy with the corresponding problem in electrostatics. To examine this question more fully, while at present making no attempt to account for the magnetic properties of material media, it will be assumed that when an isotropic medium is placed in a magnetic field it acquires a certain magnetic moment per unit volume. This is termed the ***intensity of magnetization*** in the medium, and is denoted by the symbol J. [This term is discussed more fully in Chap. XI.] For most media the direction of $\vec{J}$ coincides with that of the magnetic field $\vec{H}$.

Magnetic field strength and magnetic induction.—The magnetic field strength at a point in air [strictly speaking, in a vacuum], is defined as the force per unit positive pole on a small positive pole placed at the point. When it is desired to measure the force on such a pole inside a magnetizable substance, and all substances are magnetizable although the degree of magnetization varies enormously, a cavity must first be made in the specimen so that the small pole may be introduced into it. Now the walls of the cavity will exhibit magnetic polarity which will contribute to the total force on the small pole in the cavity. The contribution will be determined, in part at least, by the shape of the cavity, which must therefore be carefully specified if the physical interpretation of this force is to have a definite meaning.

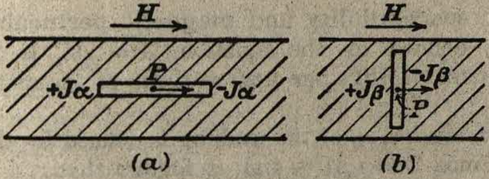


Fig. 4·08.—Magnetic induction.

Let us consider the force per unit positive pole on a small positive pole, δm, at the point P, Fig. 4·08 (*a*), near the centre of a cylindrical cavity, whose diameter is small compared with its length, and whose axis is in the direction of the magnetization at P. The induced magnetism will appear only on the ends of this cavity. If J is the intensity of magnetization, and α the cross-section of the cavity, the charges of magnetism at the ends of the cavity will be $J\alpha$ and $-J\alpha$ respectively. The force on the small pole δm at P due to the magnetism on the walls of the cavity is $\frac{1}{\mu_0}\left(\frac{J\alpha}{r_1^2} + \frac{J\alpha}{r_2^2}\right)\delta m$, where r_1 and r_2 are the distances of δm from the ends of the cavity. This is zero, since both r_1 and r_2 are very long compared with the width of the cavity. The force per unit positive pole at P is therefore due to the magnetizing field, H.

Now consider the force on δm when this is at P a point within a cavity whose length is small compared with its diameter; the cavity resembles a disc and is shown in section in Fig. 4·08 (*b*). Again let the axis of the cylinder be parallel to the magnetizing field. Let β be the area of each plane face of the disc. It is only on these faces that induced magnetism will appear. Now the contribution to the force per unit positive pole on δm due to these induced charges of magnetism is $\frac{4\pi J}{\mu_0}$, a result obtained from analogy with the corresponding problem in electrostatics [cf. p. 26].

The total force per unit positive pole on the small pole in the cavity is $H + \frac{4\pi J}{\mu_0}$, and is a measure of the ***magnetic induction***, B of the material, which, by analogy with the corresponding problem in electrostatics, cf. p. 26, is defined by

$$B = \mu_0 H + 4\pi J.$$

The unit of magnetic induction is the *gauss*.

Magnetic susceptibility and magnetic permeability.—The quantity χ, defined by the equation, $J = \chi H$, is termed the ***magnetic susceptibility per unit volume*** of the material of the specimen.

The ***permeability***, μ, of the medium is defined by the equation $B = \mu H$. Since $B = \mu_0 H + 4\pi J$, it follows that

$$\mu = \mu_0 + 4\pi\chi.$$

Since the magnetic induction B is related to the magnetic field strength H in the same way as electric induction or displacement is to electric field strength, it follows that the magnetic field strength at a point at a distance r from a pole of strength m in a medium whose permeability is μ, is given by

$$H = \frac{m}{\mu r^2}, \quad \text{or} \quad \vec{H} = \hat{r}\frac{m}{\mu r^2}.$$

The magnetic induction is

$$B = \mu H = \frac{m}{r^2}, \quad \text{or} \quad \vec{B} = \hat{r}\frac{m}{r^2}.$$

Similarly, the magnetic potential due to a pole of strength m and surrounded by an isotropic medium with permeability μ, is given by

$$\Omega = \frac{m}{\mu r}.$$

Gauss' theorem.—This states that the flux of magnetic induction across a closed surface, with a unique outward-drawn normal at every point on it, is always zero,

i.e. $$\int \vec{B}\cdot\hat{n}\,dS = 0.$$

To establish this theorem for isotropic media, since $\vec{B} = \mu_0\vec{H} + 4\pi\vec{J}$, we may rewrite the above equation in the form

$$\mu_0 \int \vec{H}\cdot\hat{n}\,dS + 4\pi \int \vec{J}\cdot\hat{n}\,dS = 0.$$

Now the whole of the volume enclosed by the surface may be divided into cylindrical elements of volume δv, and each with its axis parallel to the $\vec{J}$. The magnetic moment of an element will be $\vec{J}\delta v$ and thus each element may be replaced by an ideal magnet which will consist of two equal and opposite poles at the appropriate distance apart and, of necessity, be situated in a vacuum. From the analogy existing between $\vec{H}$ and $\vec{E}$, we infer at once that

$$\mu_0 \int \vec{H}\cdot\hat{n}\,dS = 4\pi\Sigma(m),$$

where $\Sigma(m)$ is the sum of all the poles enclosed by S. Now since poles only occur in pairs, equal in magnitude and opposite in sign, the only cylindrical elements which contribute to $\Sigma(m)$ are those which are intersected by the Gaussian surface. When this occurs one pole of the ideal magnet will lie within that surface; two possibilities arise.

Consider therefore, as shown in Fig. 4·09, the two elementary magnets M_1 and M_2, each being cut by S. In the first instance the negative pole lies within S whereas in the second it is the positive pole which is thus situated. For M_1 let δS_1 be the area of the intersection and suppose the normal to it makes an angle α_1 with $\vec{J}$. If l_1 is the axial length of this magnet, its magnetic moment is $Jl_1(\cos\alpha_1\,\delta S_1)$, so that for it $m_1 = J\cos\alpha_1\,\delta S_1 = \vec{J}\cdot\hat{n}_1\,\delta S_1$. The strength of the enclosed pole is therefore $-\vec{J}\cdot\hat{n}_1\,\delta S_1$.

For M_2 the cross-section intercepted by S has an area $\delta S_2\cos(\pi-\alpha_2)$; its magnetic moment is therefore $Jl_2\cos(\pi-\alpha_2)\,\delta S_2$. Its positive pole, which is now the one enclosed by S, has a strength given by

$$\begin{aligned} m_2 &= J\cos(\pi-\alpha_2)\,\delta S_2 = -J\cos\alpha_2\,\delta S_2 \\ &= -\vec{J}\cdot\hat{n}_2\,\delta S_2. \end{aligned}$$

Hence the sum of all the poles within S is

$$0 - \int \vec{\mathrm{J}} \cdot \hat{n}\, d\mathrm{S}.$$

$$\therefore\ \mu_0 \int \vec{\mathrm{H}} \cdot \hat{n}\, d\mathrm{S} = 4\pi\left[0 - \int \vec{\mathrm{J}} \cdot \hat{n}\, d\mathrm{S}\right],$$

i.e.

$$\int \vec{\mathrm{B}} \cdot \hat{n}\, d\mathrm{S} = 0.$$

The unit of magnetic flux is the ***maxwell.***

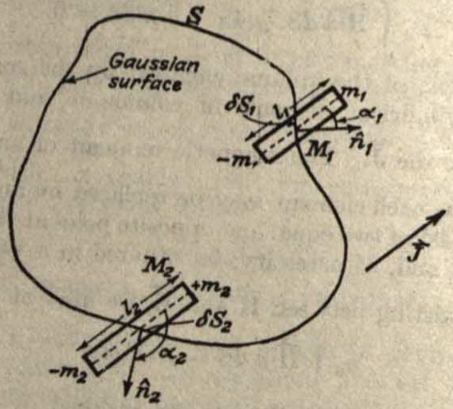


FIG. 4·09.—Gauss' theorem in magnetostatics.

On the flux of magnetic induction from an isolated magnetic pole.—If a Gaussian surface surrounds an isolated magnetic pole of strength m, the flux across the surface is $4\pi m$ maxwells. Thus the flux from a pole of unit strength is 4π maxwells.

If the surface is not closed but has a definite boundary, Γ, the flux of magnetic induction across Γ due to a point pole of strength m is $m\omega$, where ω is the solid angle that the boundary subtends at m. The proof of this statement follows at once from the fact that the flux across an element of surface $\delta\mathrm{S}$ is

$$\delta\Phi = \vec{\mathrm{B}} \cdot \hat{n}\, \delta\mathrm{S} = \frac{m}{r^2}\hat{r} \cdot \hat{n}\, \delta\mathrm{S} = m\, \delta\omega,$$

where $\delta\omega$ is the solid angle that $\delta\mathrm{S}$ subtends at m. Hence

$$\Phi = m\omega.$$

The importance of these statements will appear later but, in passing, it should be noted that when the flux from any system is known, its effective pole-strength may be obtained by dividing the flux by 4π.

EXAMPLES IV

4·01. A deflexion magnetometer is situated with its centre at a point O where the earth's horizontal field is H_0. A bar magnet of moment M and magnetic length $2l$ is suitably orientated and placed with its centre at a point P in the same horizontal plane as O, the axis of the magnet being parallel to OP and making an angle θ with the magnetic meridian. Derive the condition that the magnetometer needle should set at an angle of 90° with H_0.

Discuss how a method making use of the above analysis could be used to determine a value for the magnetic length of the magnet.

4·02. Two magnets, A and B, are suspended horizontally in turn from the same torsion head. To twist A through 30° requires a twist of the torsion head of 270°. To twist B through 30° the torsion head must be rotated through 360°. Compare the magnetic moments of A and B.

Obtain a *general* expression for the comparison of the magnetic moments of two magnets by the method suggested in the first part of this question. $\left[0{\cdot}73,\ \dfrac{\phi_1 - \theta_1}{\phi_2 - \theta_2} = \dfrac{M_A \sin\theta_1}{M_B \sin\theta_2}.\right]$

4·03. A bar magnet of overall length 10 cm. (2λ) lies in the magnetic meridian and there is a neutral point on its axis 16 cm. from its centre. When the magnet is reversed the neutral point is 11·5 cm. from its centre and on a line bisecting at right angles the length of the magnet. The semi-magnetic length of the magnet $= l = \alpha\lambda$. Take, in turn, $\alpha = 0$, $\alpha = 1$ and assuming H_0, the horizontal component of the earth's magnetic field, to be 0·2 oersted, calculate values for the magnetic moment of the magnet. If M_0 and M'_0 are the calculated values of the magnetic moment when $\alpha = 0$, plot the points $(M_0, 0)$, $(M'_0, 0)$. If M_1 and M'_1 are the corresponding values when $\alpha = 1$, plot the points $(M_1, 1)$ and $(M'_1, 1)$. Join by a straight line the points $(M_0, 0)$ and $(M_1, 1)$; also join $(M'_0, 0)$ and $(M'_1, 1)$. From the diagram determine a value for α for which the calculated magnetic moments will be equal.

Explain how a more accurate value for α may be obtained. [0·65.]

4·04. A small compass needle is placed at a distance r from the centre of a bar magnet (m, l), (*a*) in the tangent A position and (*b*) in the tangent B position of Gauss. If θ and ϕ are the deflexions of the needle in the two instances respectively, show that

$$\frac{\tan\theta}{\tan\phi} = n + \frac{l^2}{r^2}\,\frac{n(n+1)(n+5)}{6},$$

if the law of force is such that the field at distance r in air from a pole of strength m is given by $H = \dfrac{m}{r^n}$, where n is a constant, and terms beyond $\dfrac{l^2}{r^2}$ are neglected.

CHAPTER V

TERRESTRIAL MAGNETISM AND ATMOSPHERIC ELECTRICITY

The magnetic elements.—To specify completely the earth's magnetic field at any point it is necessary to state its strength and its direction in space.

To describe this direction the ***angle of declination*** is first given; this is the angle between the geographical meridian and the magnetic meridian at the station concerned. Let this angle be θ. Then the angle between the horizontal in the plane containing the magnetic meridian and the actual direction of the field at the point considered is stated; this is termed the ***angle of dip***, ϕ.

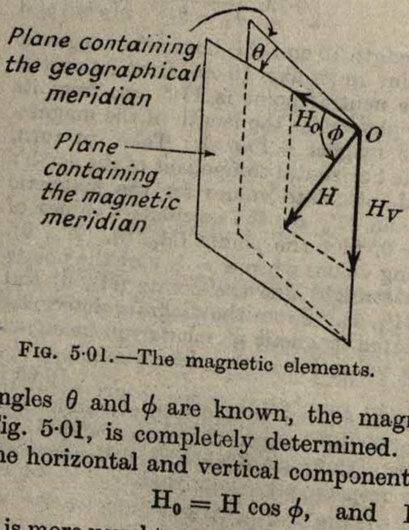

Plane containing the geographical meridian
Plane containing the magnetic meridian
θ
H₀
φ
O
H
H_V

FIG. 5·01.—The magnetic elements.

Thus when H, the total field strength, together with the angles θ and ϕ are known, the magnetic state at a point O, cf. Fig. 5·01, is completely determined. Since, however, H_0 and H_V, the horizontal and vertical components of H, are given by

$$H_0 = H \cos \phi, \quad \text{and} \quad H_V = H \sin \phi,$$

it is more usual to regard H_0, θ and ϕ as the (independent) ***magnetic elements*** at a given station.

Measurement of the angle of dip.—For details concerning the structure of a dip circle, the method of using it, and the elimination of the more important errors that arise when a dip circle is used, the author's 'Intermediate Physics' may be consulted.

Calculation of the dip in a given latitude (ideal case).—This is an important problem since the earth's magnetic field may, as a first approximation, be regarded as due to a 'small' magnet at its centre. We shall therefore suppose that the negative pole of the small magnet, with magnetic moment M, points to the geographic north as shown in Fig. 5·02. Let P be a point on the surface in latitude λ (south). If $\theta = \frac{1}{2}\pi - \lambda$, the magnetic potential at P is given by $\Omega = \dfrac{M\cos\theta}{\mu_a r^2}$, where μ_a is the permeability of air. Hence the vertical component of the magnetic field at P is

$$H_r = -\frac{\partial\Omega}{\partial r} = \frac{2M\cos\theta}{\mu_a r^3}, \quad \text{[cf. p. 103]}$$

$$= \frac{2M\sin\lambda}{\mu_a r^3}.$$

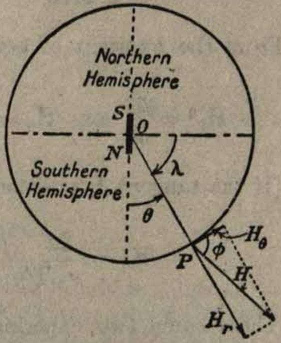


FIG. 5·02.—The dip in a given latitude (ideal case).

The horizontal component of the magnetic field at P is H_θ, where

$$H_\theta = -\frac{1}{r}\frac{\partial\Omega}{\partial\theta} = \frac{M\sin\theta}{\mu_a r^3} = \frac{M\cos\lambda}{\mu_a r^3}.$$

If ϕ is the angle which the resultant magnetic field, H, at a station in the 'southern hemisphere' makes with H_0, i.e. with H_θ [ϕ is the angle of dip], then

$$\tan\phi = \frac{H_r}{H_\theta} = 2\cot\theta = 2\tan\lambda.$$

[In the 'northern hemisphere', $\theta = (\frac{1}{2}\pi + \lambda)$; the vertical component is negative, i.e. it is directed towards O. Also,

$$\tan\phi = -2\tan\lambda,$$

but, by convention, $0 < \phi < \dfrac{\pi}{2}$, so that the minus sign is neglected.]

Experimental determination of H_0.—To determine the absolute value of the horizontal component of the earth's magnetic field two experiments are necessary. In the first or deflexion experiment the value of $\dfrac{M}{H_0}$ is ascertained by using a magnet in the tangent A [or B] position of Gauss. If θ is the mean deflexion of the magnetometer needle, and μ_a, r and l have their usual significance, we have

$$\frac{M}{H_0} = \frac{\mu_a(r^2 - l^2)^2}{2r}\tan\theta = \alpha \text{ erg.oersted.}^{-2} \text{ (say)} \quad . \quad \text{(i)}$$

The second or oscillation experiment consists of a determination of MH_0 by suspending the magnet in the position where the magnetometer was situated and observing its period T. Then, with the notation already explained,

$$T = 2\pi\sqrt{\frac{I}{MH_0}}, \quad \text{or} \quad MH_0 = \frac{4\pi^2 I}{T^2} = \beta \text{ erg. (say)} \qquad \text{(ii)}$$

From the quotient of equations (i) and (ii) we have,

$$H_0^2 = \frac{\beta}{\alpha}, \quad \text{or} \quad H_0 = \frac{2\pi}{\mu_a^{\frac{1}{2}} T (r^2 - l^2)\sqrt{\frac{2rI}{\tan\theta}}} \text{ oersted.} \quad . \qquad \text{(iii)}$$

[If the tangent B position is used, the final equation reduces to

$$H_0 = \frac{2\pi}{\mu_a T}\sqrt{\frac{I}{(r^2 + l^2)^{\frac{3}{2}} \tan\theta}} \text{ oersted.]}$$

These same two experiments also enable us to determine the absolute value of M, for the square root of the product of the first two equations gives

$$M = \sqrt{\alpha\beta} = \frac{\pi\mu_a^{\frac{1}{2}}(r^2 - l^2)}{T}\sqrt{\frac{2.I}{r}} \tan\theta \text{ erg.oersted.}^{-1} \; . \qquad \text{(iv)}$$

In all experimental work on these lines the errors are never so small that μ_a may not be taken as unity.

Example.—A bar magnet, A, suspended horizontally by a fine wire which is attached to a torsion head, sets in the magnetic meridian when the wire is free from torsion. The magnet is deflected 30° from the magnetic meridian when the torsion head is turned through 180°. When A is replaced by another magnet B, the latter is deflected through 60° when the torsion head is rotated through 510°. Compare the magnetic moments of A and B.

Assuming the moments of inertia of the magnets about their axes of rotation to be equal, compare their periods when they oscillate with small amplitudes about their rest positions in the magnetic meridian before the torsion head is rotated.

If b is the couple per radian twist in the suspension, we have, with the usual notation,

$$M_A H_0 \sin 30° = b\left(\frac{180 - 30}{180}\right) \times \pi,$$

and

$$M_B H_0 \sin 60° = b\left(\frac{510 - 60}{180}\right) \times \pi.$$

$$\therefore \frac{M_B}{M_A} = 1{\cdot}73_2.$$

Now, in general, the period T is given by $T = 2\pi\sqrt{\frac{I}{MH + b}}$, since $(MH + b)$ is the restoring couple per unit angular displacement of the magnet. Hence

$$T_A = 2\pi\sqrt{\frac{I}{M_AH_0 + (M_AH_0 \sin 30)\frac{6}{5\pi}}} = 2\pi\sqrt{\frac{I \times \pi}{\pi + 0{\cdot}6}}.\frac{1}{\sqrt{M_AH_0}},$$

and

$$T_B = 2\pi\sqrt{\frac{I}{M_BH_0 + (M_BH_0 \sin 60)\frac{2}{5\pi}}} = 2\pi\sqrt{\frac{I \times \pi}{\pi + \frac{\sqrt{3}}{2}.\frac{2}{5}}}.\frac{1}{\sqrt{M_BH_0}}.$$

∴ after some reduction, we find,

$$\frac{T_A}{T_B} = \sqrt{\frac{M_B}{M_A}}.\sqrt{\frac{\pi + 0{\cdot}346}{\pi + 0{\cdot}6}} = 1{\cdot}27.$$

Example.—Calculate the magnetic latitude at which a line of force comes down to the earth's surface if its distance above the earth's surface at the magnetic equator is 120 kilometres. [It may be assumed that the earth is a sphere of radius $6{\cdot}4 \times 10^3$ km. and its external magnetic field is identical with that which would be produced by a dipole at its centre.]

Let a line of force (outside the earth) be as shown in Fig. 5·03 (*a*).

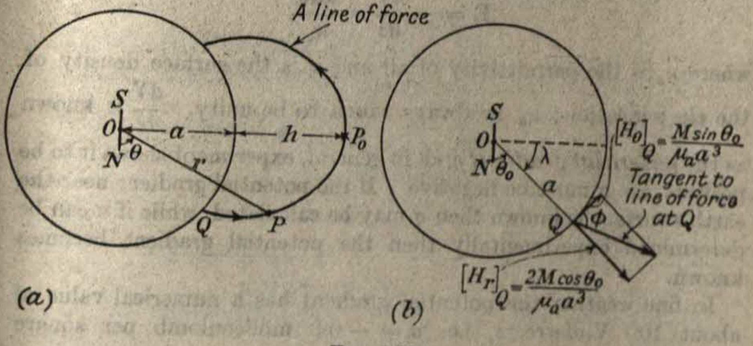

A line of force
S
O
N
θ
a
h
P_0
r
Q
P
(a)
S
O
N
θ_0
λ
a
Q
Φ
$[H_\theta]_Q = \frac{M \sin\theta_0}{\mu_0 a^3}$
Tangent to line of force at Q
$[H_r]_Q = \frac{2M\cos\theta_0}{\mu_0 a^3}$
(b)

FIG. 5·03.

If P is a point on this line, then its polar coordinates are (r, θ), as indicated. Let P_0 be the point on the same line of force at which the tangent is parallel to NS, the dipole. If h is the height of P_0 above the surface of the earth, radius a, then P_0 is the point $\{(a + h), \frac{1}{2}\pi)\}$. Since the equation to a line of force associated with a dipole may be written

$$r = A \sin^2 \theta,$$

where A is a constant, cf. p. 21, we have

$$(a + h) = A \sin^2 \tfrac{1}{2}\pi = A,$$

i.e.

$$r = (a + h) \sin^2 \theta.$$

Let Q be the point in the southern hemisphere at which the above line of force intersects the earth's surface; let its coordinates be (a, θ_0)

as shown in Fig. 5·03 (*b*). Then

$$a = (a + h) \sin^2 \theta_0, \text{ or } \sin^2 \theta_0 = \frac{6\cdot4}{6\cdot52}, \text{ i.e. } \theta_0 = 82^\circ\ 12'.$$

The magnetic latitude required is $\frac{1}{2}\pi - \theta_0$, i.e. $\lambda = 7^\circ\ 48'$.

[If ϕ is the angle of dip at Q and μ_a the permeability of air (assumed constant),

$$\tan \phi = \frac{2\text{M} \cos \theta_0}{\mu_a a^3} \div \frac{\text{M} \sin \theta_0}{\mu_a a^3} = 2 \cot \theta_0$$

$$\therefore \quad \phi = 15^\circ\ 19'.]$$

Direct measurement of the surface density of the earth's charge.—In 1752 LEMONNIER discovered the existence of an electric field in the air above the earth's surface. The direction of this field is such that the charge on the earth's surface must be negative. This is known as the 'fine-weather' charge since, during a thunderstorm, the sign of the charge may be reversed. If V and $\text{V} + \delta\text{V}$ are the potentials at points at distances z and $z + \delta z$, respectively, above the earth's surface, as shown in Fig. 5·04 (*a*), then the electric field strength E, considered positive in the direction of z increasing, is given by

$$\text{E} = -\frac{d\text{V}}{dz} = \frac{4\pi\sigma}{\kappa_a},$$

where κ_a is the permittivity of air and σ is the surface density of the electrification; κ_a is always taken to be unity. $\frac{d\text{V}}{dz}$ is known as the ***potential gradient*** and, in general, experiment shows it to be positive, i.e. σ must be negative. If the potential gradient near the earth's surface is known then σ may be calculated, while if σ can be determined experimentally then the potential gradient becomes known.

In fine weather the potential gradient has a numerical value of about 100 V.metre.$^{-1}$, i.e. $\sigma = -0\cdot9$ millicoulomb per square kilometre.

Now the charge on any surface due to the earth's electric field is so small that a sensitive electrometer must be used to measure it. The instrument devised by C. T. R. WILSON * to measure σ is shown in Fig. 5·04 (*b*). G is a gold-leaf electroscope provided with an earthed outer case E and an inner one I, maintained at a constant potential of about 50 V. by means of a battery B_1. These cases are made from cylindrical brass tubing and the ends are covered with glass plates, silvered on the inside. A horizontal strip of the silvering is removed, where necessary, so that the gold-leaf may be observed with a low power microscope. The gold-leaf is only 0·1 mm. wide and when it is earthed its potential differs from that of the inner case by 50 V. If the potential of the leaf is raised the deflexion will diminish since the difference in potential between the leaf and the case is decreased.

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* *Proc. Camb. Phil. Soc.*, **13**, 184 and 363, 1906.

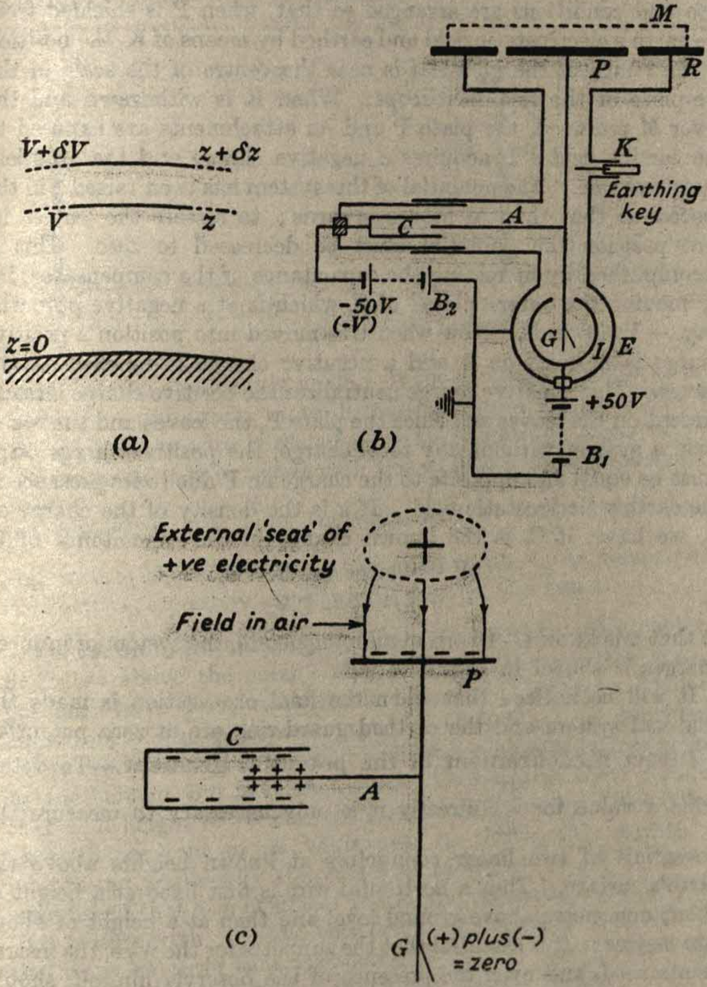

FIG. 5·04.—Wilson's apparatus for measuring σ.

P is a flat circular metal plate of radius r, known as the 'test-plate', and connected to the electroscope as indicated. This plate is mounted flush with the earth's surface and is surrounded by an earthed guard-ring, R. C is a variable cylindrical air-condenser, with its inner plate A, in the form of a brass rod, joined to the gold-leaf system; the outer plate of the cylindrical air-condenser C is kept at a potential of $-$ 50 V. by means of the battery B_2. This condenser is the 'compensator'. K is an earthing key.

To begin an experiment the capacitance of C is made practically zero and conditions are arranged so that, when P is shielded from the earth's electrostatic field and earthed by means of K, the position of the image of the gold-leaf is near the centre of the scale in the eye-piece of the tele-microscope. When K is withdrawn and the cover M removed, the plate P and its attachments are exposed to the earth's field; P acquires a negative charge and the gold-leaf a positive one. The potential of this system has been raised * in the process so that the leaf moves inwards; to restore the leaf to its zero position the potential must be decreased to zero. This is accomplished by increasing the capacitance of the compensator, i.e. by moving the outer 'plate' of C, which is at a negative potential (say $-$ V) inwards. Now when C is moved into position a positive charge is induced on A and a negative charge is repelled to the leaves. This negative charge neutralizes the positive charge already present on the leaves and since the plate P, the leaves and the rod A form a system without any total charge, the positive charge on A must be equal and opposite to the charge on P due to its presence in the earth's electrostatic field. If σ is the density of the charge on P, we have, if C_s is the known change in the capacitance of C,

$$C_sV + \text{charge on } P = 0,$$

or

$$\sigma . \pi r^2 = -C_sV,$$

so that σ is known. Shorn of all refinements, the system of induced charges is shown in Fig. 5·04 (c).

It will be noticed that when the final observation is made the gold-leaf system and the earthed guard-ring are at zero potential.

Direct measurement of the potential gradient.—To determine a value for $\dfrac{dV}{dz}$ directly it is only necessary to measure the potentials of two linear conductors at known heights above the earth's surface. Thus a horizontal wire is first fixed at a height of about one metre above ground level and then at a height of about two metres. It is essential that the supports for the wire, the instruments used, and even the presence of the observer himself, should not appreciably affect the distribution of the equipotential surfaces. In Fig. 5·05, two vertical rods support the horizontal wire W, small pieces of amber, A_1 and A_2, serving as insulators. The length of the wire should exceed three times the height at which it is to be used. S is a speck of radioactive material placed near to the centre of the wire in order that this may acquire very quickly the potential of the air in its neighbourhood. When the wire is first erected, or disconnected from earth, its potential is different from that of the surrounding air. An electric current flows until this difference of potential is annulled; the radioactive body, by its spontaneous emission of electrons, causes the potential of the wire to increase rapidly to the equilibrium value. If the radioactive material is not used, the equilibrium state is reached more slowly. The final potential of the wire is recorded by the calibrated electroscope G. The experiment is repeated with the wire at a different height and $\frac{dV}{dz}$ calculated. By using the same wire at two different heights any uncertainty as to the exact position at which the ionizing agent is effective, especially on a windy day, is eliminated.

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* The electric field in the air may be considered as due to an external positively charged body.

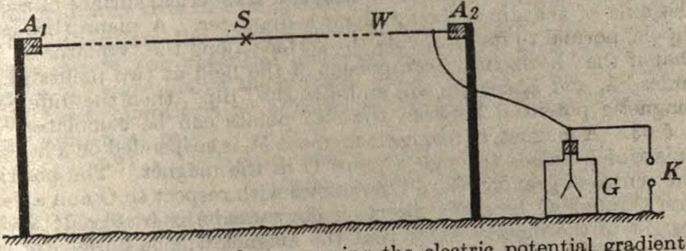

FIG. 5·05.—Apparatus for measuring the electric potential gradient near to the earth's surface.

Variation of potential gradient with height.—At considerable distances above the earth's surface observations of the potential gradient have been made with balloons fitted with 'potential equalizers of the radioactive type'. Unfortunately, considerable errors are inherent for the field is distorted by any charge acquired by the balloon, but nevertheless it is known with some confidence that at a height of 10 km. the electric field strength has fallen to about 2 per cent of its value near to the earth's surface. By integrating graphically the values of $(dV/dz)\delta z = \delta V$, it has been established that at a height of 15 km. the potential is 0·3 MV. above that of the earth. Above this height the natural conductivity of the air is so large that the potential is very nearly that of the 'Heaviside layer', 80 km. above the earth's surface.

If there is a space charge of density ρ in the air and the equipotential surfaces are horizontal, Poisson's equation is

$$\frac{d}{dz}\left(\frac{dV}{dz}\right) = -\frac{4\pi\rho}{\kappa_a},$$

so that ρ may be determined if κ_a is taken as unity. The results obtained are not very reliable.

EXAMPLES V

5·01. Describe the methods by which the variation of the earth's magnetic field during the day at a land station may be investigated.

5·02. Obtain an expression for the potential due to a magnetic dipole in air.

A magnetic dipole lies at the centre of a spherical surface of radius a, the axis of the dipole being along a diameter. A plane through the dipole, normal to its axis, cuts the surface along the 'equator'. Show that if the 'horizontal' component of the field at two points in 'latitudes' λ_1 and λ_2 are known, and also the 'dip', then the difference in magnetic potential between the two points can be calculated.

5·03. A magnet, of magnetic moment M, is suspended on a horizontal axis which passes through a point O in the magnet. The position of the centre of gravity, G, of the magnet with respect to O and a vertical line through O (positive direction downwards) is (r, θ). If W is the weight of the magnet show that the magnet will be horizontal if H_v, the vertical component of the earth's magnetic field, satisfies the relation

$$MH_v = Wr \sin \theta.$$

Show that when H_v becomes $H_v + \Delta H_v$, the magnet moves through an angle $\Delta\theta$ given by

$$H_v \, \Delta\theta = \Delta H_v \tan \theta.$$

5·04. Explain the meaning of the term potential (V) in a field of force and show that the field strength along a line element δs is equal to $-\frac{\partial V}{\partial s}$.

Determine the potential and the radial and azimuthal components of field strength at a point P in the field due to a *short* magnet of moment $\overrightarrow{M}$.

Express the results in terms of the distance r from the centre of the magnet to the point and of the angle θ between r and the axis of the magnet.

If the earth's magnetism be supposed due to a short magnet placed at the centre of the earth, deduce the relation between the angle of dip and the latitude. Criticize the result by a consideration of the actual field of the earth's magnetism.

5·05. The surface of the earth may be taken as that of a sphere of radius $6{\cdot}4 \times 10^8$ cm., and the earth's magnetic field may be regarded as due to a dipole situated at the centre, O, of the sphere. P is a point on the surface such that OP makes an angle $\frac{\pi}{6}$ with the axis of the doublet. If, at P, $H_0 = 0{\cdot}17$ oersted, show that the magnetic moment of the equivalent dipole is $8{\cdot}9 \times 10^{25}$ erg.oersted.$^{-1}$ and at P the angle of dip is $\tan^{-1} 2\sqrt{3}$. What is the value of H_v at P?

[0·59 oersted.]

5·06. Obtain an expression for the potential at a point r, θ due to a magnetic dipole (i.e. a short magnet). Find the field strength at the same point and show that the equation to a line of force is $r = A \sin^2 \theta$, where A is a constant.

Calculate the magnetic latitude at which a line of force comes down to the earth's surface if its distance above the earth's surface at the magnetic equator is equal to the earth's radius. [45°.]

5·07. Explain how the electric potential at a point in air may be measured.

Calculate a value for the radius of a water drop which, carrying a charge equal to that of the electron ($-4{\cdot}80 \times 10^{-10}$ e.s.u.) floats in the earth's electric field when the numerical value of its vertical component is 150 volt.metre^{-1}. Is the general direction of the field upwards or downwards in this instance? [$8{\cdot}4 \times 10^{-6}$ cm., downwards.]

5·08. Calculate the charge per unit area of the earth's surface at a place where the potential increases by 200 V. for an increase of 1 metre in height above the surface. Also calculate the stress. How would these quantities be affected if the potential decreased at the above numerical rate?

[$-5{\cdot}31 \times 10^{-4}$ e.s.u. cm.$^{-2}$, $1{\cdot}77 \times 10^{-6}$ dyne.cm.$^{-2}$; charge altered in sign, stress unchanged.]

5·09. Experiments show that in air at standard pressure the rate of increase of potential with height is 90,000 volt.km.$^{-1}$, i.e. 3×10^{-3} e.s.u. cm.$^{-1}$. Calculate values for the maximum charge on and the potential at the surface of the earth, if this can be regarded as an isolated conducting sphere surrounded by air at standard pressure. [The earth's radius may be taken as $6{\cdot}4 \times 10^{6}$ metre.]

[$-4{\cdot}1 \times 10^{5}$ coulomb., $-17{\cdot}4 \times 10^{10}$ volt.]

5·10. On a certain day the potential gradient at the earth's surface was 150 V.metre.$^{-1}$ and at a height of one kilometre it was 50 V.metre.$^{-1}$. With the usual notation prove that at a height z above ground level

$$\frac{dV}{dz} = -\frac{4\pi}{\kappa_a}\int \rho\, dz + A,$$

where A is an arbitrary constant and κ_a is the permittivity of air; its numerical value may be taken as unity. Hence show that the total charge in a column of air 1 km. high and 1 metre.^{2} in cross-section was 2·64 e.s.u.

Also show that (*a*) the upward force on the charge within a layer of cross-sectional area 1 cm.^{2} and thickness δz is $-(\rho\, \delta z)\frac{dV}{dz}$, and (*b*) if ρ is constant,

$$\frac{dV}{dz} = \frac{1}{200} - \frac{4\pi\rho z}{\kappa_a}.$$

Hence, show that the upward force on a column of 1 cm.^{2} cross-sectional area and height z_1 cm. is

$$-\rho z_1 \times \text{mean value of } \left(\frac{dV}{dz}\right).$$

Finally, prove that if the charge in a column of height 1 km. is uniformly distributed its effect is to increase the atmospheric pressure by about 4×10^{-7} dyne.cm.$^{-2}$.

5·11. Water from a supply at a potential V volt. falls as small droplets of radius r through a small hole into a hollow sphere of radius a; this sphere is insulated and initially at zero potential. If the drops continue to fall until the sphere is half-filled, to what potential is it raised? $\left[\frac{1}{2}\frac{a^2}{r^2}\text{V volt.}\right]$

CHAPTER VI

THE MAGNETIC FIELD DUE TO A STEADY ELECTRIC CURRENT

Magnetic shells.—A magnetic shell is a thin sheet of magnetic material magnetized everywhere at right angles to its surface. Consider an element of such a shell, t being its thickness and δS the area of each face. If J is the intensity of magnetization in the material of the shell, then $J(\delta S . t)$ is the magnetic moment of the element of the shell considered. The magnetic moment of the element per unit area of one of its faces is therefore Jt and this is called the ***strength, ϕ, of the shell***; thus $\phi = Jt$. The sequel will show that the only magnetic shells with any physical importance are those whose strength ϕ is constant throughout; such are known as ***uniform magnetic shells.***

Magnetic potential due to a uniformly magnetized shell in a vacuum.—It has already been shown [cf. p. 103] that the magnetic potential, due to a magnetic dipole, at a point (r, θ) with respect to the dipole and in a vacuum is $\frac{M \cos \theta}{\mu_0 r^2}$, where μ_0 is the permeability of a vacuum and M the magnetic moment of the dipole.

To calculate the magnetic potential at an external point due to a uniform shell we imagine the shell divided up into a large number of dipoles whose poles lie on the positive and negative faces of the shell.

(a) First consider the magnetic potential at a point P on the positive side of the shell. Let δS, Fig. 6·01 (a), be the surface area of the 'end' of a dipole formed from a uniformly magnetized shell, so that if J is the intensity of magnetization in the material of the shell in the region considered, the dipole will have pole strengths $\pm J\,\delta S$. If t is the thickness of the shell and hence the distance between the poles of the dipole, the magnetic potential at P, due to the dipole, is given by

$$\delta\Omega_P = \frac{M \cos \theta}{\mu_0 r^2} = \frac{J\,\delta S\,t \cos \theta}{\mu_0 r^2}$$

$$= (Jt)\frac{\cos \theta\,\delta S}{\mu_0 r^2} = \frac{\phi}{\mu_0}\delta\omega,$$

where $\phi = Jt$, the strength of the shell, and $\delta\omega$ is the solid angle which δS subtends at P.

$$\therefore\ \Omega_P = \int \frac{\phi}{\mu_0}\, d\omega.$$

Since the shell is uniformly magnetized ϕ is constant and we have therefore,

$$\Omega_P = \frac{\phi}{u_0}\int d\omega = \frac{\phi}{\mu_0}\omega.$$

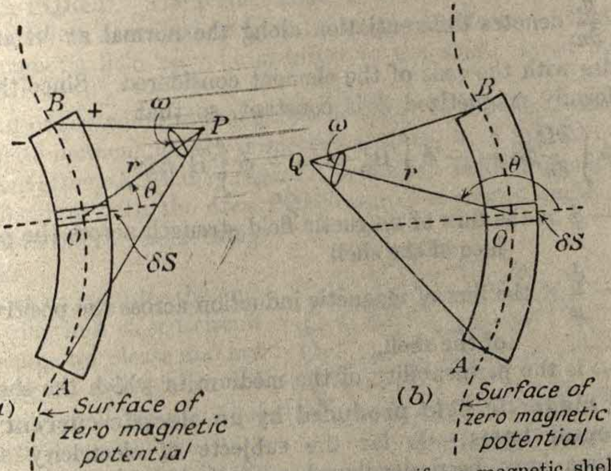
B
+
−
ω
P
r
θ
O
δS
A
(a)
Surface of zero magnetic potential
Q
ω
B
r
θ
O
δS
A
(b)
Surface of zero magnetic potential

FIG. 6·01.—Magnetic potential due to a uniform magnetic shell.

(b) Now consider a point Q on the negative side of the shell. If Q lies in the region shown in Fig. 6·01 (*b*), we have in a similar way

$$\delta\Omega_Q = \frac{M\cos\theta}{\mu_0 r^2} = \frac{\phi\cos\theta\,\delta S}{\mu_0 r^2} = -\frac{\phi}{\mu_0}\delta\omega,$$

$$\left[\because\ \delta\omega = \frac{\delta S\cos(\pi-\theta)}{r^2} = -\frac{\delta S\cos\theta}{r^2}\right]$$

$$\therefore\ \Omega_Q = \int \frac{\phi}{\mu_0}\, d\omega = -\frac{\phi}{\mu_0}\omega,$$

since the shell is uniform.

The meaning of this equation is that if we approach the shell and bring up a small magnetic pole from an infinite distance so that we finally arrive at a point on the negative side of the shell, the work done against the field per unit positive pole is $-\dfrac{\phi}{\mu_0}\omega$.

Potential energy of a uniform magnetic shell.—First consider the potential energy of a dipole with poles $\pm\, m$ and length

δs situated in a magnetic field. If Ω is the magnetic potential at the negative end of the dipole, its potential energy will be

$$-m\Omega + m\left[\Omega + \frac{\partial \Omega}{\partial s}\delta s\right] = \mathrm{M}\frac{\partial \Omega}{\partial s} = \delta \mathrm{W}, \text{ say.}$$

Now consider an element of a uniformly magnetized shell of strength ϕ and lying in a magnetic field. Then $\mathrm{M} = \mathrm{J}\delta\mathrm{S}.t = \phi.\delta\mathrm{S}$.

$$\therefore\ \delta \mathrm{W} = \phi\delta\mathrm{S}\frac{\partial \Omega}{\partial s} = \phi\frac{\partial \Omega}{\partial n}\delta\mathrm{S},$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the normal at $\delta\mathrm{S}$ and this coincides with the axis of the element considered. Since the shell is uniformly magnetized ϕ is constant, so that

$$\mathrm{W} = \phi\int\frac{\partial \Omega}{\partial n}d\mathrm{S} = -\phi\int \mathrm{H}_n\, d\mathrm{S} = -\phi\int \vec{\mathrm{H}}\cdot\hat{n}d\mathrm{S}$$

$= -\phi\ \times$ the flux of magnetic field strength across the positive face of the shell

$= -\frac{\phi}{\mu}\ \times$ the flux of magnetic induction across the positive face of the shell,

where μ is the permeability of the medium in which the shell lies.

The magnetic field produced by an electric current: experimental basis.—So far the subjects of electricity and of magnetism have been developed as entirely separate groups of physical phenomena. Although the mathematical treatment in the two cases has been on parallel lines, we have not had occasion to deal with any physical links connecting the two series of phenomena. The first definite link was discovered by OERSTED in 1820. Oersted's discovery was the fact that a steady* electric current produced a steady magnetic field in its neighbourhood.

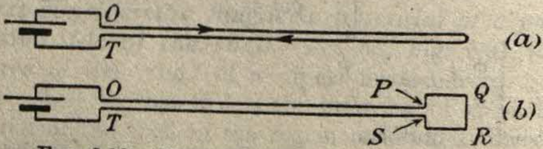

FIG. 6·02.—Magnetic effect due to a current.

The nature of this field can be investigated in a simple manner. We first double back upon itself a wire in which a steady current is flowing—cf. Fig. 6·02 (*a*). It is found that no magnetic field is produced. Next we open out the end into a small plane loop PQRS as in Fig. 6·02 (*b*). It is found that at distances from the loop which are great compared with its linear dimensions, and assuming the loop lies in air, or better in a vacuum, such a loop exerts the same magnetic effects as a magnetic dipole of which the axis is perpendicular to the plane PQRS and the magnetic moment is jointly proportional to the strength of the current and to the area of the loop. The single current flowing in the circuit OPQRST is equivalent to two currents of equal strength, one flowing in OPST and the other flowing in PQRSP. The former effect is shown by the preliminary experiment first described to have no magnetic effects, so that the whole magnetic field may be ascribed to the small closed circuit PQRS.

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* Such currents, in the past, have been called direct currents; they may also be called **zero frequency currents** and we speak of z.f. currents and z.f. voltages.

Instead of regarding the magnetic field as due to a small magnet of magnetic moment jointly proportional to the area PQRS and to the current strength, we may regard it as due to a small thin magnetic shell coinciding with the area PQRS and of strength simply proportional to the current flowing in PQRS.

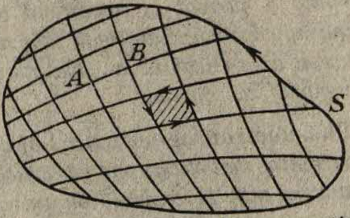

FIG. 6·03.—An equivalent magnetic shell.

Next let us consider the current flowing in a closed circuit of any shape we please and not necessarily in one plane; at present we shall assume that the circuit is situated in a vacuum. Let us cover in the closed circuit by an area of any kind having the circuit for its boundary and let us cut up this area into infinitely small meshes by two families of lines as indicated in Fig. 6·03. A current of strength *i* flowing round the boundary circuit is exactly equivalent to a current of strength *i* flowing round each mesh in the same sense as the current in the boundary. For if we imagine this latter system of currents in existence, any line in the interior such as AB will have two currents flowing in it, one from each of the two meshes it separates and these currents will be equal but flow in opposite directions. Thus all the currents in the lines which have been introduced in the interior of the circuit will annihilate one another as regards the total effect, while the currents in those parts of the meshes which coincide with the original circuit just combine to reproduce the original current flowing in this circuit.

Thus the original circuit is equivalent, as regards its magnetic effects, to a system of currents, one in each mesh. By making the meshes sufficiently small we may regard each mesh as plane so that the magnetic effect of a current circulating in it is known; the magnetic effect of the current in a single mesh is that of a magnetic shell of strength proportional to the current and coinciding in position with the mesh. Thus, by addition, we find that the whole system of currents produces the same magnetic effect as a single magnetic shell whose periphery coincides with that of the circuit and of strength proportional to the current. This shell produces the same magnetic effect as the original single current and is therefore termed the ***equivalent magnetic shell.*** Experiment shows that in the small circuit the positive direction of the magnetization in the shell bears to the direction of the current the same relation as advance bears to rotation in a right-handed screw.

We thus arrive at the conclusion that a steady current flowing in any closed circuit produces the same magnetic field as a certain magnetic shell known as 'the equivalent magnetic shell'; this shell may be any magnetic shell having the circuit for its boundary, its strength being uniform and proportional to that of the current. This fact was stated by Ampère in 1823 as follows :—*Every conductor carrying a current is equivalent to a simple magnetic shell, whose bounding edge coincides with that of the conductor and whose moment per unit area—that is the strength of the shell—is proportional to the strength of the current.*

The electromagnetic unit of current.—*The electromagnetic unit of current is that steady current which when flowing in a closed circuit in a vacuum may be replaced by an equivalent magnetic shell of strength unity.*

On the work done per unit positive magnetic pole in carrying a small magnetic pole round a closed circuit.—Let ABC, Fig. 6·04, be a circuit situated in a vacuum and carrying a current i e.m.u. This circuit may be replaced by any uniform magnetic shell $[\phi = i]$ whose boundary coincides with that of the circuit, in so far as the magnetic field outside the region not occupied by the shell is concerned. Let ADB be such a shell. Consider two points P_1 and P_2 very close to the magnetic shell, but on opposite sides of it. We shall assume that P_1 is on the positive side of the shell. Then the magnetic potential at P_1 is $\phi\omega_1$, where ϕ is the strength of the shell, and ω_1 the solid angle subtended by the boundary of

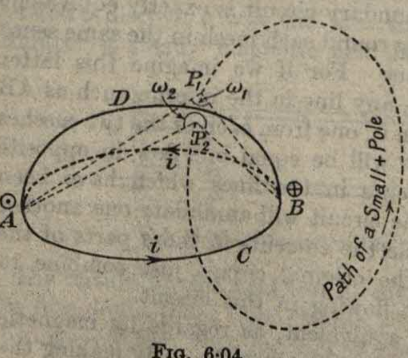

FIG. 6·04.

the shell at P_1. At P_2 the potential is $-\phi\omega_2$. Since we contemplate moving a small positive pole δm from P_2 to P_1 by the path indicated, there will be an increase in potential energy associated with the pole δm, i.e. work will be done against the magnetic field. This increase, provided the field is not affected by the small pole, is

$$[\phi\omega_1 - (-\phi\omega_2)]\delta m = \phi(\omega_1 + \omega_2)\delta m.$$

Hence the work done per unit positive pole against the field in taking a small positive pole from P_2 to P_1 by a path not cutting the shell is

$$\phi(\omega_1 + \omega_2).$$

In the limit when the shell is infinitely thin, its strength remaining equal to ϕ however, the points P_1 and P_2 practically coincide, and we have

$$\omega_1 + \omega_2 = 4\pi.$$

Hence the work done against the field per unit pole, which is the increase in magnetic potential, is $4\pi\phi$.

[It should be noted that if the magnetic pole is moved from P_1 to P_2 by a path not intersecting the shell, the work is done by the field.]

If we now pass from P_1 to P_2 through the shell, the direction of the field is reversed and the work done per pole unit is $-4\pi\phi$. It is not necessary to establish this analytically, for the magnetic potential at a point due to a shell must be single-valued, so that the total work done when the path is completely closed is zero. Otherwise useful work could be obtained by allowing a magnetic pole to move round a closed path drawn in a magnetic field; this is contrary to experience.

When we are dealing with the work done in threading a closed circuit, however, there is no actual shell present, and the work done in passing from P_1 to P_2 to complete the path is zero. Under these circumstances, the work done per unit positive magnetic pole in passing from P_2 to P_2 via P_1, i.e. in completely threading the circuit, which is assumed to be in a vacuum, is $4\pi\phi = 4\pi i$.

In this instance the magnetic potential at a point will be a multi-valued function, i.e. its potential may be any of a series whose values differ by multiples of $4\pi i$. This is not contrary to the principle of the conservation of energy, for a current is not a static effect, and the circuit is not in the same condition as it was initially, when it has been threaded by a magnetic pole.

The equivalence of a current and a magnetic shell in any medium.—Suppose that Γ, Fig. 6·05 (*a*), is a closed circuit of any shape and that it lies in free space, i.e. in a vacuum. Let PAP be the path of a small magnetic pole taken from P to P via A and in the direction indicated, i.e. the magnetic potential at the end of the path is greater than at the beginning. We shall denote this path by the symbol [PAP]. For our present purpose this path must be such that it threads the circuit, i.e. the path is linked with the circuit. Then it has been established [cf. p. 129] that the work done per unit pole against the field in taking a small pole along [PAP], and in the direction indicated, is $4\pi i$, where i is the current in Γ.

Now suppose that the circuit lies in an extended medium of permeability μ. Then the medium is polarized and we can consider it to be made up of dipoles orientated in a manner determined by the resultant magnetic field at any point. A few of these dipoles

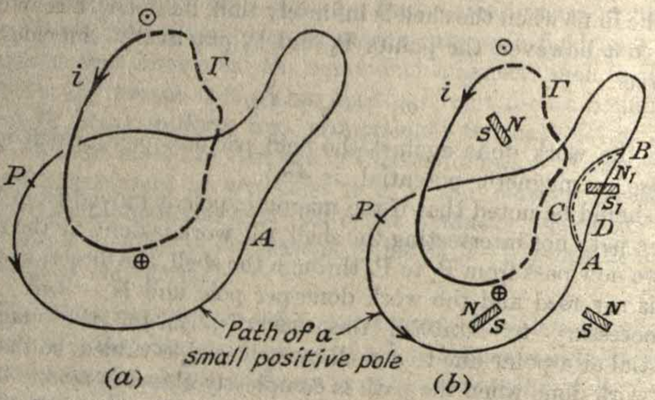

Fig. 6·05.—Equivalent magnetic shells (a) in free space, (b) in a medium.

are shown in Fig. 6·05 (b). Let [PAP] be the path of a small pole taken from P to P and let it pass through the dipole N_1S_1. Let [ACB] be a path in free space alternative to the path [ADB] through the dipole. For simplicity it will be assumed that the path from P to A and from B to P is in free space. Now the work done per unit pole in taking a small pole from P to P in free space, and in the direction indicated, is $4\pi i$. To find the work done when the path is in a material medium we use the fact that the work done when the path is [ADB] and then [BCA] is zero, since the forces on the small pole are reversed when it traverses the dipole. Thus the work done per unit pole along the path

$$[PA] + [ADB] + [BP]$$

is equal to the work done per unit pole along

$$[PA] + [ADB] + [BCA] + [ACB] + [BP].$$

But the work done along the closed path [ADBCA] is zero, i.e. for the paths represented by the second and third terms in the above equation the work done is zero and hence the total work done in the complete circuit is the same as if the path were entirely in free space. It follows at once that the work done in a complete circuit is independent of the path and therefore of the medium. This indicates that the strength of the magnetic field due to a current is independent of the permeability of the medium in which the circuit lies.

But the magnetic field and hence the magnetic potential due to a magnetic shell is reduced μ-times when the shell lies in a medium of permeability μ; thus when the current i flows in a circuit situated in a medium of permeability μ, the strength of the equivalent magnetic shell is given by $\phi = \mu i$.

The magnetic field due to a circular current at a point on its axis.—Let P, Fig. 6·06 (*a*), be the point considered. Let i be the current in electromagnetic units, O the centre of the circle and a its radius. The magnetic potential at P, a point on the positive side of the equivalent magnetic shell, is therefore $i\omega$, where ω is the solid angle subtended at P by the boundary of the circle. Now the solid angle ω is formed by the revolution of the diagram about the axis OP. It is defined as the area of the spherical cap ABC, Fig. 6·06 (*b*), divided by the square of the distance BP. Let BP $= r$. Then

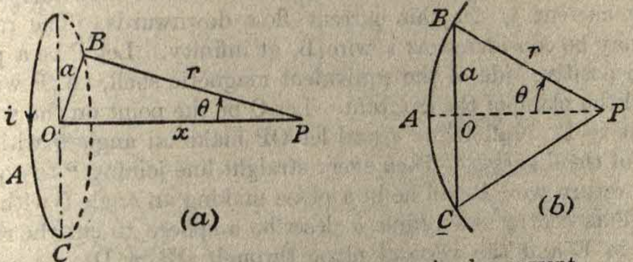

FIG. 6·06.—Magnetic field due to a circular current.

$$\omega = \frac{2\pi r^2(1-\cos\theta)}{r^2} = 2\pi(1-\cos\theta) = 2\pi\left[1 - \frac{x}{(a^2+x^2)^{\frac{1}{2}}}\right].$$

The magnetic potential at P is therefore given by

$$\Omega = 2\pi i\left[1 - \frac{x}{(a^2+x^2)^{\frac{1}{2}}}\right].$$

If H_x and H_y are the components of the magnetic field at P, we have

$$H_x = -\frac{\partial\Omega}{\partial x} = \frac{2\pi i a^2}{(a^2+x^2)^{\frac{3}{2}}},$$

and

$$H_y = -\frac{\partial\Omega}{\partial y} = 0.$$

[In passing it may be noted that conditions of symmetry require $H_y = 0$ at points on the axis OP.]

Thus
$$H = H_x = \frac{2\pi i a^2}{(a^2 + x^2)^{\frac{3}{2}}}.$$

The magnetic field at O, the centre of the circle, is found by putting $x = 0$ in the above equation. We have
$$\Big[H\Big]_{x=0} = \frac{2\pi i}{a}.$$

If there were N turns of thin wire each of radius a the magnetic field would be increased N-fold everywhere, i.e.
$$H = \frac{2\pi N i a^2}{(a^2 + x^2)^{\frac{3}{2}}}, \text{ and } \Big[H\Big]_{x=0} = \frac{2\pi N i}{a}.$$

The magnetic field due to a linear current.—Let A, Fig. 6·07 (a), be a portion of an infinite vertical wire carrying a steady current i; let this current flow downwards. The return wire may be considered as a wire B, at infinity. Let P be a point on the positive side of the equivalent magnetic shell, i.e. P will lie behind the plane of the diagram. Let O be the point on the wire A nearest to P. Call $OP = r$ and let OP make an angle θ with the plane of the diagram. Then every straight line joining P to a point in the return wire B will lie in a plane making an angle θ with PO. With P as centre and radius a describe a sphere to cut the above plane in E and the vertical plane through OP in D.

Then if P is joined to every point on the wires, i.e. to every point on the periphery of the equivalent magnetic shell which may replace the circuit, these lines will intersect the above sphere and cut off an area in the form of a 'lune'—it is shown shaded in Fig. 6·07 (a).

To determine the solid angle which a lune subtends at the centre of its associated sphere let us consider the lune which lies between the planes COL and COM, Fig. 6·07 (b). Then δS, the area of the shaded element, with sides of length $a\delta\beta$, and $(a \sin \beta)\theta$, is given by
$$\delta S = a^2\theta \sin \beta \, \delta\beta.$$

Hence for the complete lune,
$$S = 2\int_0^{\frac{1}{2}\pi} a^2\theta \sin \beta \, d\beta = 2a^2\theta.$$

∴ Solid angle which the lune subtends at O is given by
$$\omega = \frac{2a^2\theta}{a^2} = 2\theta.$$
$$\therefore \Omega_P = 2i\theta.$$

Hence $\quad H_r = -\frac{\partial \Omega}{\partial r} = 0, \quad$ and $\quad H_\theta = -\frac{1}{r}\frac{\partial \Omega}{\partial \theta} = -\frac{2i}{r}.$

The magnetic field is therefore of magnitude $\frac{2i}{r}$ and acts in the direction opposite to that of the unit vector $\hat{\theta}$ at P. In the diagram this field is shown by the vector $\vec{\mathrm{H}}$. [The unit vectors at P are shown in Fig. 6·07 (*d*).]

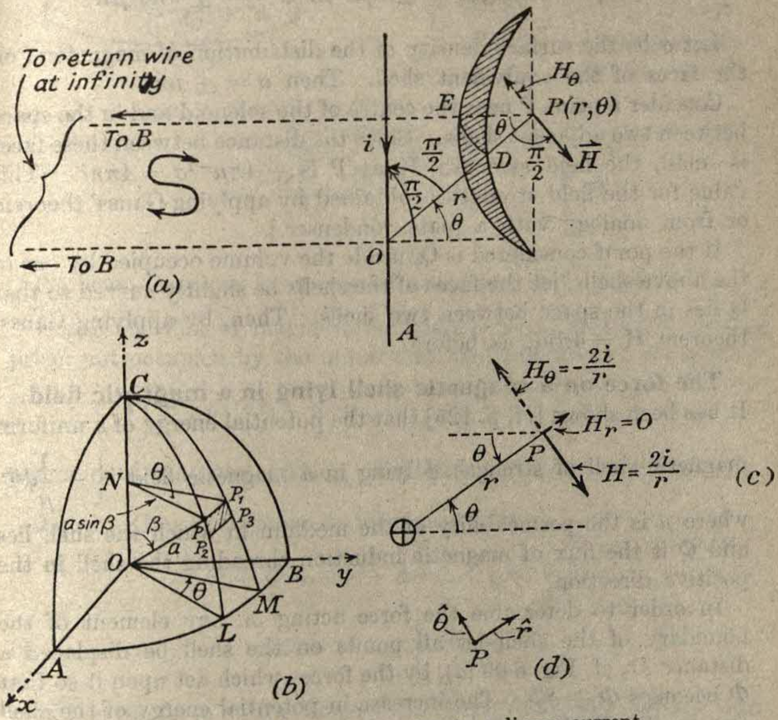

FIG. 6·07.—Magnetic field due to a linear current.

The magnetic field inside a long straight solenoid wound uniformly.—Consider such a solenoid having n turns per unit length and each turn having a diameter $2a$, as shown in Fig. 6·08. Let i be the current. Then each turn of the wire and the current in it may be replaced by its equivalent magnetic shell. Let us assume that these shells are normal to the axis of the solenoid, and opposing faces of adjacent shells are very close together. If μ is the permeability

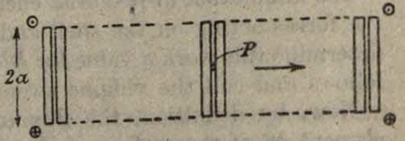

FIG. 6·08.—Magnetic field inside a long straight solenoid wound uniformly.

of the core the strength of these equivalent shells is μi, i.e. the magnetic moment per unit area of a face is μi. The magnetic moment of a shell is therefore $\pi\mu a^2\mu i$. But the thickness of the shell is $\frac{1}{n}$.

$$\therefore \text{ pole strength} = \pm\,\pi\mu a^2\mu i \div \frac{1}{n} = \pm\, n\pi a^2\mu i.$$

Let σ be the surface density of the distribution of magnetism on the faces of the equivalent shell. Then $\sigma = \pm\,\mu ni$.

Consider a point P near the centre of the solenoid and in the space between two adjacent shells. Since the distance between these faces is small, the field strength, H, at P is $+\,4\pi\mu^{-1}\sigma = 4\pi ni$. [This value for the field strength is obtained by applying Gauss' theorem or from analogy with a plate condenser.]

If the point considered is Q, inside the volume occupied by one of the above shells, let the faces of the shells be slightly curved so that Q lies in the space between two shells. Then, by applying Gauss' theorem $H = 4\pi ni$, as before.

The force on a magnetic shell lying in a magnetic field.—It has been shown [cf. p. 125] that the potential energy of a uniform magnetic shell of strength ϕ lying in a magnetic field is $-\frac{1}{\mu}\phi\Phi$, where μ is the permeability of the medium in which the shell lies and Φ is the flux of magnetic induction threading the shell in the positive direction.

In order to determine the force acting on any element of the boundary of the shell let all points on the shell be displaced a distance δx, cf. Fig. 6·09 (*a*), by the forces which act upon it so that Φ becomes $\Phi + \delta\Phi$. The increase in potential energy of the shell due to the displacement is

$$-\frac{1}{\mu}[\phi(\Phi + \delta\Phi) - \phi\Phi] = -\frac{1}{\mu}\phi\,\delta\Phi.$$

The diminution in potential energy is equal to the work done by the forces acting on the shell and due to the magnetic field. To determine this work a value for $\delta\Phi$ must be found. We proceed as follows and call the volume swept out by the shell a 'prism'.

Now, by definition, the flux of magnetic induction across an element δS of the surface of the prism not occupied by the initial and final positions of the shell is

$$\vec{B}\cdot\hat{n}\,\delta S,$$

where B is the magnetic induction at δS and $\hat{n}$ the unit vector along the outward drawn normal at δS. Since $\hat{n}\,\delta S = [\overrightarrow{\delta s}\wedge\overrightarrow{\delta x}]$, this flux is

$$\vec{B}\cdot\overrightarrow{\delta s}\wedge\overrightarrow{\delta x} = \vec{B}\wedge\overrightarrow{\delta s}\cdot\overrightarrow{\delta x},$$

by the rules for computing a triple scalar product.

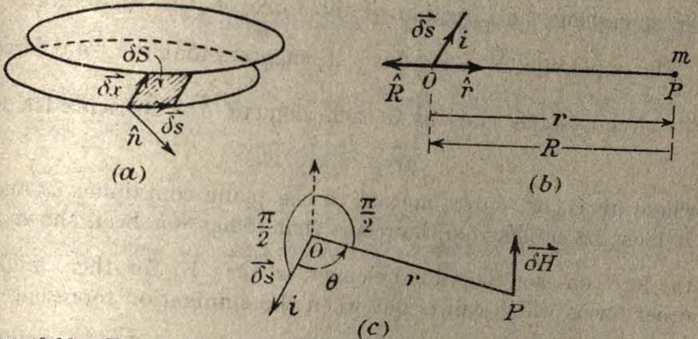
δS
$\overrightarrow{\delta x}$
$\overrightarrow{\delta s}$
$\hat{n}$
(a)
$\overrightarrow{\delta s}$
i
m
$\hat{R}$
O
$\hat{r}$
P
r
R
(b)
$\frac{\pi}{2}$
$\frac{\pi}{2}$
O
$\overrightarrow{\delta s}$
θ
r
$\overrightarrow{\delta H}$
i
P
(c)

FIG. 6·09.—Force on an element of a magnetic shell; Laplace's law.

Hence the flux of magnetic induction across the surface of the prism not occupied by the initial and final positions of the shell is

$$\iint \vec{B}\wedge\overrightarrow{ds}\cdot\overrightarrow{dx} = \oint \vec{B}\wedge\overrightarrow{ds}\cdot\int\overrightarrow{dx} = \overrightarrow{\delta x}\cdot\oint\vec{B}\wedge\overrightarrow{ds},$$

where $\oint$ denotes integration round the periphery of the shell.

But by Gauss' theorem in magnetostatics

$$-\Phi + \Phi + \delta\Phi + \overrightarrow{\delta x}\cdot\oint\vec{B}\wedge\overrightarrow{ds} = 0,$$

or

$$\delta\Phi = -\overrightarrow{\delta x}\cdot\oint\vec{B}\wedge\overrightarrow{ds}.$$

If $\vec{F}$ is the force on the whole shell, $\vec{F}\cdot\overrightarrow{\delta x}$ is the work done during the displacement $\overrightarrow{\delta x}$ of the shell and this work is the diminution in potential energy of the shell. Thus

$$\vec{F}\cdot\overrightarrow{\delta x} = \frac{1}{\mu}\phi\,.\,\delta\Phi = -\frac{\phi}{\mu}\overrightarrow{\delta x}\cdot\oint\vec{B}\wedge\overrightarrow{ds},$$

i.e.

$$\vec{F} = -\frac{\phi}{\mu}\oint\vec{B}\wedge\overrightarrow{ds} = \frac{\phi}{\mu}\oint[\overrightarrow{ds}\wedge\vec{B}].$$

On the force on an element of a conductor carrying a current and lying in a magnetic field.—It has just been shown that the force on a uniform magnetic shell lying in a magnetic field is given by

$$\vec{F} = \frac{\phi}{\mu}\oint[\overrightarrow{ds}\wedge\vec{B}].$$

If the shell is replaced by a circuit whose form coincides with the boundary of the shell and if $\phi = \mu i$, where i is the current, we have

$$\vec{F} = i \oint [\vec{ds} \wedge \vec{B}].$$

This equation is a generalized form of ***Fleming's left hand rule*** for expressing the direction of the force on the conductor.

The expression $\vec{F} = i \oint [\vec{ds} \wedge \vec{B}]$ suggests that the total force on the circuit is the same as if each element $\vec{\delta s}$ experienced a force

$$\vec{\delta F} = i[\vec{\delta s} \wedge \vec{B}],$$

where $\vec{\delta F}$ is, of course, normal to the plane containing $\vec{\delta s}$ and $\vec{B}$. It must be emphasized, however, that it has not been shown that the force on each individual element is $i[\vec{\delta s} \wedge \vec{B}]$, for there might be other terms which cancel out when the summation represented by the integral is effected. The force on an element $\vec{\delta s}$ of a conductor cannot be measured experimentally but if its value is assumed to be $i[\vec{\delta s} \wedge \vec{B}]$ we can determine the force on a moving charge as follows and, in this instance, the correctness of our conclusions can be tested experimentally.

Let a succession of point charges each equal to q e.s.u. and distributed uniformly along the line of motion, move with a velocity u in a direction normal to a magnetic field with induction $\vec{B}$. If there are n charges per unit length of path, the current in e.m.u. is $i = \frac{nq\,u}{(c)}$, where c is the velocity of light in a vacuum. Thus the force δF on the charges within a length δs is given by

$$\delta F = iB\,\delta s = \frac{nq\,u}{(c)}B\,\delta s.$$

Since the total number of point charges in a length δs is $n\,\delta s$, the force on each such charge is $\frac{q}{(c)}uB$, and the effect of this force can be studied experimentally—cf. Vol. VI. The complete agreement found justifies the splitting up of the force on a complete circuit into forces $i[\vec{\delta s} \wedge \vec{B}]$ on each element $\vec{\delta s}$.

Laplace's law.—If the magnetic induction $\vec{B}$ arises from a single magnetic pole m at a point P at a distance $\vec{R}$ from $\vec{\delta s}$, then

$$\vec{\delta F} = i\left[\vec{\delta s} \wedge \left(\frac{m}{R^2}\hat{R}\right)\right] = i\frac{m}{R^2}[\vec{\delta s} \wedge \hat{R}],$$

where $\hat{R}$ is a unit vector in the direction indicated in Fig. 6·09 (b).

Since action and reaction are equal and opposite, the force on the pole m due to the element of the circuit considered is given by

$$-i\frac{m}{R^2}[\overrightarrow{\delta s}\wedge \hat{R}] = i\frac{m}{r^2}[\overrightarrow{\delta s}\wedge \hat{r}],$$

where $\hat{r} = -$ R, i.e. $\hat{r}$ is a unit vector in the direction of r measured from $\overrightarrow{\delta s}$ to P.

The strength of the magnetic field at P and due to the current in $\overrightarrow{\delta s}$ is therefore given by

$$\overrightarrow{\delta H} = \frac{i}{r^2}[\overrightarrow{\delta s}\wedge \hat{r}].$$

The direction of this field is indicated in Fig. 6·09 (*c*).

The last equation is known as ***Laplace's law*** and from it we have at once,

$$\delta H = \frac{i\,\delta s \sin\theta}{r^2},$$

where θ is the angle between $\overrightarrow{\delta s}$ and $\hat{r}$. [Cf. also p. 150.]

The magnetic field at P due to a current i in a finite length of wire is given by

$$\overrightarrow{H} = i\int\frac{\overrightarrow{ds}\wedge\hat{r}}{r^2} = i\int\frac{\overrightarrow{ds}\wedge\overrightarrow{r}}{r^3},$$

where the integration extends along that finite portion of the circuit concerned.

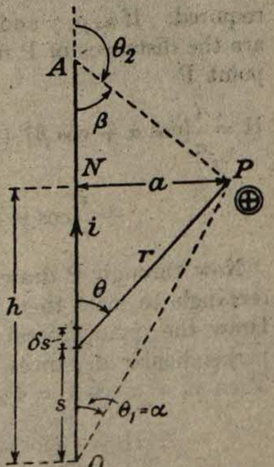

FIG. 6·10.—Magnetic field due to a linear current of finite length.

Magnetic field due to a linear current of finite length.—Let P, Fig. 6·10, be the point at which the field strength due to a steady current in a straight conductor of length OA is required. O will be taken as the origin from which distances along the conductor are measured. If i is the current, the field at P due to the element δs is given by

$$\delta H = \frac{i\,\delta s \sin\theta}{r^2},$$

where θ is the angle indicated. If $a = NP$, $h = ON$, then

$$s = h - a\cot\theta.$$

$$\therefore\ \delta s = a\,\text{cosec}^2\,\theta\,\delta\theta.$$

Since $r = a\,\text{cosec}\,\theta$, we have

$$\delta H = \frac{ia\,\text{cosec}^2\,\theta\,.\,\delta\theta\,.\sin\theta}{a^2\,\text{cosec}^2\,\theta} = \frac{i}{a}\sin\theta\,.\,\delta\theta.$$

If θ_1 and θ_2 are the angles indicated, the magnetic field at P due to the current in the wire OA is given by

$$H = \frac{i}{a}\int_{\theta_1}^{\theta_2} \sin\theta \, d\theta = \frac{i}{a}[\cos\theta_1 - \cos\theta_2],$$

$$= \frac{i}{a}[\cos\alpha + \cos\beta],$$

where $\alpha = \theta_1$ and $\beta = \pi - \theta_2$.

Magnetic field at any point in the plane of a rectangular framework carrying a steady current.—Let P, Fig. 6·11, be the point inside the rectangle at which the field strength is required. If α, β, γ and δ are the angles indicated, and a, b, c and d are the distances of P from the sides of the rectangle, then at the point P

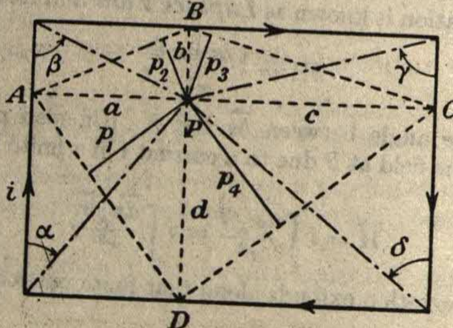

Fig. 6·11.—Magnetic field within a rectangular framework.

$$H = \frac{i}{a}[\cos\alpha + \cos\beta] + \frac{i}{b}\left[\cos\left(\frac{\pi}{2} - \beta\right) + \cos\left(\frac{\pi}{2} - \gamma\right)\right]$$

$$+ \frac{i}{c}[\cos\gamma + \cos\delta] + \frac{i}{d}\left[\cos\left(\frac{\pi}{2} - \delta\right) + \cos\left(\frac{\pi}{2} - \alpha\right)\right].$$

Now through P draw straight lines parallel to the sides of the rectangle to meet them in the points A, B, C and D respectively. Draw the quadrilateral ABCD and let p_1, p_2, p_3 and p_4 be the perpendicular distances of P from the sides of the quadrilateral. Then $p_1 = a\cos\alpha = d\sin\alpha$, etc.

$$\therefore\ H = \frac{i}{p_1}(\cos^2\alpha + \sin^2\alpha) + \ldots + \ldots + \ldots,$$

$$= i\left[\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}\right].$$

To verify the above formula let three of the wires 'move to infinity', the other wire becoming infinitely long but remaining at a distance a from P. Then $H = \frac{2i}{a}$, because $p_1 = p_2 = a$ and only the reciprocals of p_3 and p_4 vanish.

Magnetic field strength at any point on the axis of a circular current.—Let O, Fig. 6·12, be the centre of a wire in the form of a circular loop of radius a and carrying a current i. Let P be the point $(x, 0)$ referred to rectangular axes with O as origin. It is required to calculate the field strength at P.

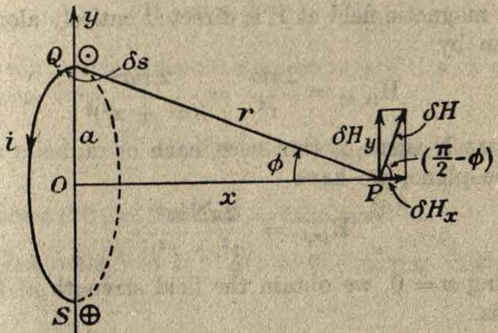

FIG. 6·12.—Magnetic field at any point on the axis of a circular current.

For simplicity, and yet without any loss of generality, consider the magnetic field at P due to the current in an element δs situated at Q, the point $(0, a)$. Let QP $= r$ and $O\hat{P}Q = \phi$. Then since the angle between $\overrightarrow{\delta s}$ and $\hat{r}$ is $\frac{\pi}{2}$, the magnetic field δH at P is, by Laplace's formula,

$$\frac{i\,\delta s \sin \frac{\pi}{2}}{r^2} = \frac{i\,\delta s}{r^2},$$

and $\overrightarrow{\delta H}$ lies in the plane xOy and makes an angle $(\frac{1}{2}\pi - \phi)$ with the x-axis.

Now resolve $\overrightarrow{\delta H}$ into two rectangular components $\overrightarrow{\delta H_x}$ and $\overrightarrow{\delta H_y}$. Then

$$\delta H_x = \frac{i\,\delta s \sin\phi}{r^2}; \quad \delta H_y = \frac{i\,\delta s \cos\phi}{r^2}.$$

In the same way it may be shown that an element δs at S, the point (0, $-a$) will produce at P a field given by

$$\delta H_x = \frac{i\,\delta s \sin\phi}{r^2}; \quad \delta H_y = -\frac{i\,\delta s \cos\phi}{r^2}.$$

By pairing off all the elements into which the wire may be divided in the above manner, we get

$$H_x = \int dH_x = \frac{i \sin\phi}{r^2}\oint ds = \frac{2\pi a i \sin\phi}{r^2},$$

and

$$H_y = \int dH_y = 0.$$

Thus the magnetic field at P is directed entirely along the x-axis and is given by

$$H_{(x,\,0)} = \frac{2\pi i a^2}{r^3} = \frac{2\pi i a^2}{(a^2 + x^2)^{\frac{3}{2}}}.$$

If there are N turns of thin wire, each of radius a and all lying in the same plane, we have

$$H_{(x,0)} = \frac{2\pi N i a^2}{(a^2 + x^2)^{\frac{3}{2}}}.$$

By putting $x = 0$, we obtain the field strength at the centre of the coil, viz.

$$H_{(0,\,0)} = \frac{2\pi N i}{a}.$$

The magnetic field strength at a point on the axis of a short solenoid.—Let O, Fig. 6·13 (a), be the centre of a short solenoid,

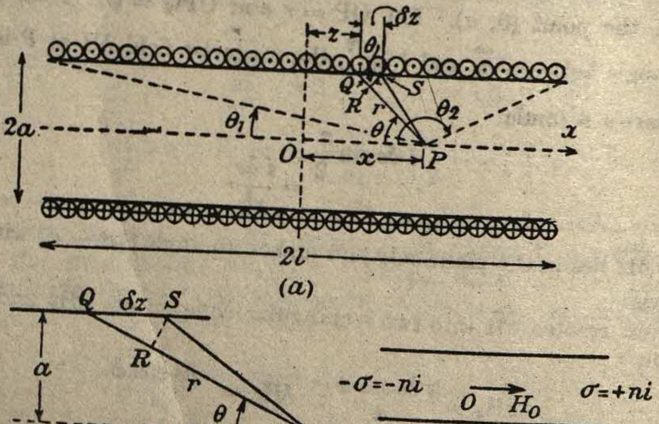

FIG. 6·13.—Magnetic field at a point on the axis of a short solenoid.

length $2l$, diameter $2a$, and having n turns per unit length. Let Ox be the axis of the coil and let P, where OP $= x$, be the point on the axis at which the magnetic field strength is required. Let the position of any point Q on the surface of the solenoid be referred to by its distance z from a plane through O normal to Ox.

Consider the contribution to the magnetic field at P made by the current in that portion of the solenoid lying between planes normal to the x-axis and at distances z and $z + \delta z$ from the plane through O normal to Ox; by symmetry the field is axial. Let δH_P be this contribution and let the above planes cut the upper trace of the solenoid in points Q and S. Then $\delta z =$ QS and if QP $= r$, we have

$$\delta H_P = \frac{2\pi(n\,\delta z)ia^2}{r^3},$$

since this portion of the solenoid may be regarded as a circular coil with $n\,\delta z$ turns.

Now $\dfrac{r\,\delta\theta}{\delta z} = \sin\theta = \dfrac{a}{r}$, cf. Fig. 6·13 (*b*), so that the above expression becomes $\delta H_P = 2\pi ni\dfrac{a}{r}\delta\theta = 2\pi ni \sin\theta\,\delta\theta$.

Hence the magnetic field at P due to the current in the complete solenoid is given by

$$\begin{aligned} H_P &= 2\pi ni\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta, \qquad [\theta_1 \text{ and } \theta_2 \text{ are the angles indicated}] \\ &= 2\pi ni[\cos\theta_1 - \cos\theta_2] \\ &= 2\pi ni\left[\frac{l+x}{\{(l+x)^2+a^2\}^{\frac{1}{2}}} + \frac{l-x}{\{(l-x)^2+a^2\}^{\frac{1}{2}}}\right]. \end{aligned}$$

At the centre of the solenoid, we have

$$\begin{aligned} H_O = H_{x=0} &= 2\pi ni\left[\frac{2l}{(l^2+a^2)^{\frac{1}{2}}}\right] = 4\pi ni\left[\frac{1}{\left(1+\frac{a^2}{l^2}\right)^{\frac{1}{2}}}\right] \\ &\simeq 4\pi ni\left[1 - \frac{1}{2}\frac{a^2}{l^2}\right], \qquad \text{if } \left(\frac{a}{l}\right)^4 \to 0. \end{aligned}$$

When $\left(\dfrac{a}{l}\right)^4 \to 0$, the same formula may be established using the properties of a uniform magnetic shell. For a length δz of the solenoid the effective current is $(ni\,\delta z)$, so that the strength of the equivalent magnetic shell is given by

$$\phi = \mu_a ni\,\delta z, \qquad \text{i.e. } J = \mu_a ni \text{ and } \sigma = \pm\,\mu_a ni,$$

where μ_a, J and σ have their usual meanings.

If we consider poles of strengths $\pm(\pi a^2)\sigma = \pm\,(\pi a^2)\mu_a ni$ at the ends of the solenoid each will tend to reduce the field that would exist at O if the solenoid were infinitely long, and we have, cf. Fig. 6·13 (c),

$$H_{x=0} = 4\pi ni - \frac{\pi a^2\mu_a(ni)}{\mu_a l^2} - \frac{\pi a^2\mu_a(ni)}{\mu_a l^2},$$

$$= 4\pi ni\left[1 - \frac{1}{2}\frac{a^2}{l^2}\right],$$

as above, and the same method of calculation may be used to calculate the strength of the magnetic field at points on the axis of the solenoid.

Circuital form of Laplace's law.—Laplace's law (sometimes called Ampère's law), which has been established theoretically on p. 136, was originally a working hypothesis. It states that

$$\delta H = \frac{i\,\delta s\sin\theta}{r^2}, \text{ or } \overrightarrow{\delta H} = \frac{i}{r^2}[\overrightarrow{\delta s}_\wedge \hat{r}] = \frac{i\,\delta s}{r^2}[\hat{s}_\wedge \hat{r}],$$

where the symbols are as defined earlier. To put this law into an alternative and sometimes more convenient form we proceed as follows.

Consider the circuit shown in Fig. 6·14 (a) and in which a steady current i flows. An equivalent magnetic shell is shown in Fig. 6·14 (b). Let Γ be a closed path linked with the circuit. By convention, a point moving along this path is said to move in the positive direction when its motion bears to the direction of the current with which it is linked the same relation as does rotation to translation in a right-handed screw motion.

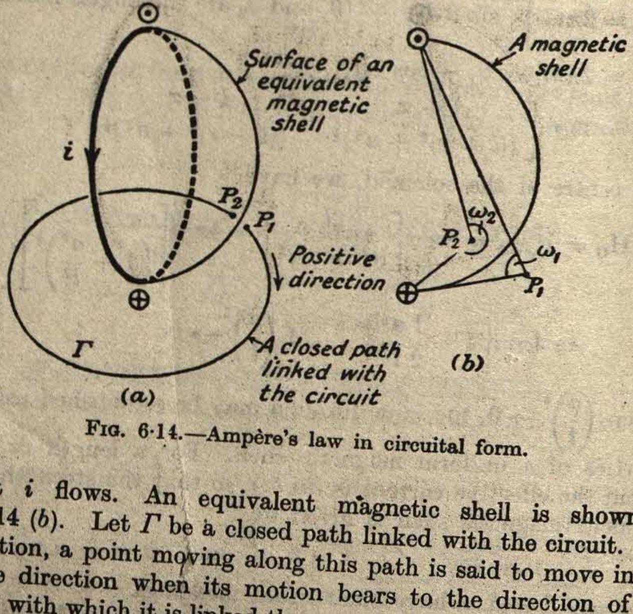
⊙
Surface of an equivalent magnetic shell
i
P₂
P₁
Positive direction
⊕
Γ
A closed path linked with the circuit
(a)
⊙
A magnetic shell
ω₂
P₂
ω₁
P₁
⊕
(b)

FIG. 6·14.—Ampère's law in circuital form.

Let P_1 and P_2 be points on the positive and negative sides respectively of the equivalent magnetic shell and let ω_1 and ω_2 be the solid angles which the boundary of the shell subtends at these two points respectively. Then, cf. p. 124 et seq.

$$\Omega_{P_1} = i\omega_1, \quad \text{and} \quad \Omega_{P_2} = -i\omega_2.$$

Hence the increase in magnetic potential in passing from P_1 to P_2 is

$$\Big[\Omega\Big]_{P_1}^{P_2} = -i\omega_2 - i\omega_1 = -i(\omega_1 + \omega_2)$$
$$= -4\pi i,$$

when P_1 and P_2 coincide and no actual shell is present.

But $$\Big[\Omega\Big]_{P_1}^{P_2} = \int_{P_1}^{P_2} d\Omega = \int_{P_1}^{P_2} \frac{d\Omega}{ds} ds = \int_{P_1}^{P_2} (-H_s) ds,$$

where H_s is the component of the magnetic field at an element ds of the path Γ. Since $H_s = \vec{H} \cdot \hat{s}$, we have

$$\Big[\Omega\Big]_{P_1}^{P_2} = -\int_{P_1}^{P_2} \vec{H} \cdot \hat{s}\, ds.$$

Hence, when P_1 and P_2 coincide

$$\oint \vec{H} \cdot \hat{s}\, ds = \oint \vec{H} \cdot \vec{ds} = 4\pi i.$$

This is ***Ampère's circuital relation.***—It is important to note that the path is considered to be traversed in the positive direction when its sense is the same as that of the rotation of a right-handed screw advancing in the direction of the current.

The magnetic field due to a linear current.—Let i be the current of zero frequency in a uniform solid infinite linear conductor of radius a.

(a) First consider the magnetic field $\vec{H}$ at a point P, Fig. 6·15 (a), outside the conductor. By symmetry this field will be in a plane normal to the conductor and have the same numerical value at all points on a circle of radius r, passing through P and having its centre on the axis of the conductor. The positive direction of the circular path is shown by the arrow. Then, by Ampère's circuital relation,

$$4\pi i = \oint \vec{H} \cdot \hat{s}\, ds = H \oint ds = 2\pi r H.$$

$$\therefore\ H = \frac{2i}{r}.$$

This equation is valid for all values of a, provided $r > a$, and so gives the field strength due to a current in a long straight wire.

(b) Next consider the field at a point inside the conductor; its direction relative to that of the current is shown conventionally in Fig. 6·15 (*b*). If this is at a distance r from the axis of the wire,

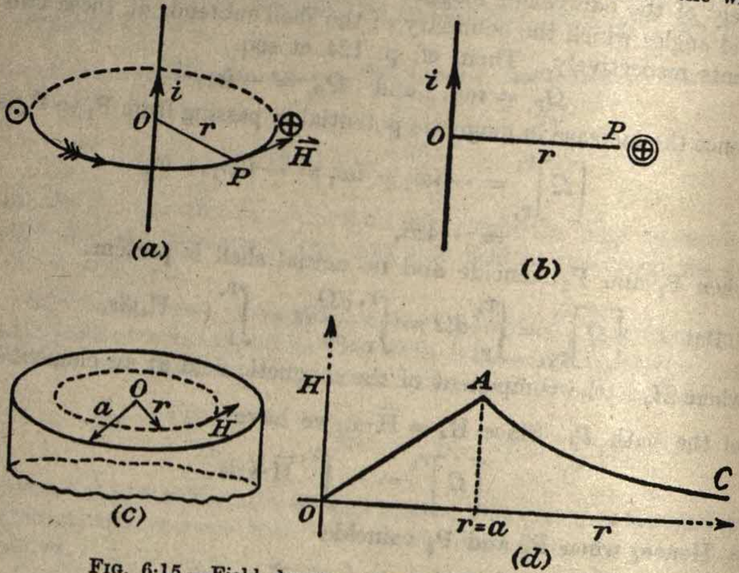

FIG. 6·15.—Field due to a long straight conductor.

where $0 < r < a$, the current embraced by a circular path of radius r, cf. Fig. 6·15 (*c*), is $i\left(\frac{r^2}{a^2}\right)$. Hence by Ampère's theorem

$$\oint \vec{H}\cdot\vec{ds} = 2\pi r H = 4\pi i\left(\frac{r^2}{a^2}\right).$$

$$\therefore H = \frac{2i}{a^2}r.$$

The manner in which the field strength varies with the distance from the axis of the conductor is shown in Fig. 6·15 (*d*).

Experimental demonstration of the magnetic field within a material carrying an electric current.—The apparatus, shown in Fig. 6·16 (*a*), consists essentially of a glass tube A, 40 cm. × 10 cm., fitted with brass end-plates B and C, and almost completely filled with a concentrated aqueous solution of copper sulphate; a resin-type glue, known as 'Araldite', is used to fix the tube in position and prevent leakage. G is a light glass rod with a piece of thin brass strip attached near its centre; this strip carries two small but powerful magnets N_1S_1 and N_2S_2, arranged as shown. In order that the glass rod and its attachments may rotate freely, bearings P_1 and P_2 are fitted at its ends; these are made from a rod of polytetrafluoroethylene (p.t.f.e.) and known commercially as 'Fluon'. This substance is resistant to nearly all liquids, has a low coefficient of friction and is self lubricating. To facilitate erection of the apparatus the lid C has an aperture O through which the bearing P_1 may be passed, and is also provided with a short length of wide brass tubing D. The lower end of D is closed with a plate and this has a central aperture through which G may pass; smaller holes allow the liquid to enter the tube A when it is poured through O. G is placed with P_2 on its 'Perspex' * (polymethyl methacrylate) support and its upper end passed through the central hole in D; with care G may be made to enter P_1 which has been drilled so that it fits snugly over the glass rod.

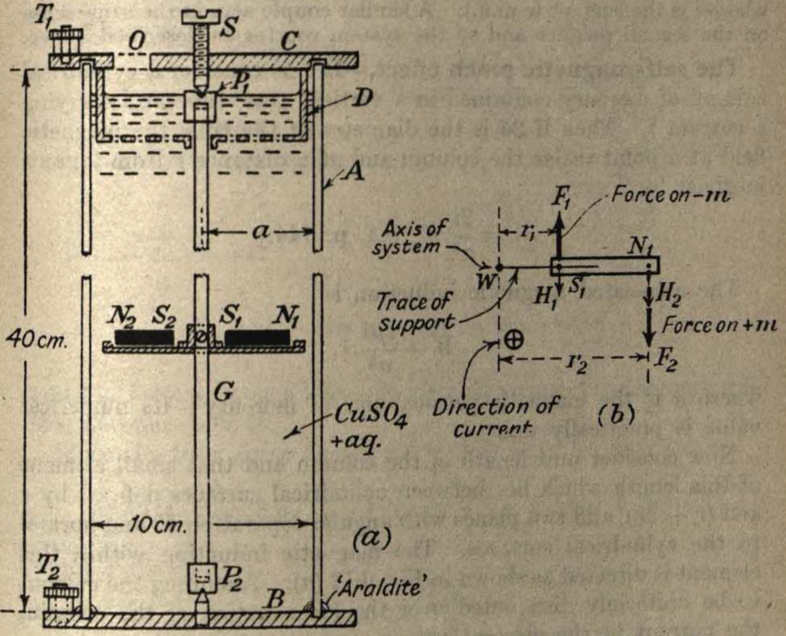

FIG. 6·16.—To show the existence of a magnetic field within a conductor carrying a current. (Designed at R.H.C.)

When a current of about 5 A. is passed through the solution, the suspended system rotates; the motion is reversed when the current is reversed.

To account for the couple due to the magnetic field associated with the current, let us consider the forces on one of the magnets; if the current flows vertically downwards, cf. Fig. 6·16 (*b*), and the poles of

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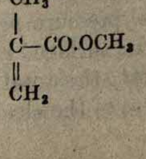
* A synthetic thermoplastic polymer; it softens at a temperature which is conveniently low for working, and yet articles made from this material can be used for many everyday purposes. Methyl methacrylate has the formula



$$\begin{array}{l} CH_3 \\ | \\ C\text{—}CO.OCH_3 \\ \| \\ CH_2 \end{array}$$

the magnet are at distances r_1 and r_2 from the central axis of the system, the forces on them are respectively $F_1 = mH_1$ and $F_2 = mH_2$, where m is the pole strength of the magnet and $H_1 = \frac{2ir_1}{a^2}$ and $H_2 = \frac{2ir_2}{a^2}$. Thus the couple on this magnet is given by

$$\Gamma = 2im\left(\frac{r_1^2 - r_2^2}{a^2}\right),$$

where i is the current (e.m.u.). A similar couple acts in the same sense on the second magnet and so the system rotates as described above.

The self-magnetic pinch effect.—Let us consider a cylindrical column of mercury contained in a vertical glass tube and carrying a current i. Then if $2a$ is the diameter of the tube the magnetic field at a point *inside* the column and at a distance r from its axis is given by

$$H = \frac{2i}{a^2}.r. \quad \text{[Cf. p. 144.]}$$

The associated magnetic induction is

$$B = \frac{2\mu i}{a^2}.r,$$

where μ is the magnetic permeability of mercury; its numerical value is practically unity.

Now consider unit length of the column and that small element of this length which lies between cylindrical surfaces defined by r and $(r + \delta r)$, and two planes with angular separation $\delta\theta$ and normal to the cylindrical surfaces. The magnetic induction within this element is directed as shown in Fig. 6·17 (a). Assuming the current to be uniformly distributed over the cross-section of the column, the current in the element is

$$\frac{i}{\pi a^2}.r.\delta r.\delta\theta.$$

The force on this element, due to the magnetic field within the column, is given by

$$\delta F = \left(\frac{2\mu i}{a^2}.r\right)\left(\frac{i}{\pi a^2}.r\ \delta r\ \delta\theta\right) \times 1$$

$$= \frac{2\mu i^2}{\pi a^4}.r^2\ \delta r\ \delta\theta,$$

and is directed inwards.

Let us now assume that the effect of such forces is to create within the column an *excess* pressure p on the surface 'r' and an excess pressure $p + \delta p$ on the surface '$r + \delta r$'. Since the element we are considering is at rest, the total force on it must be zero. The pressures on the surfaces of the element normal to the plane of the diagram are shown in Fig. 6·17 (*b*), the mean pressures on the flat vertical sides being taken as $p + \frac{1}{2}\delta p$. The forces on these sides, cf. Fig. 6·17 (*c*), will have a resultant

$$2[(p + \tfrac{1}{2}\delta p)(\delta r \times 1).\tfrac{1}{2}\delta\theta],$$

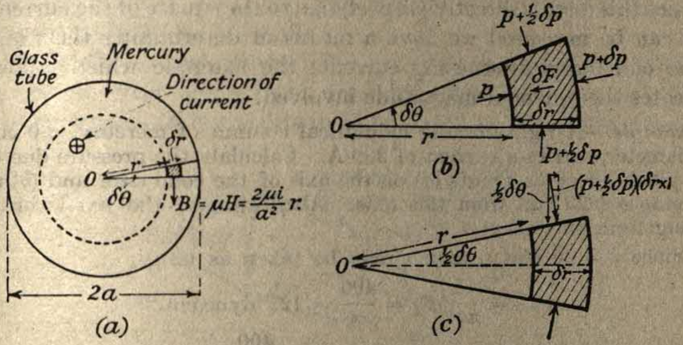


FIG. 6·17.—The self-magnetic pinch effect.

and this force acts outwards. Neglecting terms of the second order this becomes $p\ \delta r\ \delta\theta$.

Since the total force on the element is zero, we have

$$pr\ \delta\theta + p\ \delta r\ \delta\theta = \frac{2\mu i^2}{\pi a^4}.r^2\ \delta r\ \delta\theta + (p + \delta p)(r + \delta r)\ \delta\theta,$$

which leads to

$$\frac{dp}{dr} = -\frac{2\mu i^2}{\pi a^4}.r.$$

Integrating with respect to r, we have

$$p = -\frac{\mu i^2}{\pi a^4}r^2 + \mathrm{C},$$

where C is an integration constant; its value is determined by assuming that at $r = a$, $p = 0$, a procedure justified by the fact that outside the surface $r = a$ there is no current and hence no force.

$$\therefore\ \mathrm{C} = \frac{\mu i^2}{\pi a^4}.a^2,$$

so that

$$p = \frac{\mu i^2}{\pi a^4}(a^2 - r^2).$$

This pressure arises from the so-called ***self-magnetic pinch effect;*** its existence was first pointed out by NORTHRUPP in 1907.

From these considerations it follows that the force outwards on a disc covering the end of the containing tube is given by

$$F = \int_0^a \frac{\mu i^2}{\pi a^4}(a^2 - r^2).2\pi r\,dr,$$

which reduces to

$$F = \tfrac{1}{2}\mu i^2.$$

Since this force is directly proportional to the square of the current, if it can be measured we have a means of determining the r.m.s. value of a large alternating current; the example which follows indicates the orders of magnitude involved.

Example.—A right circular cylindrical column of mercury, 2·0 cm. in diameter, carries a current of 200 A. Calculate the pressure due to the pinch effect at points (*a*) on the axis of the conductor and (*b*) at a distance 0·25 cm. from this axis. Also estimate the axial thrust arising from this effect.

(*a*) Since $i = 20$ e.m.u. and μ may be taken as unity,

$$p_{r=0} = \frac{i^2}{\pi a^4}(a^2) = \frac{400}{\pi} = 127 \text{ dyne.cm.}^{-2}$$

(*b*)

$$p_{r=0{\cdot}25\text{ cm.}} = \frac{i^2}{\pi a^4}(a^2 - r^2) = \frac{400}{\pi}[1^2 - (\tfrac{1}{4})^2]$$

$$= \frac{375}{\pi} = 119 \text{ dyne.cm.}^{-2}$$

(*c*) The axial thrust is $\frac{1}{2}\mu i^2 = 200$ dyne.

A practical application of the pinch effect.—In some of the nuclear reactors which have been built in an attempt to provide electrical power, it was desired to make use of the fusion of light elements. In such a reaction the mass of the products is less than the total mass of the original materials, and the energy which appears is the equivalent of the matter which has 'vanished'. A well-known reaction of this type occurs at the centre of the Sun, where heavy hydrogen, or deuterium, atoms are fused together to form helium according to the equation

$${}_1H^2 + {}_1H^2 \rightarrow {}_2He^3 + {}_0n^1 + 3{\cdot}27 \text{ MeV}.$$

Since deuterium forms about one part in 5000 of natural hydrogen there is a copious supply of this isotope in the oceans covering parts of the Earth, and this implies that there is almost an inexhaustible supply of fuel at hand provided an efficient method of causing a reaction of the above type to take place can be found. Its great advantage over a fission reaction to obtain power is that in a fusion process no appreciable amount of radioactive material, with the increasingly difficult problem of its disposal, is likely to be produced.

Whether or not two colliding particles fuse is determined by their relative velocity; in a hot, completely-ionized gas the nuclear reactions take place between the most energetic particles and it is known that temperatures exceeding 3 × 10^6 deg. K. must be reached if a reactor of this type is to become a practical proposition. At such temperatures the isotopes of hydrogen are completely ionized; an ionized gas is known as 'plasma'. From a thermonuclear point of view it is desirable to ionize the gas within a low-pressure discharge tube and then use the pinch effect to keep the plasma away from the walls.

The main obstacle encountered in experimental work involving the pinch effect arises from the instability of the system. Two possible modes of instability are illustrated in Fig. 6·18; they are referred to as 'sausage' and 'kink' instabilities respectively. Such instabilities are most undesirable since conduction losses become appreciable if the plasma strikes the wall of the containing vessel from which impurities are given off; these also increase radiation losses thus lowering the temperature still further. To stabilize the discharge LUNDQUIST applied an axial magnetic field inside the plasma; this field is obtained by passing a current through a solenoid wound on the discharge tube.

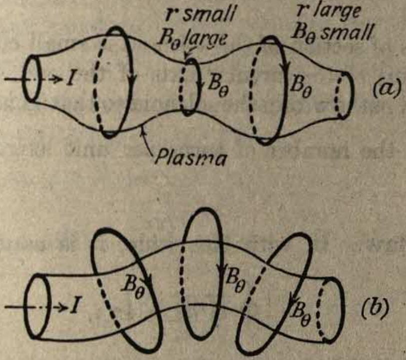


FIG. 6·18.—Instabilities in a pinched discharge (*a*) sausage-type, (*b*) kink-type.

The pinch effect serves to keep the plasma from the walls of the tube and at the same time the temperature of the gas must be raised to an astronomical value; the necessary energy is supplied by using the Joule heating effect or by compressing the gas adiabatically. The energy associated with the Joule heating is given primarily to the electrons which constitute the current. In due course their energy is shared with positive ions in collision processes but many collisions must occur with any positive ion before it acquires the necessary energy; this is because an electron can only lose a fraction of its energy in an encounter with a massive ion. It is because this heating time is long that attention has been paid to the possibility of compressing a plasma adiabatically.

Preliminary reports from ZETA [Zero Energy Thermonuclear Assembly] are not too encouraging; they appear to indicate that the necessary high temperatures have not yet been attained.

A ring or toroidal solenoid.—Suppose we have a toroidal solenoid wound uniformly with N turns. The lines of force will be circles in planes normal to the linear axis of the toroid and have their centres on this axis. If H is the field at any point on a circular path of length s, lying inside the coil and having its centre on the linear axis of the coil, then by Ampère's theorem

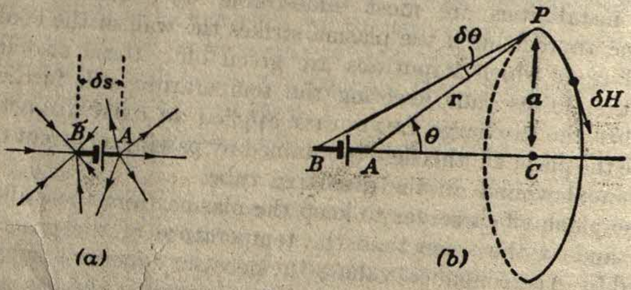
$$Hs = 4\pi Ni, \quad \text{or} \quad H = 4\pi\frac{N}{s}i.$$

If the radius of section of the solenoid is small compared with R, the mean radius of the circular axis of the coil, then s is nearly constant for all paths within the solenoid so that H is constant. Also $\frac{N}{s}$ becomes n, the number of turns per unit length, and we have $H = 4\pi ni$.

Laplace's law. If, with Heaviside, it is assumed that

$$\oint \vec{H}\cdot\vec{ds} = 4\pi\, i,$$

then the magnetic field due to an element of current may be calculated as follows. Consider an element AB, Fig. 6·19 (a), of *insulated* wire and length δs containing a source of e.m.f. and immersed in a *conducting* medium of infinite extent. Current may be imagined to flow out with radial symmetry from A to infinity and to flow back similarly from infinity to B. By symmetry, or by inference from experiments on straight wires, the lines of force of the magnetic field will be circles centred on the axis which is the prolongation



(a) (b)

FIG. 6·19.

of BA. If a is the radius of one of these circles, cf. Fig. 6·19 (*b*), the *outward* current from A through it will be

$$\frac{i}{4\pi r^2} \times 2\pi r^2(1 - \cos\theta),$$

where θ is the angle indicated. This current is

$$\tfrac{1}{2}i(1 - \cos\theta).$$

The *return* current to B and through the same circle is

$$\tfrac{1}{2}i[1 - \cos(\theta - \delta\theta)],$$

where $\delta\theta$ is the angle which AB subtends at a point on the circle. Hence the *resultant* outward current through the circle is

$$\tfrac{1}{2}i[\cos(\theta - \delta\theta) - \cos\theta] = \tfrac{1}{2}i\sin\theta\,.\,\delta\theta.$$

If $\delta\mathrm{H}$ is the field strength at any point on the circle, then by Ampère's circuital relation, cf. p. 142,

$$2\pi a\,.\,\delta\mathrm{H} = 4\pi \times \tfrac{1}{2}i\sin\theta\,.\,\delta\theta.$$

$$\therefore\ \delta\mathrm{H} = \frac{i\sin\theta\,\delta\theta}{a} = \frac{i\,\delta\theta}{r}$$

$$= \frac{i\,.\,\delta s\sin\theta}{r^2}. \quad \left[\because\ \delta\theta = \frac{\delta s\cos(\tfrac{1}{2}\pi - \theta)}{r}.\right]$$

Example. Find the force and torque on a small magnet of moment M lying in a plane normal to a long straight wire carrying a steady current.

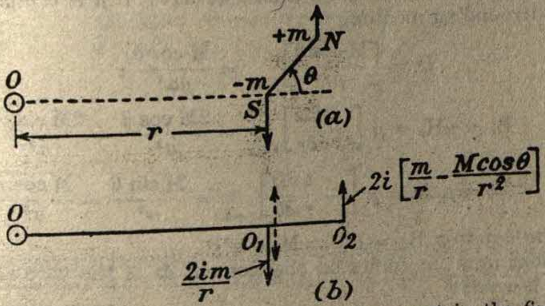


FIG. 6·20.—Force and torque on a small magnet in the field of a linear current.

Let NS, Fig. 6·20 (*a*), be the small magnet of length δs situated as indicated. Then the force on S and in the direction of the field due to the current is equal to $-m\left(\frac{2i}{r}\right)$; on N the force is $2im\left[\frac{1}{r} + \frac{\partial}{\partial s}\left(\frac{1}{r}\right)\delta s\right]$, which is $2i\left[\frac{m}{r} - \frac{\mathrm{M}\cos\theta}{r^2}\right]$.

If the lines of action of these two forces pass through points O_1 and O_2 at distances r and $r + \delta s(\cos\theta)$ respectively from the wire, then at O_1 let us add two equal and opposite forces each being of magnitude $2i\left[\frac{m}{r} - \frac{M\cos\theta}{r^2}\right]$, as indicated in Fig. 6·20 (*b*). Then the small magnet is acted upon by a force $\frac{2Mi\cos\theta}{r^2}$ in a direction opposite to the field at O_1 due to the current and an anticlockwise couple of magnitude

$$2i\left[\frac{m}{r} - \frac{M\cos\theta}{r^2}\right]\delta s.\cos\theta = \frac{2Mi\cos\theta}{r},$$

since $M\,\delta s \longrightarrow 0$.

Example.—If the magnetic field of the earth (radius a) may be regarded as that due to a suitably orientated magnet at the centre of the earth, and the induction associated with the magnetic field at the equator denoted by B, show that a wire surrounding the earth along the parallel of latitude λ south and carrying a steady current i from west to east, would experience a resultant force towards the south pole of the heavens, equal to

$$6\pi a i B\sin\lambda\cos^2\lambda.$$

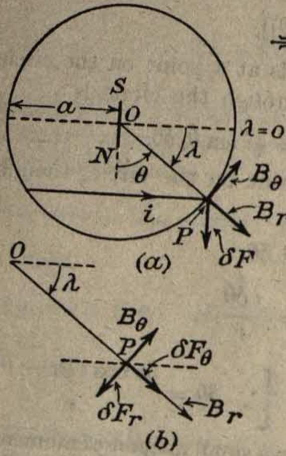


FIG. 6·21.

Let M be the magnetic moment of the magnet placed at the centre, O, Fig. 6·21 (*a*), of the earth and with its axis pointing south. At P, a point on the wire, with the usual notation, we have, if μ is the permeability of the surrounding medium,

$$\Omega = \left[\frac{M\cos\theta}{\mu r^2}\right]_{r=a} = \frac{M\cos\theta}{\mu a^2};$$

$$B_r = \mu H_r = \mu\left[-\frac{\partial\Omega}{\partial r}\right]_{r=a} = \frac{2M\cos\theta}{a^3} = \frac{2M\sin\lambda}{a^3},$$

and $$B_\theta = \mu H_\theta = \mu\left[-\frac{1}{r}\frac{\partial\Omega}{\partial\theta}\right]_{r=a} = \frac{M\sin\theta}{a^3} = \frac{M\cos\lambda}{a^3}.$$

At the equator $B_\theta = B$, $\therefore M = a^3B$.

Consider now the forces on the element δs of the wire and situated at P. We have

$$\delta F_r = \text{force due to } B_r = (i\,\delta s)B_r,$$

and this force acts in the sense opposite to that of B_θ;

also $$\delta F_\theta = \text{force due to } B_\theta = (i\,\delta s)B_\theta,$$

and this force acts along B_r as shown in Fig. 6·21 (*b*).

$\therefore$ Resultant force on the wire, directed towards the south pole of the heavens, is

$$2\pi i a\cos\lambda\,B_r\cos\lambda + 2\pi a i\cos\lambda\,B_\theta\sin\lambda,\quad \left[\because \oint ds = 2\pi a\cos\lambda\right]$$

and since $B_r = 2B \sin \lambda$ and $B_\theta = B \cos \lambda$, the force required is $6\pi a i B \sin \lambda \cos^2 \lambda$.

Instead of finding δF_r and δF_θ, we may proceed as follows :—the magnetic induction associated with the magnetic field at P in the plane of the wire and also perpendicular to the wire is

$$B_r \cos \lambda + B_\theta \sin \lambda = \frac{3M}{a^3} \sin \lambda \cos \lambda = 3B \sin \lambda \cos \lambda.$$

$$\therefore\ \delta F = i\, \delta s \left[\frac{3M}{a^3} \sin \lambda \cos \lambda\right] = 3Bi \sin \lambda \cos \lambda\, \delta s.$$

$$\therefore\ F = 3Bi \sin \lambda \cos \lambda \oint ds$$

$$= 6\pi a i B \sin \lambda \cos^2 \lambda. \quad [\because \oint ds = 2\pi a \cos \lambda.]$$

The induction at P also has a component parallel to NS but the force due to this vanishes when the whole wire is considered.

Example.—Find the force due to a small magnet on a circular coil carrying a steady current, the plane of the coil being normal to the axis of the magnet and its centre on the axis.

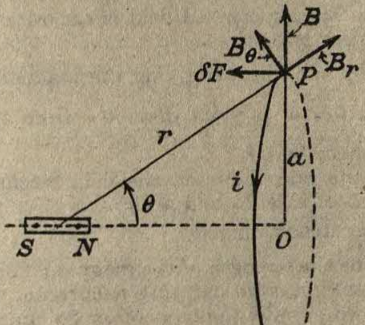


FIG. 6·22.

Since the magnetic induction at a point (r, θ) due to a small magnet of magnetic moment M at $(0, 0)$ and with its axis along the line $\theta = 0$, has for its components

$$B_r = \frac{2M \cos \theta}{r^3}, \quad \text{and} \quad B_\theta = \frac{M \sin \theta}{r^3},$$

the induction, B, in the plane of the wire and normal to it, cf. Fig. 6·22. is given by

$$B = B_r \sin \theta + B_\theta \cos \theta = \frac{M}{r^3}[3 \sin \theta \cos \theta].$$

The force δF on each element of the wire is therefore parallel to

NS and the resultant force in this direction is given in magnitude by

$$F = li\mathrm{B} = 2\pi a i\frac{\mathrm{M}}{r^3}[3 \sin \theta \cos \theta] \qquad [a = \text{radius of circle.}]$$

$$= 6\pi \mathrm{M}i\left[\frac{a^2(r^2 - a^2)^{\frac{1}{2}}}{r^5}\right].$$

N.B. The components of the induction normal to the plane of the wire produce no resultant effect.

EXAMPLES VI

6·01. Sketch the magnetic field in any plane normal to, and due solely to, two long parallel wires carrying the same steady current but in opposite directions.

Calculate the force on 60 cm. of either wire when their distance apart is 6 cm. and the current 15 amp. Indicate the direction of the force, stating the rule you apply.

If a uniform magnetic field of strength 2 oersteds is now superposed in a direction at right angles to the plane of the wires, find the force experienced by a third current-bearing wire placed parallel to and exactly midway between the first two wires. You are to assume that at a midway point the superposed field is opposite in direction to the field due to the first two wires. [45 dyne., 0.]

6·02. State fully the formula giving the magnetic field strength of a current element $i\,\overrightarrow{\delta s}$ at a point distant $\vec{r}$ from the element, if θ is the angle between $\overrightarrow{\delta s}$ and $\vec{r}$.

Hence deduce the ratio between the field strengths at the centres of a square circuit of side $2a$ and a circular circuit of radius a, each carrying the same steady current. $[2\sqrt{2} : \pi = 0{\cdot}90.]$

6·03. Determine the strength of the magnetic field in the neighbourhood of a long straight wire carrying a current. Imagine two such conductors are fixed parallel to one another $2a$ cm. apart and form the lead and return of a circuit carrying a current of I amperes. Determine the resultant field strength due to the current at a point in the plane of the wires and lying between them at a distance x cm. from one wire. By means of a graphical construction show where the field is a minimum and find its value when the current is 10 A. and the wires are 20 cm. apart.

$$\left[\frac{0{\cdot}4\mathrm{I}a}{x(2a - x)},\ x = a,\ 0{\cdot}4 \text{ oersted.}\right]$$

6·04. Define *intensity of magnetization.*

Explain what is meant by a uniformly magnetized shell. Use the conception of a uniform magnetic shell to obtain an expression *either* for the field strength due to a linear current *or* at the centre of a short solenoid carrying a steady current.

6·05. A circular coil of radius a and carrying a steady current is placed with its plane horizontal in the earth's magnetic field. Obtain an expression for the couple which acts on the coil. $[\pi a^2 i \mu_a \mathrm{H}_0.]$

6·06. Two coaxial circular coils each contain N closely wound turns of wire of radius a; their planes are parallel and at a distance $2x$ apart. When a current i circulates in the same sense in each coil, the magnetic field strength at the centre of the system is given by

$$H = \frac{Ai}{a\left(B + \frac{x^2}{a^2}\right)^{1\cdot5}},$$

where A and B are constants for the system. Calculate values for A and B, and describe how these values could be obtained experimentally when H_0, the horizontal component of the earth's magnetic field is known. [$A = 2\pi N$, $B = 1$.]

6·07. Describe the arrangement of coils designed by Helmholtz for the production of a uniform magnetic field. Justify the arrangement. Calculate the percentage change in the strength of the magnetic field due to a steady current in a Helmholtz coil system in going from its centre to the centre of either coil. [5·4 per cent decrease.]

6·08. Find the magnetic field strength due to a steady current in a single circular turn of wire at a point on the axis at a distance x from the centre, the radius of the coil being a. Two similar circular coils are arranged coaxially at a distance apart equal to their radius. What special property has the magnetic field at a point on the axis midway between the coils and how has this property been applied?

6·09. Two long parallel wires carry equal currents in opposite directions. A plane normal to the wires cuts them in points C_1 and C_2. P is a point on the circumference of a circle whose diameter is C_1C_2. Show that at P the field due to the currents is directed towards the centre of the circle and its magnitude is $\frac{4ia}{r_1r_2}$, where r_1 and r_2 are the distances of P from C_1 and C_2 respectively, and $C_1C_2 = 2a$.

6·10. Give the theory of the Helmholtz system of coils and describe in detail a modern method for the determination of the horizontal component of the earth's magnetic field.

6·11. Describe in detail and give the theory of a method for determining the magnetic moment of a bar magnet.

A flat circular coil consisting of 100 turns of mean radius 14 cm. is set with its plane vertical and making an angle of 85° with the magnetic meridian. Calculate the current that must pass through the coil in order that a small magnet suspended at its centre may set normally to the magnetic meridian. [Assume $H_0 = 0\cdot18$ oersted.]

[$4\cdot03 \times 10^{-2}$ amp.]

6·12. Two long vertical wires at a distance $2a$ apart convey equal currents i e.m.u. in opposite directions. Show that the magnetic field strength at a point P at distance r from each wire is $\frac{4ia}{r^2}$. If a dipole at P and in a plane normal to the two wires has a magnetic moment M and a moment of inertia I about an axis through its centre and parallel to the two wires, prove that the period of small oscillations is that of a simple pendulum of length l given by

$$4iMal = Ir^2g,$$

where g is the intensity of gravity. [Neglect the earth's magnetic field.]

6·13. Define a uniform magnetic shell and establish the relation

$$\oint \vec{H}\cdot\vec{ds} = 4\pi i.$$

A current i e.m.u. per unit width flows along the boundary separating two media whose permeabilities are μ_1 and μ_2 respectively, the direction of the current being normal to the plane containing the lines of magnetic induction. Show that the normal component of the magnetic induction on either side of the boundary is continuous but that the tangential component of the field strength suffers a discontinuity equal to $4\pi i$.

6·14. A very long, straight, thin copper ribbon of width $2t$ carries a steady current i e.m.u. Show that the magnetic induction at a point in the plane of the ribbon, and at a distance a from its central axis, is given by $B = \dfrac{2\mu i}{t} \ln \left(\dfrac{a+t}{a-t}\right)$, where μ is the permeability of the surrounding medium.

CHAPTER VII

OHM'S LAW AND ITS APPLICATIONS

The electromagnetic unit of current and of quantity of electricity.—The electromagnetic unit of current is defined as that steady current, or current of zero frequency, which, flowing in a closed circuit in a vacuum, may be replaced by an equivalent magnetic shell of strength unity.

From this unit we derive the e.m.u. of quantity of electricity which is defined as the amount of electricity flowing across any section of a conductor in one second when the steady current in the conductor is one e.m.u. of current.

The practical unit of current.—For many purposes the e.m.u. of current is too large; a current whose magnitude is one-tenth the e.m.u. of current is known as the ***ampere.*** The ampere is determined in terms of the force between two conductors carrying the same current by means of the Ayrton-Jones-Smith current balance; its structure and use are described on p. 706.

The practical unit of quantity of electricity is the ***coulomb;*** it is the amount of electricity which passes in one second across any section of a conductor in which the steady current is one ampere.

The electromagnetic unit of potential difference and the practical unit of potential difference.—*Suppose that* A *and* B *are two equipotential surfaces in a conductor through which a current is flowing from* A *to* B. *Let this current flow for such a time that one e.m.u. of quantity of electricity passes across each section of the conductor. Then if the energy liberated is one erg, the potential at* A *exceeds that at* B *by one e.m.u. of potential.*

The practical unit of potential difference is termed the ***volt.*** *If the current flows in a conductor from an equipotential surface* A *to another such surface* B *and for such a time that one coulomb passes across any section of the conductor and the energy liberated is one joule* [*the practical unit of energy*], *then the potential at* A *exceeds that at* B *by one volt.*

The relation between the e.m.u. of potential difference and the volt.—If a definite potential difference exists between two points in a conductor, the same amount of energy will always be liberated irrespective of the units used to express the quantity of electricity and the potential difference. Now when one coulomb is transported across each section of a conductor between whose ends there is a potential difference of one volt, the energy liberated is one joule, or 10^7 ergs. Since one e.m.u. of quantity of electricity is equivalent to 10 coulombs, the energy liberated when this quantity of electricity passes each cross-section of the above conductor is $10 \times 10^7 = 10^8$ ergs. But since, when one e.m.u. of electricity is transported across each section of a conductor between whose ends there is a potential difference of one e.m.u. of potential, the work done is one erg, it follows that

$$1 \text{ volt} \equiv 10^8 \text{ e.m.u. of potential.}$$

Ohm's law for metallic conductors.—***When a steady current is flowing through a conductor the potential difference between its ends is directly proportional to the current provided that the physical condition of the conductor does not change.*** Thus, if V is the potential difference and I the current, we may write

$$V = IR,$$

where R is a constant termed the ***resistance*** of the conductor. It is measured in ***ohms*** when the potential is in volts and the current in amperes.

A conductor has a resistance equal to one e.m.u. of resistance if the potential difference between its ends is one e.m.u. of potential when the current through it is one e.m.u. of current. Thus

$$1 \text{ ohm} = \frac{1 \text{ volt}}{1 \text{ ampere}} = \frac{10^8 \text{ e.m.u. of potential}}{10^{-1} \text{ e.m.u. of current}} = 10^9 \text{ e.m.u.}$$

of resistance.

The evaluation of the ohm.—The system of 'practical' units was devised originally by a committee of the British Association; the value of the ohm was determined experimentally, and standard resistance coils (German-silver) were constructed (1863). In 1881 the first International Congress of Electricians advocated a redetermination of the ohm in absolute measure; moreover, it was agreed to dispense with German-silver wire and construct standards of resistance on a plan suggested first by SIEMENS. The so-called 'practical ohm' was to be the resistance of a certain column of pure mercury of definite length and 1 mm.^{2} cross-sectional area. The original 'Siemens' unit' was a column of mercury 1 metre long and 1 mm.^{2} in section. This standard is now known to be equal to about 0·9408 ohm.

The ohm is determined in absolute measure by means of the Lorenz type of machine described on p. 698. Essentially it is a development of Faraday's wheel,* constructed on an engineering scale with high mechanical precision and from materials free from ferromagnetic impurities. The ohm can also be determined in absolute measure by means of the alternating current bridge described on p. 701. In this method use is made of the primary standard of mutual inductance designed originally by Campbell and described on p. 234: the structure of this inductance is simple and it is probably the most accurate of all electrical standards.

Standard resistors.—A section of a standard resistor suitable for use with small currents is shown in Fig. 7·01. C_1 and C_2 are thick copper leads which are held by brass nuts in the keramot lid A of a brass container B. The main part of the resistor is the manganin wire M, wound non-inductively and hard-soldered to the current leads. This coil is mounted on, but insulated from, the drum D which forms the inner portion of the container. The closed end of the loop is opened out, as shown at L, so that there shall be no sudden changes in the curvature of the wire. The wire must be well annealed to remove strains in it, but large strains exist in wires wherever there are sudden bends and these are not entirely removed by the process of annealing. The container is filled with an insulating oil which also serves to protect the wire from the atmosphere and consequent oxidation of its outer surface.

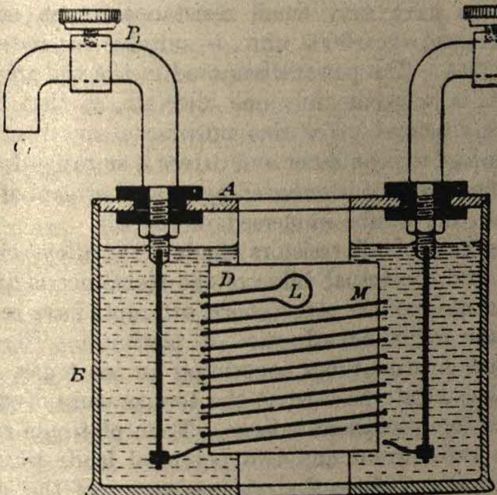


FIG. 7·01.—A standard resistor.

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* Cf. 'Intermediate Physics'.

All standard resistors are provided with potential terminals P_1, P_2, and it is the potential difference between these terminals per unit current in the resistor which is the resistance represented by the standard. C_1 and C_2 dip into mercury cups and any slight variations in the contact resistances here do not affect the value of the resistance which the assembly is intended to represent.

The whole is mounted in a thermostat at 20° C. and, for work of the highest precision, it is necessary that the resistance between the terminals P_1 and P_2 should be a maximum at this temperature in order to avoid large corrections for the change of resistance with temperature. Manganin itself does not have this property but by adding a short piece of copper wire of the correct length in series with M, this desirable feature is achieved.

Standard resistors have very small resistances when used to measure large currents, e.g. 0·0003 ohm. is suitable for measuring currents up to 2000 A. The power dissipated in the coil when the current is 2000 A. is approximately one kilowatt, so that the oil must be thoroughly stirred. For this purpose a small motor is mounted on the top of the container and drives a suitably designed paddle in the oil, and the heat is removed by a rapid stream of water passing through a coiled tube immersed in the oil.

Since the resistance of all resistors made from alloys changes slowly with time and the actual value of the resistance is affected by the humidity content of the atmosphere, attempts have recently been made to use well-annealed coils of platinum. These are hermetically sealed in glass tubes containing an inert gas. Such standard resistors can only be used at the temperature of melting ice for the temperature coefficient is large. Each platinum resistor is provided with two current and two potential leads which are used as described on p. 195, so that the resistance of the leads is eliminated from the final measurements. These resistors have now been shown not to be satisfactory standards; the temperature of the actual wire is never known with sufficient accuracy.

Resistance and conductance.—When it is stated that the resistance of a certain conductor is R ohms, we mean that a potential difference of R volts is required to send a steady current of one ampere through that conductor. Thus it is equally correct and certainly more informative to say that the resistance of the above conductor is R volt.amp.$^{-1}$.

The reciprocal of resistance is called ***conductance***, S, i.e.

$$S = \frac{1}{R}\ \text{ohm.}^{-1} = \frac{1}{R}\ \text{amp.volt.}^{-1}.$$

Resistivity.—By means of a metre bridge or post-office box it may be confirmed that the resistance, R, of a uniform wire is directly proportional to its length, l, and inversely proportional to its cross-sectional area, A, provided of course that the physical state of the wire remains constant. Hence $R = \frac{\chi l}{A}$, where χ is a constant for the material of the wire. It is termed its ***resistivity.***

Since we may write $[\chi] = \left[\frac{RA}{l}\right]$ = [ohm. × length² ÷ length], the unit of resistivity is the ohm.cm. when the unit of length is the centimetre.

Conductivity.—The ***conductivity*** of a substance is the reciprocal of its resistivity. It is therefore expressed in ohm.$^{-1}$ cm.$^{-1}$ and is denoted by σ.

Kirchhoff's laws for steady currents.—In the study of the flow of electricity through different circuits problems arise concerning the manner in which electricity distributes itself when its path is divided; it is a matter of some importance to be able to determine how the electricity will flow through a network of conductors in which junctions exist. KIRCHHOFF gave two laws which enable solutions to be obtained, provided each link in the network obeys Ohm's law. Since the currents are steady there can be no accumulation of electricity at any point so that the sum of all the currents arriving at any junction must be equal to the sum of all the currents which leave it. This is the first law. If we introduce the convention that currents flowing into a junction are considered positive so that those leaving it are negative, Kirchhoff's first law states that ***in any network of wires the algebraic sum of the currents which meet at a point is zero,*** i.e. $\Sigma\,(I) = 0$.

Now let the different junctions be denoted by A, B, C . . . as in Fig. 7·02 (*a*); let the potentials at these junctions be V_A, V_B, V_C . . .

Let I_{AB} be the current flowing from A to B, R_{AB} being the resistance between these two points.

Let us apply Ohm's law to each conductor in any set of conductors which form a possible path starting from A and finally returning to A. We have

$$\begin{aligned} V_A - V_B &= I_{AB}R_{AB}, \\ V_B - V_C &= I_{BC}R_{BC}, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ V_N - V_A &= I_{NA}R_{NA}. \end{aligned}$$

By addition we get

$$0 = \Sigma\,(IR).$$

If cells are present at different points in the circuit selected the above treatment needs a slight modification. Consider one link, say AB, of the complete path and let it contain a cell of e.m.f. E_1, cf. Fig. 7·02 (*b*); P and Q are the terminals of this cell.

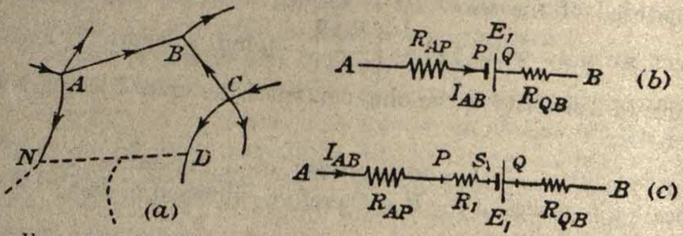

FIG. 7·02.—Kirchhoff's laws for steady currents in a network.

Since the cell has internal resistance R_1 we may replace the actual cell by a resistance R_1 and a source of e.m.f. E_1 without internal resistance as in Fig. 7·02 (*c*). Then

$$V_A - V_P = I_{AB}R_{AP},$$
$$V_P - V_S = I_{AB}R_1,$$
$$V_S - V_Q = -E_1,$$

and

$$V_Q - V_B = I_{AB}R_{QB}.$$

By addition we get for this link

$$V_A - V_B = I_{AB}R_{AB} - E_1,$$

where $R_{AB} = R_{AP} + R_1 + R_{QB}$.*

Since an equation of this type applies to every link of a closed circuit, we have

$$0 = \Sigma\,(IR) - \Sigma\,(E),$$

which is Kirchhoff's second law. In words it states ***the algebraic sum of the electromotive forces in any closed circuit or loop is equal to the algebraic sum of the products of the resistances of each link of the circuit or loop and the currents of zero frequency flowing through them, provided that for each link Ohm's law is obeyed,*** i.e. $\Sigma\,(E) = \Sigma\,(IR)$.

Maxwell's cyclic currents.—The above method of determining the current in any part of a circuit becomes complicated when the circuit has many branches. Maxwell suggested the 'cyclic current' device to simplify the problem. He imagined that a specified cyclic current flowed in each loop, and considered all clockwise currents as positive. The current in any branch is thus the algebraic difference between the cyclic currents in the loops it separates.

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* This equation assumes the law for the effective resistance of several resistors in series; the proof is independent of Kirchhoff's laws and may therefore be used. In contrast to this, the proof of the law for resistances in parallel depends on Kirchhoff's first law.

Use will be made of both Kirchhoff's laws and of Maxwell's cyclic currents at a later stage in this chapter.

The Wheatstone bridge network of conductors.—The Wheatstone bridge network of four resistances is one of the oldest and, at the same time, one of the most commonly used methods for comparing resistances, i.e. one resistance can be measured in terms of three others ; it is for use with currents of zero frequency. The 'bridge' is represented diagrammatically in Fig. 7·03 (*a*), and by

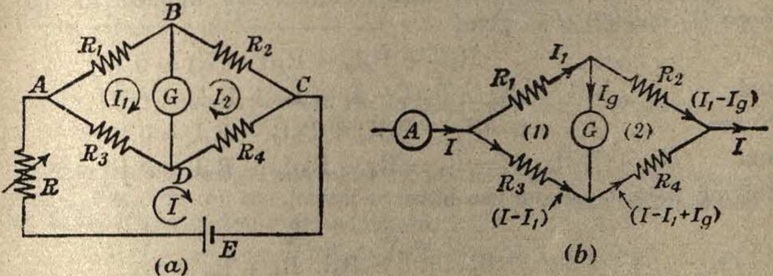

FIG. 7·03.—Elementary theory of a Wheatstone bridge network.

considering the cyclic currents I, I_1 and I_2, the necessary equations are obtained in as symmetrical a form as possible. The resistance R, whose value need not be known when it is only a matter of comparing resistances, should be added so that large currents are never used ; R includes the resistance of the cell of e.m.f. E which supplies current to the bridge. If R_g is the resistance of the galvanometer, the equations for the bridge are

$$\left.\begin{aligned} RI + R_3(I - I_1) + R_4(I - I_2) - E &= 0,\\ R_1I_1 + R_g(I_1 - I_2) + R_3(I_1 - I) &= 0,\\ \text{and}\qquad R_2I_2 + R_4(I_2 - I) + R_g(I_2 - I_1) &= 0,\end{aligned}\right\}$$

which when rearranged give

$$\left.\begin{aligned} (R + R_3 + R_4)I - R_3I_1 - R_4I_2 - E &= 0,\\ - R_3I + (R_1 + R_g + R_3)I_1 - R_gI_2 &= 0,\\ \text{and}\qquad - R_4I - R_gI_1 + (R_g + R_2 + R_4)I_2 &= 0.\end{aligned}\right\}$$

These equations may be solved and an expression for

$$(I_1 - I_2) = I_g,$$

the current through the galvanometer, obtained. It is given by

$$I_g = \frac{E(R_2R_3 - R_1R_4)}{RR_g[R_1+R_2+R_3+R_4] + R(R_1+R_3)(R_2+R_4) + R_g(R_1+R_2)(R_3+R_4) + R_1R_2(R_3+R_4) + R_3R_4(R_1+R_2)}.$$

It is seldom necessary to obtain this expression since the condition for the bridge to be balanced, i.e. for $I_g = 0$, can be obtained very readily by expressing I_g in terms of I, the current supplied by the cell. If necessary, this current I can be measured by means of a milliammeter.

Simplified theory of the Wheatstone bridge network of conductors.—Let the four resistors be arranged as in Fig. 7·03 (*b*) and let the currents in the various arms be as indicated. In order to halve the number of unknowns use is made of Kirchhoff's first law, i.e. the currents are marked so that at each junction their algebraic sum is zero. Kirchhoff's second law applied to the loops (1) and (2) then gives

$$\left.\begin{aligned} R_1I_1 + R_gI_g - R_3(I - I_1) &= 0, \\ \text{and} \quad R_2(I_1 - I_g) - R_4(I - I_1 + I_g) - R_gI_g &= 0. \end{aligned}\right\}$$

Hence

$$\left.\begin{aligned} (R_1 + R_3)I_1 + R_gI_g - R_3I &= 0, \\ \text{and} \quad (R_2 + R_4)I_1 + (-R_2 - R_4 - R_g)I_g - R_4I &= 0. \end{aligned}\right\}$$

Solving these equations we have

$$\frac{I_1}{\begin{vmatrix} R_g, & -R_3I \\ -R_2 - R_4 - R_g, & -R_4I \end{vmatrix}} = \frac{I_g}{\begin{vmatrix} -R_3I, & R_1 + R_3 \\ -R_4I, & R_2 + R_4 \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} R_1 + R_3, & R_g \\ R_2 + R_4, & -R_2 - R_4 - R_g \end{vmatrix}}$$

$$\therefore\ I_g = \frac{[-R_3(R_2 + R_4) + R_4(R_1 + R_3)]}{\begin{vmatrix} R_1 + R_3, & R_g \\ R_2 + R_4, & -R_2 - R_4 - R_g \end{vmatrix}}\,.\,I.$$

Hence I_g is zero when the numerator of the above fraction is zero, i.e. when

$$R_1R_4 = R_2R_3,$$

which is the well-known condition for the bridge to be balanced.

Example.—In a Wheatstone bridge which is nearly balanced the resistances of the arms are 100·01, 10, 1000, and 100 ohms. The cell has an e.m.f. of 2 volts and negligible internal resistance. The galvanometer has a resistance of 10 ohms and sensitivity 0·5 cm. per 0·1 μA. Calculate values for the galvanometer deflexions in the two possible arrangements of cell and galvanometer and indicate the direction of the current through the galvanometer in each instance.

First let the bridge be arranged as in Fig. 7·04 (*a*). Then the effective resistance of the bridge is 100 ohms so that the current from the cell is 0·02 amp. If I_1 and I_g are the currents indicated, by applying Kirchhoff's second law to the two loops we have

$$100I_1 + 10I_g - 1000(\tfrac{1}{50} - I_1) = 0,$$

or

$$110I_1 + I_g - 2 = 0, \qquad \text{(i)}$$

and

$$10(I_1 - I_g) - 100{\cdot}01(\tfrac{1}{50} - I_1 + I_g) - 10I_g = 0,$$

or

$$5500{\cdot}5I_1 - 6000{\cdot}5I_g - 100{\cdot}01 = 0 \qquad \text{(ii)}$$

Eliminating I_1 from (i) and (ii) we have

$$100{\cdot}0091 - 6050I_g - 100{\cdot}01 = 0.$$

Thus $$I_g = -1{\cdot}49 \times 10^{-7} \text{ amp.},$$

the negative sign indicating that the current flows from D to B through the galvanometer.

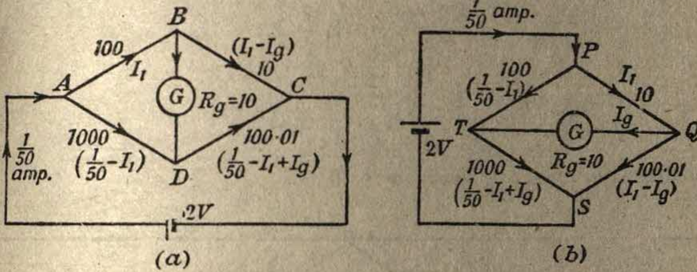

FIG. 7·04.—Current through the galvanometer when a Wheatstone bridge is slightly out of balance.

When the cell and galvanometer are interchanged so that the circuit is as in Fig. 7·04 (*b*), the current from the cell is again 0·02 amps and we have, if the currents are as shown,

$$10I_1 + 10I_g - 100(\tfrac{1}{50} - I_1) = 0, \quad . \quad \text{(iii)}$$

and $$100{\cdot}01(I_1 - I_g) - 1000(\tfrac{1}{50} - I_1 + I_g) - 10I_g = 0 \quad . \quad \text{(iv)}$$

Thus $$I_1 = \frac{(0{\cdot}2 - I_g)}{11},$$

and hence from equation (iv),

$$100{\cdot}01(0{\cdot}2 - 12I_g) - 11{,}000\left(\frac{1}{50} - \frac{0{\cdot}2 - I_g}{11} + I_g\right) - 110I_g = 0.$$

After some reduction we find

$$I_g = 1{\cdot}50 \times 10^{-7} \text{ amp.},$$

so that the current through the galvanometer flows from Q to T.

The galvanometer deflexions will therefore be 0·75 cm. in each instance.

Now although the current through the galvanometer is practically independent of the arrangement of the cell and the galvanometer yet it must be noted that in the first instance the current through the 100·01 ohms coil, which is presumably the coil whose resistance is to be measured, is much less than in the second arrangement. If the material from which this coil is constructed has a large resistance-temperature coefficient, then a correction for the effect of the Joule heating that takes place will have to be made. The procedure will be mentioned in connexion with platinum thermometry—cf. p. 191.

Example.—The arms AB, BC, CD and DA of a Wheatstone bridge contain coils of resistances 2, 3, 2 and 3 ohms respectively. The point A is maintained at a potential of 2 volts while C is at zero potential. The points B and D are joined by a galvanometer of resistance 10 ohms. Show that the sum of the potentials at B and D is 2 volts and determine the magnitude, and direction, of the galvanometer current.

The bridge is shown diagrammatically in Fig. 7·05; the currents in the different branches are also indicated. If V_B and V_D denote the potentials at B and D respectively, we have

$$I = I_1 + I_2 = \frac{2 - V_B}{2} + \frac{2 - V_D}{3}.$$

Also

$$\frac{V_B - 0}{3} + \frac{V_D - 0}{2} = (I_1 - I_g) + (I_2 + I_g)$$

$$= I_1 + I_2 = I.$$

$$\therefore \quad \frac{2 - V_B}{2} + \frac{2 - V_D}{3} = \frac{V_B}{3} + \frac{V_D}{2},$$

which gives

$$V_B + V_D = 2.$$

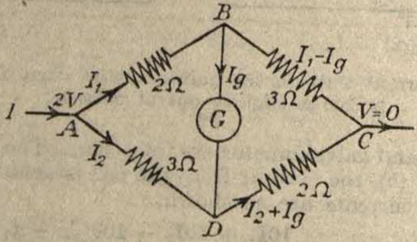

Fig. 7·05.

Applying Kirchhoff's second law to the loops ABD and BCD, we have

$$2I_1 + 10I_g - 3I_2 = 0. \quad . \quad . \quad \text{(i)}$$

and

$$3(I_1 - I_g) - 2(I_2 + I_g) - 10I_g = 0,$$

or

$$3I_1 - 2I_2 - 15I_g = 0. \quad . \quad . \quad \text{(ii)}$$

Eliminating I_g from equations (i) and (ii), we find

$$12I_1 - 13I_2 = 0 \quad . \quad . \quad . \quad \text{(iii)}$$

To obtain another equation connecting I_1 and I_2, we note that the drop in potential between A and C is 2V, whether we go via B or D. Thus

$$2 = 2I_1 + 3(I_1 - I_g),$$

and

$$2 = 3I_2 + 2(I_2 + I_g).$$

Eliminating I_g from these equations, we get

$$10 = 10I_1 + 15I_2 \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Hence

$$I_1 = \frac{13}{31} \text{ amp., } I_2 = \frac{12}{31} \text{ amp.}$$

$$\therefore \quad I_g = \frac{1}{31} \text{ amp.} = 0{\cdot}032 \text{ amp.}$$

Example.—In a certain metal rectifier the current is found to be proportional to the square of the voltage, i.e. it does not obey Ohm's law. The rectifier is connected in the arm CD of the usual Wheatstone bridge network; each of the other arms has a resistance of 20 ohms.

Current is supplied to the bridge so that 2V. is the fall in potential between the points A and C; the bridge is found to be balanced. The current to the bridge is then raised so that the drop in potential across the bridge is 6V. when this is again balanced. If this balance is effected by altering the resistance R in the arm BC of the bridge, obtain a value for it.

When the potential drop across the bridge is 2V., the current through the rectifier is given by

$$I = \frac{2}{20 + 20} = \frac{1}{20} = a.1^2,$$

since the potential difference between C and D is 1V. and the current through the rectifier is proportional to the square of the voltage across it; a is a constant. Hence $a = 0{\cdot}050$ amp.volt.$^{-2}$.

In the second instance the bridge is balanced when the resistance of the rectifier is equal to R. Since the current through the rectifier (or R) is $\frac{6}{20 + R}$, the drop in potential across it is $\frac{6R}{20 + R}$. Hence

$$\frac{6}{20 + R} = a.\left(\frac{6R}{20 + R}\right)^2.$$

$$\therefore \quad 20 + R = 0{\cdot}050(6R)^2,$$

or

$$(20 + 3R)(10 - R) = 0.$$

$\therefore$ R = 10 ohms is the only solution with a physical significance.

On the sensitivity of a Wheatstone bridge.—This method of dealing with the problem of the sensitivity of a Wheatstone bridge is due to CALLENDAR. The conditions of sensitivity in measuring resistance by a Wheatstone bridge have been discussed by MAXWELL and by HEAVISIDE. They start with the assumption that the battery power available is limited by the internal resistance of the cells and give rules for obtaining the maximum current through the galvanometer with this implied limitation. This limitation was no doubt a consideration in many kinds of telegraph testing in the early days of such work, but it is rarely applicable in modern laboratory practice and never in dealing with electrical resistance thermometers. In the majority of resistance measurements, and more particularly in electrical thermometry, the limiting condition is imposed by the heating effect of the current on the resistance to be measured, so that the resistances to form the bridge should be chosen and arranged to give the greatest sensitivity for a given limiting value of the current through the resistance to be measured. The resistance of the battery is quite immaterial provided that it can be arranged to give the limiting current required.

Let Fig. 7·06 (*a*) represent a Wheatstone bridge network in which I is the current through the resistance to be measured, r, while, when the bridge is not balanced, I_g is the current through the galvanometer of resistance R_g. The resistance 'in series' with r is nR and the current through it is $I + I_g$; the resistance 'in parallel' with r on the same side of the galvanometer circuit is $m\mathrm{R}$ and the current through it is $\mathrm{I}_1 + \mathrm{I}_g$. The resistance in the arm of the bridge conjugate to r is $nm\mathrm{R}$ and the current therein is I_1. The currents are assumed to flow in the directions indicated so that the conditions of continuity are satisfied. The bridge is balanced when $r = \mathrm{R}$.

Applying Kirchhoff's second law to the loops ABD and BCD, respectively, we have

$$nm\mathrm{R}\mathrm{I}_1 - \mathrm{R}_g\mathrm{I}_g - n\mathrm{R}(\mathrm{I} + \mathrm{I}_g) = 0,$$

and

$$m\mathrm{R}(\mathrm{I}_1 + \mathrm{I}_g) - r\mathrm{I} + \mathrm{R}_g\mathrm{I}_g = 0.$$

Eliminating I_1 from these equations, we have

$$\frac{\mathrm{I}_g}{\mathrm{I}} = \frac{n(r - \mathrm{R})}{(n+1)\mathrm{R}_g + n(1+m)\mathrm{R}} = \frac{r - \mathrm{R}}{\left(1 + \frac{1}{n}\right)\mathrm{R}_g + (1+m)\mathrm{R}}.$$

This ratio may be regarded as a measure of the sensitivity of the arrangement. It is at once evident that, for a given defect of balance as measured by $r - \mathrm{R}$, or Δr (say), the ratio $\frac{\mathrm{I}_g}{\mathrm{I}}$ will be a maximum when n is as large as possible and m as small as possible. Thus

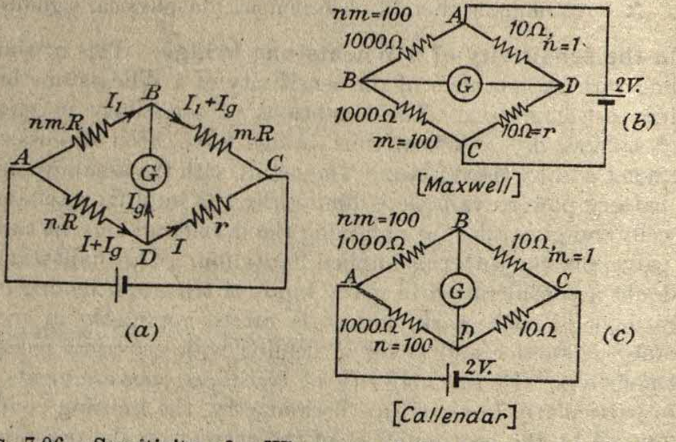

FIG. 7·06.—Sensitivity of a Wheatstone bridge network of resistors.

$$\left[\frac{\mathrm{I}_g}{\mathrm{I}}\right]_{\substack{n\to\infty \\ m\to 0}} = \frac{r - \mathrm{R}}{\mathrm{R}_g + \mathrm{R}}.$$

The advantage which is gained by making $n \longrightarrow \infty$ and $m \longrightarrow 0$

is not very great for

$$\left[\frac{I_g}{I}\right]_{n=m=1} = \frac{r - R}{2(R_g + R)},$$

or the sensitivity is still half as great as in the limiting case.

Maxwell has given a rule regarding the best arrangement of the resistances to form a Wheatstone bridge network. It is as follows :—***Of the two resistances, that of the battery and that of the galvanometer, connect the greater resistance to join the two greatest to the two least of the four other resistances.*** This rule is seldom applicable to accurate resistance measurements and when applied in platinum thermometry has often led to disastrous results. The following example is illustrative.

Suppose that it is required to measure the resistance of a platinum thermometer ($r = 10$ ohms), with a post-office box having ratio arms 10 and 1000 ohms. Suppose that the galvanometer has a resistance $R_g = 10$ ohms and the current is supplied by a 2-volt cell of negligible internal resistance. The connexions according to Maxwell are indicated in Fig. 7·06 (*b*). Comparing the two diagrams we find $n = 1$ and $m = 100$. Suppose that the balance is slightly disturbed by a change $\Delta r = (r - R)$. Then

$$\frac{I_g}{I} = \frac{\Delta r}{1030} \quad \text{and} \quad I \fallingdotseq 0{\cdot}1 \text{ amp.} \tag{i}$$

If, however, the battery and galvanometer connexions are interchanged as shown in Fig. 7·06 (*c*), $n = 100$ and $m = 1$. Then

$$\frac{I_g}{I} = \frac{\Delta r}{30}, \quad \text{and} \quad I = 0{\cdot}002 \text{ amp.} \tag{ii}$$

The actual value of the current through the galvanometer in (i) is greater than in (ii) in the ratio of 3 to 2. Maxwell's rule is so far justified. The current through the resistance to be measured, however, is fifty times greater in (i) than in (ii), and the heating effect of the current is 2500 times greater. With an ordinary platinum thermometer the first arrangement would give a rise in temperature of about 5 deg. C., which would be fatal for any purpose of measurement. The second arrangement, with nearly the same sensitivity, would give a rise of about 0·002 deg. C.; this can usually be measured and a correction applied.

The essential point to observe is that the ratio of the galvanometer current to that through the thermometer, viz. $\frac{I_g}{I}$, which is the true measure of the sensitivity in this instance, is 34 times greater in (ii) than in (i). Hence the second arrangement is by far the better where the heating effect of the current on the resistance to be measured is the primary consideration. Thus we have Callendar's rule: ***Connect the battery so as to make the resistance in 'series' with the thermometer greater than the resistance in 'parallel'.***

In the practical use of platinum thermometers we have the further restriction that n must be equal to unity, because it is necessary to compensate the changes of resistance of the leads by the equal changes of resistance of a pair of compensating leads on the opposite side of the galvanometer-contact on the bridge-wire. The loss in sensitivity, as compared with the case of n very large, does not amount to more than 30 per cent and is of no consequence as compared with the trouble and uncertainty involved in measuring the resistance of the leads separately at each observation.

It is now necessary to consider what is the best value for the resistance of the galvanometer which is to be used in the network. It will be assumed that when the coil is changed the size of it and the distribution of mass in it are not changed. Now for a given current, the deflexion is directly proportional to the number of turns in the coil when the size and position of the turns is not altered by changing the number. Thus when a single wire of cross-sectional area S is replaced by N turns, each of cross-section $\frac{S}{N}$, the deflexion for the same current is increased N times. If l is the length of wire in the single turn and χ the resistivity of its material, the resistance of the coil will have increased from $\frac{\chi l}{S}$ to $\frac{\chi(Nl)}{\left(\frac{S}{N}\right)}$, i.e. the resistance increases N^2 times, i.e. $R_g \propto N^2$. The deflexion for a given current is proportional to N, i.e. to $R_g^{\frac{1}{2}}$; it is also proportional to the current and hence to $I_g R_g^{\frac{1}{2}}$. But

$$I_g R_g^{\frac{1}{2}} = \frac{I(r-R)R_g^{\frac{1}{2}}}{\left(1+\frac{1}{n}\right)R_g + (1+m)R}.$$

Hence for any given values of I and Δr, where $\Delta r = r - R$, the galvanometer deflexion, θ, is proportional to

$$\frac{R_g^{\frac{1}{2}}}{\left(1+\frac{1}{n}\right)R_g + (1+m)R},$$

i.e.
$$\theta = \frac{kR_g^{\frac{1}{2}}}{\left(1+\frac{1}{n}\right)R_g + (1+m)R},$$

where k is a constant.

Now

$$\frac{d\theta}{dR_g}=\frac{\frac{1}{2}kR_g^{-\frac{1}{2}}}{\left(1+\frac{1}{n}\right)R_g+(1+m)R}+\frac{kR_g^{\frac{1}{2}}(-1)\left(1+\frac{1}{n}\right)}{\left[\left(1+\frac{1}{n}\right)R_g+(1+m)R\right]^2}.$$

For a maximum deflexion this is zero.

$$\therefore\ \frac{1}{2R_g^{\frac{1}{2}}}-\frac{\left(1+\frac{1}{n}\right)R_g^{\frac{1}{2}}}{\left(1+\frac{1}{n}\right)R_g+(1+m)R}=0.$$

$$\therefore\ R_g=\left[\frac{1+m}{1+\frac{1}{n}}\right]R,$$

i.e. R_g is equal to the effective resistance of the total resistance in ABD and the total resistance in BCD, when these are arranged in parallel and R is written for r.

This result agrees with that given by Maxwell and shows that in the majority of instances which occur in practice, R_g should be of the same order of magnitude as R, being restricted to the limits 0·5R and 2R if n is never less than one or m never greater than one.

The Carey Foster bridge for measuring R–S where R and S are two nearly equal resistances.—The difference between two resistances which are nearly equal may be determined with tolerable

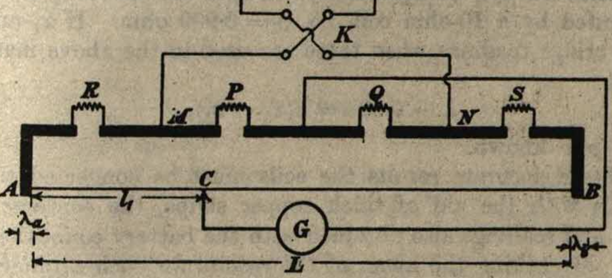

FIG. 7·07.—The Carey Foster bridge.

accuracy by a form of the metre bridge due to CAREY FOSTER. A uniform wire AB, Fig. 7·07, is stretched across a scale in cm., etc., and the two resistances R and S are placed in the outer gaps of a copper bar whose extremities are joined to AB. P and Q are two resistances placed in the inner gaps of this bar. These should be nearly equal, but it is not necessary to know their actual values. A cell, reversing key, K, and galvanometer are connected as indicated. If the difference in resistance between R and S is not large a point C on the wire AB may be found where the galvanometer deflexion is zero. Let AC $= l_1$, AB $=$ L, the total length of the wire, while λ_a and λ_b are the end-corrections expressed as lengths of bridge wire. Then P and Q may be regarded as two arms of a Wheatstone bridge network, while R and S and the resistances of the circuit between them and C constitute the other two arms. When the bridge is balanced we have

$$\frac{P}{Q} = \frac{R + \rho(\lambda_a + l_1)}{S + \rho(\lambda_b + L - l_1)},$$

where ρ is the resistance per cm. of the wire. Adding unity to each side of this equation, we get

$$\frac{P + Q}{Q} = \frac{R + S + \rho(\lambda_a + \lambda_b + L)}{S + \rho(\lambda_b + L - l_1)}.$$

The coils R and S are now interchanged and a new balance point on the bridge wire is determined. Let this be at distance l_2 from A. Proceeding as before we obtain an equation in which only the right-hand denominator is affected; its new value must be equal to the old, i.e.

$$R + \rho(\lambda_b + L - l_2) = S + \rho(\lambda_b + L - l_1).$$

$$\therefore\ R - S = \rho(l_2 - l_1).$$

This equation expresses the difference between R and S in terms of the resistance per unit length of the bridge wire and we note that the end corrections λ_a and λ_b do not appear in it.

To determine ρ, R is replaced by a 1-ohm coil and S by a 1-ohm coil shunted by a 10-ohm coil, i.e. S $= 0{\cdot}909$ ohm. If x_1 and x_2 are the bridge readings when these are used in the above manner, we have

$$1 - 0{\cdot}909 = \rho(x_2 - x_1),$$

so that ρ is known.

To obtain accurate results the coils must be connected to the terminals with the aid of thick copper strips, the contacts well cleaned and readings also obtained with the battery connexions reversed. By taking the mean of the results for each arrangement errors due to parasitic e.m.fs. (thermoelectric, etc.) will be eliminated.

Example.—In order to find the small difference between a standard resistance S of 1·000105 ohms and a nominal ohm X, the resistances are connected in a Wheatstone network with other nominal ohm coils P and Q forming the ratio arms so that PX equals QS when the bridge is balanced. To obtain a balance it is necessary to shunt P with

10,600 ohms; upon removing this shunt and interchanging X and S, Q must be shunted with 20,200 ohms to restore the balance. Calculate a value for X.

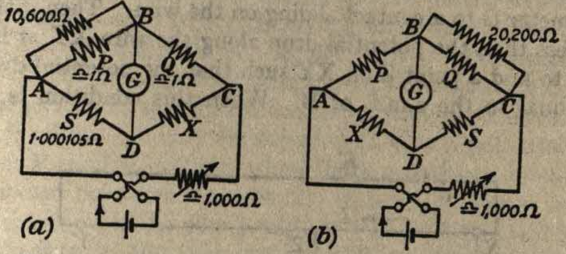

FIG. 7·08.—Determination of the small difference in resistance for two standard resistors. [The bridge actually measures the ratio of their resistances.]

The two arrangements are shown in Fig. 7·08 (*a*) and (*b*). When the bridges are balanced, we have

$$\begin{aligned}\frac{1 + 1{\cdot}05 \times 10^{-4}}{\mathrm{X}} &= \frac{\left(\dfrac{\mathrm{P} \times 1{\cdot}06 \times 10^4}{\mathrm{P} + 1{\cdot}06 \times 10^4}\right)}{\mathrm{Q}}\\ &= \frac{\mathrm{P}}{\mathrm{Q}}\cdot\frac{1{\cdot}06 \times 10^4}{\mathrm{P} + 1{\cdot}06 \times 10^4}\\ &= \frac{\mathrm{P}}{\mathrm{Q}}\cdot\frac{1{\cdot}06 \times 10^4}{1 + 1{\cdot}06 \times 10^4} \qquad [\because\ \mathrm{P} \simeq 1]\\ &\ \frac{\mathrm{P}}{\mathrm{Q}}\cdot\frac{1}{1 + \dfrac{1}{1{\cdot}06 \times 10^4}},\end{aligned}$$

and, similarly,

$$\begin{aligned}\frac{\mathrm{X}}{1 + 1{\cdot}05 \times 10^{-4}} &= \frac{\mathrm{P}}{\left(\dfrac{\mathrm{Q} \times 2{\cdot}02 \times 10^4}{\mathrm{Q} + 2{\cdot}02 \times 10^4}\right)}\\ &= \frac{\mathrm{P}}{\mathrm{Q}}\left(1 + \frac{1}{2{\cdot}02 \times 10^4}\right). \qquad [\because\ \mathrm{Q} \simeq 1]\end{aligned}$$

Eliminating $\frac{\mathrm{P}}{\mathrm{Q}}$ from these equations, we find

$$\left(\frac{1 + 1{\cdot}05 \times 10^{-4}}{\mathrm{X}}\right)^2 = \frac{1}{\left(1 + \dfrac{1}{1{\cdot}06 \times 10^4}\right)\left(1 + \dfrac{1}{2{\cdot}02 \times 10^4}\right)}.$$

$$\begin{aligned}\therefore \mathrm{X} &= (1 + 1{\cdot}05 \times 10^{-4})\sqrt{\left(1 + \frac{1}{1{\cdot}06 \times 10^4}\right)\left(1 + \frac{1}{2{\cdot}02 \times 10^4}\right)}\\ &= (1 + 1{\cdot}05 \times 10^{-4})\left\{1 + \frac{1}{2}\left(\frac{1}{1{\cdot}06} + \frac{1}{2{\cdot}02}\right)10^{-4}\right\}\\ &= 1 + (1{\cdot}05 + 0{\cdot}047 + 0{\cdot}025)10^{-4}\\ &= 1{\cdot}000112 \text{ ohm.}\end{aligned}$$

The principle of a potentiometer.—Suppose that XY, Fig. 7·09 (*a*), is a wire connected to a cell A so that there is a fall in potential from X to Y along the wire. Let B be another cell with its positive pole connected to X and its negative pole through a galvanometer G to a contact sliding on the wire. Then if the e.m.f. of B is less than the potential drop along the wire XY, it is always possible to find a point Z on XY such that the potential drop along XZ is equal to the e.m.f. of B. When this condition is satisfied there is no current flowing through the galvanometer and this, in turn, implies that the current in XZ is equal to the current in ZY.

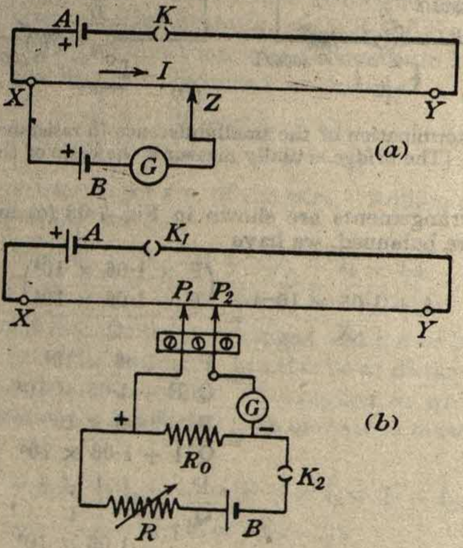

FIG. 7·09.—The principle of a potentiometer [for use with z.f. currents].

We therefore conclude that when no current flows through the galvanometer every unit length of the potentiometer wire has the same current flowing through it and has the same potential difference between its ends, provided that

(*a*) the wire is uniform,

(*b*) its properties are not affected by altering its electrical potential.

If these conditions are satisfied the potential difference between any two points in the potentiometer wire is directly proportional to the length of wire between them.

In the case of heat flow along a uniform and perfectly lagged bar the heat flow depends on the temperature gradient and on the thermal conductivity which, in general, is not a constant.

When the thermal conductivity is a function of temperature the temperature gradient, even in a perfectly lagged bar, will not be constant. In the electrical case it is easy to test whether there is any analogous effect and at the same time detect any lack of uniformity in the wire by the following experiment. P_1 and P_2 are two probes—cf. Fig. 7·09 (*b*)—rigidly fixed at a convenient distance apart. A cell B supplies current to two resistors R_0 and R in series and the probes P_1 and P_2 are connected through a galvanometer to the ends of R_0. R may be adjusted to give a potential difference between P_1 and P_2 so that when these probes are in contact with two selected points on the wire XY there is no current through the galvanometer. If the wire is uniform this condition will be satisfied for all points on the wire separated by a distance equal to that between the probes.

Now it must be emphasized that in the potentiometer there is only one constant current and so any question as to the validity of Ohm's law for metallic conductors does not arise. Thus a potentiometer, in preference to a quadrant electrometer, may be used to test the validity of Ohm's law, since it is so much more sensitive; a tangent galvanometer must be used to measure the current since other current measuring instruments are based on the assumption that Ohm's law for metallic conductors is true.

The potentiometer in practice.—In actual laboratory practice it is advisable to insert an adjustable resistance R in series with the potentiometer wire AB, Fig. 7·10 (*a*). In addition, a resistance of about 1000 ohms should be placed in series with the galvanometer in order to prevent large currents being taken from the cell under investigation. This is very desirable if the e.m.f. of a cell does not recover when a large current has been taken from it. Let $AP_1 = l_1$ where P_1 is such that the p.d. across AP_1 is equal to the e.m.f. of E_1. Then $E_1 = \kappa l_1$, as before. When E_1 is replaced by a second cell E_2, we shall have $E_2 = \kappa l_2$. Now it may happen that l_1 and l_2 are relatively short in comparison with AB. The experiment is repeated, R being adjusted so that if E_2 is the cell with the larger e.m.f. the value of l_2 is greater than about two-thirds the length of the potentiometer wire. Then the ratio $l_1 : l_2$ may be determined with greater accuracy.

In more advanced work it is always essential to maintain constant the current in the potentiometer wire and to test the constancy at frequent intervals. To do this an additional resistor R_0 is added to the main circuit, cf. Fig. 7·10 (*b*), and the value of its resistance is such that the p.d. across it is about one volt. $[SC]_1$ is a standard cell arranged as shown in series with a 10,000-ohm. resistor and the galvanometer G. The resistance R is varied until the p.d. across R_0 is exactly equal to the e.m.f. of the standard cell.

The e.m.fs. of the cells E_1 and E_2 are then compared in terms of the distances AP_1 and AP_2 on the potentiometer wire. Immediately before each balance is obtained the current in the potentiometer circuit is adjusted to its fixed value.

In work of the highest precision at least two standard cells are used. The 10,000-ohm coil in series with the first standard cell prevents large currents being taken from the cell but even small currents do affect the e.m.f. to a few parts in a million. The final test of the constancy of the current in the potentiometer wire is made by using the second standard cell and this is never in action until the bridge has been balanced almost completely.

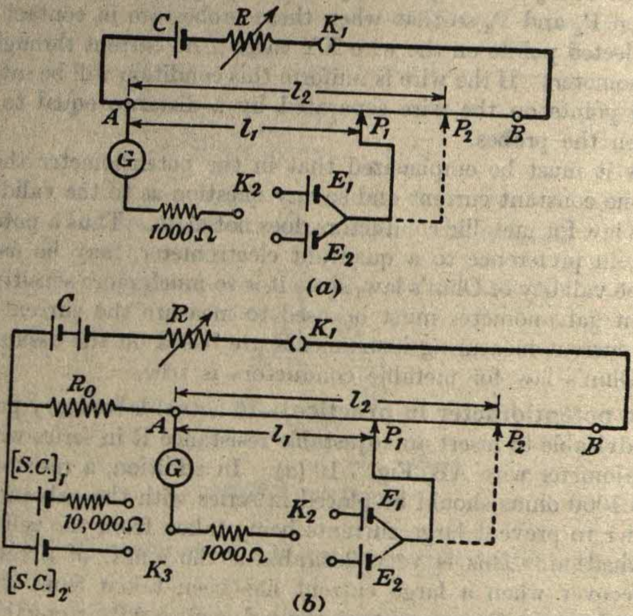

FIG. 7·10.—A simple potentiometer in practice.

Of course a simple potentiometer is not used in precision work but the general method of working is as described above. Similar features may be added to the experimental arrangements which follow later.

Hitherto it has been assumed that the potentiometer is to be used to compare e.m.fs. Such comparisons are seldom required in physics where the main use of a potentiometer is in the accurate measurement of steady currents and the comparison of resistances of the same order of magnitude.

To measure a current by means of a potentiometer.—The potentiometer is easily employed when one wishes to measure a current accurately. For this purpose the connexions are arranged as in Fig. 7·11. The wire AB, the source of constant current C, and key K_1 are as before. Suppose it is desired to measure the current in the circuit EPR, where E is a cell sending a current through a resistance P and a standard resistance coil R when the key K_2 is closed. The potential terminals, cf. p. 178, of the coil R are joined to A and to D through the galvanometer G. When there is no current through G the voltage across R is proportional to the length AD. But the voltage (V) across a coil of resistance R is given by $V = IR$, where I is the current. Hence, if $AD = l$, $IR = V = \kappa l$, where κ is a constant; it is the drop in potential per unit distance along AB. In order to obtain the value of I it is necessary to know κ. For this purpose a standard cell S is connected to a high resistance (10,000 Ω at least), a galvanometer and jockey and arranged as shown. When the deflexion of the galvanometer is zero, i.e. the jockey is at H, say, the potential difference across AH is equal to the e.m.f. of S [1·0186 volt., if S is a Weston cell]. κ is therefore known and I may be calculated. The resistance R must be such that IR is nearly one volt; then AD and AH are nearly equal.

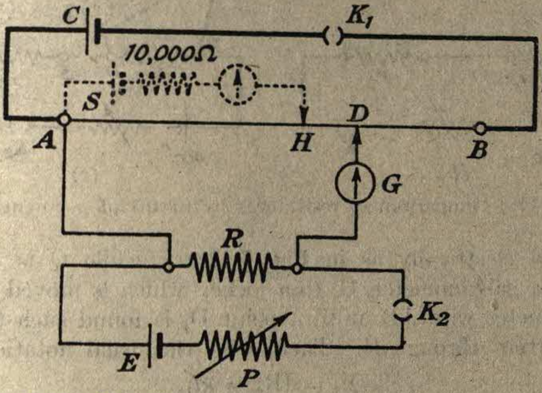

FIG. 7·11.—Use of a potentiometer to measure a current.

Comparison of resistances by means of a potentiometer.—Since the potential difference across a resistance is proportional to the product of its resistance and the current through it, the ratio of two resistances will be equal to the ratio of the potential differences across them when they are each carrying the same current. Hence, by comparing these potential differences we have a means of comparing two resistances.

To carry out this comparison of potential differences a potentiometer is set up as in Fig. 7·12 (*a*) and also a second circuit comprising the resistances R_1 and R_2 to be compared, an accumulator, adjustable resistance and key K_2, as in Fig. 7·12 (*b*). When K_2 is closed the potential differences across R_1 and R_2 will be IR_1 and IR_2 respectively, where I is the current. The points A and P are now connected electrically as in Fig. 7·12 (*c*), while Q is connected through a galvanometer, G, to a jockey which is moved along the potentiometer wire AB, until a point D_1 is found such that there is no current through G. Then, with the usual notation,

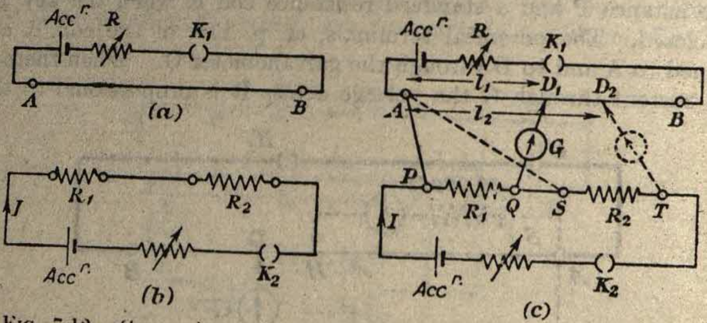

FIG. 7·12.—Comparison of resistances by means of a potentiometer.

$$V_1 = IR_1 = \kappa l_1.$$

The potential difference across R_2 is found by making the connexions shown by the dotted lines. Then

$$V_2 = IR_2 = \kappa l_2,$$

so that

$$\frac{R_1}{R_2} = \frac{l_1}{l_2}.$$

If l_1 and l_2 are small the resistance R must be increased to diminish the potential drop per unit length of the potentiometer wire and the experiment repeated. The method works equally well whether the resistances are large [10^5 ohm.] or small [10^{-2} ohm.] provided that a suitable galvanometer is selected and the current adjusted accordingly.

Resistors with potential leads.—Fig. 7·13 shows diagrammatically a resistor provided with potential leads P_1 and P_2 in addition to the usual current leads C_1 and C_2. The current enters at C_1 and leaves at C_2 and by means of a potentiometer the potential difference between the sections A and B is measured. Since no current flows at balance through the potential leads their resistance is of no consequence but the wire forming the main part of the resistor as well as the potential leads and the current terminals should be hard soldered to a thick piece of brass or copper in order that the lines of current flow shall be invariable, i.e. the resistor will have a fixed resistance at a fixed temperature. Such resistors are essential in all work of precision.

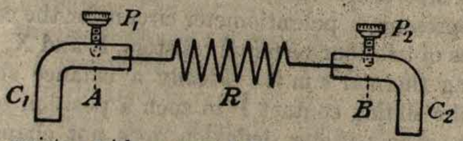

FIG. 7·13.—A resistor with potential leads [or a four-terminal resistor].

The elimination of parasitic e.m.fs. in z.f. bridges and potentiometer circuits.—When measuring or comparing very small potential differences, or using them to detect want of balance in a network, thermal e.m.fs. may be a serious source of trouble. They arise from temperature differences caused by self-heating and non-uniformity in the ambient conditions, and even the operation of dial switches can introduce thermal e.m.fs. of several microvolts. These parasitic e.m.fs. may be reduced if the entire circuit is as free as possible from junctions of dissimilar metals * with large thermoelectric

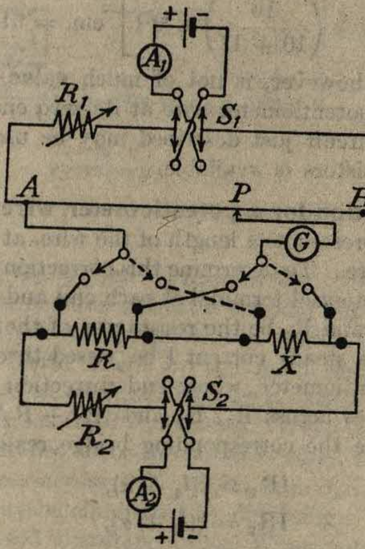

FIG. 7·14.—The elimination of parasitic e.m.fs. in z.f. bridges and potentiometer circuits.

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* Nickel-plated terminals and nickel-silver components should not be used since each material has a large thermoelectric power when in contact with copper or brass.

powers, cf. p. 271, and shielded from draughts and other causes of sporadic temperature variations.

Fig. 7·14 represents a potentiometer circuit for the comparison of the resistances of two four-terminal resistors, R and X, and a means for eliminating thermal e.m.fs. Ideally a balance should be obtained with the sliding contact P in such a position on the bridge wire that the galvanometer deflexion does not change when the switches S_1 and S_2 are reversed simultaneously. Such a condition is seldom, if ever, realized in actual experimental work but the true balance point may be obtained by proceeding as follows. Suppose that when P is on the 61·5 cm. division of the potentiometer wire the galvanometer deflexion changes by 10 divisions to the right when the currents are reversed; when P is on the 61·6 cm. mark let the corresponding change be 17 divisions to the left. The true balance point is such that

$$\text{AP} = \left[61{\cdot}5 + \left(\frac{10}{10+17}\right) \times 0{\cdot}1\right] \text{ cm.} = 61{\cdot}54 \text{ cm.}$$

Such a reading, however, is not of much value unless the end correction for the potentiometer wire at its zero end is small. To measure this the circuit just described may be used if a suitable combination of resistors is available.

The end correction for a potentiometer wire.—Let α be the end correction, expressed as a length of the wire, at the zero end of a potentiometer wire. To determine this correction let a resistor be provided with a potential terminal at each end and another near to its centre; let R_1 and R_2 be the resistances of the two portions of the resistor. Let a steady current I be passed through them. By means of the potentiometer, whose end correction is required, the potential differences across R_1, R_2 and $(R_1 + R_2)$ are compared. If l_1, l_2 and l_3 are the corresponding bridge readings, we have

$$IR_1 = k(l_1 + \alpha),$$

$$IR_2 = k(l_2 + \alpha),$$

and

$$I(R_1 + R_2) = k(l_3 + \alpha),$$

where k is a constant. Hence

$$\alpha = l_3 - (l_1 + l_2).$$

A vernier type potentiometer.—The principle of a vernier potentiometer is indicated in Fig. 7·15 (*a*); in an actual instrument dial resistors are used but here they are represented in the conventional manner. The instrument consists essentially of the main dial R_1, which has 20 coils of 10 Ω resistance each; across any two adjacent coils of R_1 is inserted a vernier dial R_2, formed from 100 coils each of 0·2 Ω resistance. R_3 is a set of 100 coils each of 0·00100 Ω

in series with R_1; this replaces the slide wire of the most simple type of potentiometer. The current through the potentiometer is made 10mA; this is effected by using a standard cell and standard resistor S with a resistance 101·86 Ω. [If S = 1018·6 Ω then the current may be made 1mA in order to alter the range of potential differences which may be measured.] Thus the main dial reads 0·1V. per step, the vernier dial 0·001V. per step, while the fine adjustment dial reads 10^{-5} V. or 10μV. per step. The zero positions on the

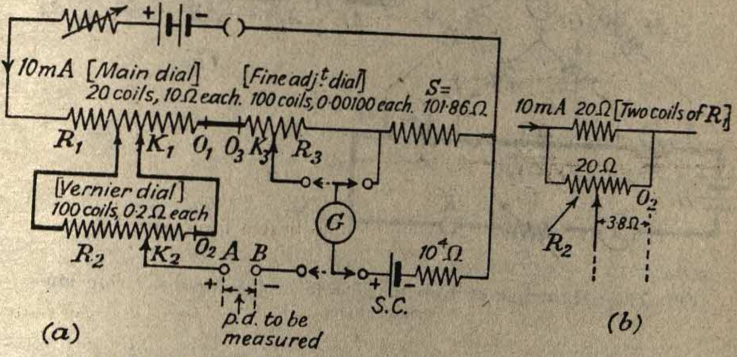

FIG. 7·15.—The principle of a vernier type potentiometer.

three dials are at O_1, O_2 and O_3 respectively. To discover what a reading on the vernier dial indicates consider Fig. 7·15 (*b*) in which two adjacent coils from R_1 are shown shunted by R_2, of total resistance 20 Ω. With 10mA in the main circuit, the current through R_2 is 5mA, so that each step on R_2 has a potential drop of $0{\cdot}2 \times 5 \times 10^{-3}$ V., i.e. 0·001V., across it. When balance is obtained the potential difference to be measured is equal to the sum of the potential differences across O_1K_1, O_2K_2 and O_3K_3; the effects of thermal e.m.f.s may be eliminated by obtaining a balance when all currents are reversed simultaneously. When potential differences of a few microvolts are to be measured the above eliminating process is not adequate and a Diesselhorst potentiometer must be used. Its significant characteristic is that all variable contacts are excluded from the galvanometer circuit but its structure is too complicated for consideration here.

The Kelvin bridge for the measurement of low resistances.—The simple Wheatstone bridge is not suitable for the measurement of resistances below 0·1 ohm because of the errors due to the network junctions which are effectively included in the bridge arms. In the Kelvin double bridge, which was invented as early as 1862, these errors are eliminated. The essentials of the bridge are shown in Fig. 7·16 (*a*). X is the unknown resistance, S a standard resistance each preferably having potentials leads. These resistances, cf. Fig. 7·16 (*b*), are joined together by a metal bar with a low resistance, ΔR (say). Four known resistances M, N, *m* and *n* are then arranged together with a low resistance galvanometer G to form

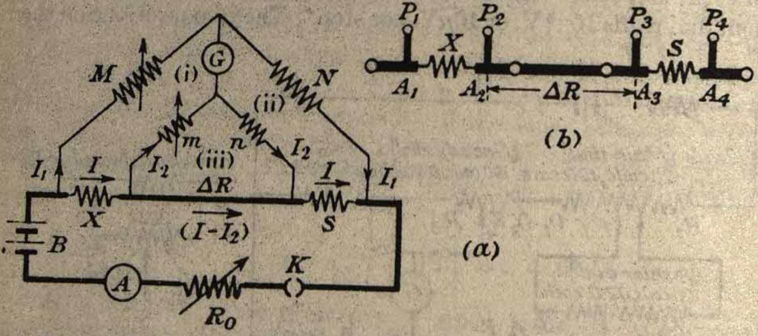

FIG. 7·16.—Principle of Kelvin's double bridge for measuring low resistances.

the bridge. Balance for the bridge is attained when, on reversing the current through it, there is no change in the galvanometer deflexion, a procedure which, as we have seen previously, eliminates the effects of parasitic e.m.fs. and doubles the sensitivity.

Let us assume that the bridge is balanced and that the currents through the various resistors are as indicated. Applying Kirchhoff's second law to the loops (i), (ii) and (iii), we find

$$MI_1 - mI_2 - XI = 0 \qquad . \quad . \quad (i)$$

$$NI_1 - SI - nI_2 = 0 \qquad . \quad . \quad (ii)$$

and

$$mI_2 + nI_2 - \Delta R(I - I_2) = 0 \qquad . \quad . \quad (iii)$$

From (iii) we have

$$I = \frac{I_2(m + n + \Delta R)}{\Delta R} \qquad . \quad . \quad . \quad (iv)$$

and from (i) and (ii),

$$\frac{MI_1}{NI_1} = \frac{mI_2 + XI}{nI_2 + SI} \qquad . \quad . \quad . \quad (v)$$

$$\therefore \frac{M}{N} = \left(X + mA\right) \div S\left[1 + \frac{n}{S}A\right], \qquad \left[\text{where } A = \frac{I_2}{I}.\right]$$

$$\therefore \frac{M}{N}S\left[1 + \frac{n}{S}A\right] - mA = X.$$

$$\therefore\ X = \frac{M}{N}S + \left[\frac{M}{N} - \frac{m}{n}\right]n.\frac{\Delta R}{m + n + \Delta R},$$

$$\left[\because\ A = \frac{I_2}{I} = \frac{\Delta R}{m + n + \Delta R}.\right]$$

This equation shows that if $\left(\frac{M}{N} - \frac{m}{n}\right) = 0$, then $X = \frac{M}{N}S$, i.e. the effect of the resistance ΔR is entirely eliminated.

If the resistances X and S are provided with potential leads as shown in Fig. 7·16 (*b*) it is at once clear that since the potential leads A_1P_1, etc., are in series with much larger resistances M, etc., their effect on the final equation is of no consequence. Also, since the current leads at A_2 and A_3 are in series with the connecting rod then ΔR is the total resistance between A_2 and A_3, and this vanishes from the final condition of balance if $\frac{M}{N} = \frac{m}{n}$.

In practice the current to the bridge must be supplied through a reversing key in order that, as explained on p. 180, small parasitic e.m.fs. in the bridge may be eliminated.

The Kelvin double bridge—alternative theory.—Fig. 7·17 shows a more symmetrical form of this bridge circuit; when it is balanced

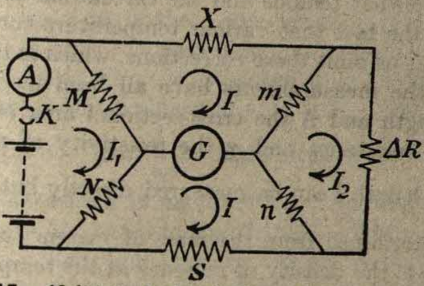

FIG. 7·17.—Kelvin's double bridge (alternative theory).

let the cyclic currents in the various loops be as shown. Applying Kirchhoff's second law, we have

$$XI + m(I - I_2) + M(I - I_1) = 0, \quad . \quad . \quad \text{(i)}$$

$$\Delta R.I_2 + n(I_2 - I) + m(I_2 - I) = 0, \quad . \quad . \quad \text{(ii)}$$

and

$$SI + N(I - I_1) + n(I - I_2) = 0. \quad . \quad . \quad \text{(iii)}$$

These equations may be re-arranged to give

$$(X + m + M)I = MI_1 + mI_2, \quad . \quad . \quad \text{(iv)}$$

$$(\Delta R + m + n)I_2 = (m + n)I, \quad . \quad . \quad . \quad \text{(v)}$$

and

$$(S + n + N)I = NI_1 + nI_2. \quad . \quad . \quad . \quad \text{(vi)}$$

Eliminating I_1 from equations (iv) and (vi), we have

$$\frac{M}{N} = \frac{(X + m + M)I - mI_2}{(S + n + N)I - nI_2}$$
$$= \frac{(X + m + M) - mA}{(S + n + N) - nA},$$

where $A = \frac{I_2}{I} = \frac{m + n}{\Delta R + m + n}$ by using (ii). Cross-multiplying and reducing, we find

$$X = \frac{M}{N}S + \frac{nM - mN}{N}(1 - A)$$
$$= \frac{M}{N}S,$$

if $\frac{M}{N} = \frac{m}{n}$. When this condition is satisfied, the term A and hence ΔR disappears and X may be calculated from the known values of M, N and S.

The resistivity of mercury.—In 1882 Rayleigh and Mrs. Sidgwick carried out a determination of the resistivity of mercury at 0° C. The final object of these experiments was to express in terms of the B.A. unit of resistance the resistances of known columns of mercury at 0° C. The comparison between a mercury resistance and a standard coil was made with the aid of a Carey Foster bridge. Although the determination is simple enough in principle, the execution is somewhat tedious and the calculation of the results is complicated by the fact that various temperature corrections have to be made. Let us omit these corrections, which is tantamount to assuming that the measurements have all been made at 0° C.

If l is the length and A the cross-sectional area of a column of mercury, R its resistance and χ the resistivity of mercury, then $R = \chi\frac{l}{A}$. The length l can be measured directly but a value for A can only be determined from the mass of mercury required to fill the tube. If ρ is the density of mercury at the temperature of the experiment, $m = \rho A l$, or

$$\chi = \frac{Rm}{\rho l^2}.$$

Apart from the temperature corrections already referred to, the simplicity of this formula is disturbed by the inevitable departure from the truly cylindrical form of the glass tubes used to contain the mercury. In general, A must be regarded as a function of the position, x, along the tube at which it is taken. Then

$$R = \chi \int_0^l \frac{dx}{A}.$$

To evaluate this integral, let the tube be divided into n sections of lengths $(\Delta l)_1$, $(\Delta l)_2$, etc., as indicated in Fig. 7·18 (*a*), and let r_k be the mean radius of cross-section for the section whose length is $(\Delta l)_k$. Then since

$$(\Delta \mathrm{R})_k = \frac{\chi(\Delta l)_k}{\pi r_k{}^2},$$

$$\mathrm{R} = \sum_{k=1}^{n} (\Delta \mathrm{R})_k = \frac{\chi}{\pi} \sum_{k=1}^{n} \frac{(\Delta l)_k}{r_k{}^2},$$

and this is practically equal to the value of the integral required.

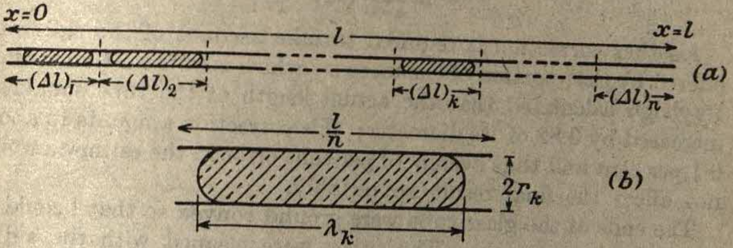

FIG. 7·18.—Calibration of a capillary tube.

In order to effect this summation let the length of the tube be divided into n *equal* parts the actual length of each section being about 5 cm. Let m be the mass of mercury required to fill the tube completely. A value for m is estimated from the mass and length of a mercury pellet which almost fills the tube. Then let a mercury pellet of length less than $\frac{l}{n}$ be introduced into the tube and its overall length measured in each section. For the kth section let λ_k be the distance indicated in Fig. 7·18 (*b*). Then λ_k is the overall length of the pellet, and $\pi\rho r_k{}^2\lambda_k$, where ρ is the density of mercury, is very nearly the mass of the mercury pellet. We may therefore write

$$\pi\rho r_k{}^2\lambda_k = \mathrm{C}_k,$$

and the question arises as to whether or not C_k varies according to the section of the tube which the pellet occupies. Strictly speaking, it does vary because the radius of the curved portion at each end of the pellet varies with the position of the pellet in the tube. This variation is small if the tube is approximately uniform in diameter, and so it is only for tubes satisfying this condition that this method of calibration is applicable. In such an instance we may write

$C_k = C$, and then we have

$$m = \pi\rho\frac{l}{n}\sum_{k=1}^{n}\frac{C}{\pi\rho\lambda_k} = \frac{l}{n}C\sum_{k=1}^{n}\left(\frac{1}{\lambda_k}\right).$$

Since $(\Delta l)_k$ is the same for all values of k, having been made equal to $\frac{l}{n}$, the expression for the resistance R becomes

$$R = \frac{\chi}{\pi}\frac{l}{n}\sum_{k=1}^{n}\left(\frac{1}{r_k^2}\right) = \frac{\chi}{\pi}\frac{l}{n}\sum_{k=1}^{n}\frac{\pi\rho\lambda_k}{C}$$
$$= \chi\frac{l^2}{n^2}\frac{\rho}{m}\sum_{k=1}^{n}\left(\frac{1}{\lambda_k}\right)\cdot\sum_{k=1}^{n}(\lambda_k).$$

Another correction is required to take account of the resistance offered by that portion of the mercury which is in the terminal cups. Rayleigh calculated that the actual length of the tube should be increased by 0·82 of its diameter; this correction amounts to about 0·1 per cent and thus even a 10 per cent error in the estimate would not affect the final result appreciably.

The ends of the glass tube were ground convex so that l could be determined accurately. The tubes were cleaned with the aid of chemical reagents and dried with air passed over calcium chloride.

The terminal cups were L-shaped pieces of vulcanite and the glass tube was held in position with a short piece of thick rubber tubing, used as a bung. Copper rods, well amalgamated, were used to connect the mercury resistance to the bridge.

These investigators found the ohm (10^9 e.m.u.) to be the resistance of a column of mercury at 0° C., one square millimetre in section, and 106·24 cm. long.

Laboratory determination of the resistivity of mercury.—In order to make use of the method outlined above for calibrating a capillary tube, the resistivity of mercury (at room temperature or at 0° C.) may be found with the aid of the apparatus shown in Fig. 7·19. The glass capillary tube is 50 cm. long and about 1 mm. in diameter. The ends of the capillary tube are slightly rounded so that its length may be quite definite. The capillary tube is fitted into glass tubes of the size and shape indicated in the diagram. C_1 and C_2 are the current leads; P_1 and P_2 are the potential leads. The potential probes consist of pyrex glass tubes into which short lengths of tungsten wire have been sealed. These tubes are partly filled with mercury and copper leads are used to make the necessary connexions to the potentiometer used to measure the resistance of the mercury column in terms of a standard resistance. The current leads are made of iron, or nickel, so that no amalgam is produced.

If the tube is kept at the temperature of melting ice a correction will have to be applied to the value of $l^2 \sum\left(\frac{1}{\lambda_k}\right)\cdot\sum(\lambda_k)$ determined experimentally, since in that determination the temperature will not have been 0° C.

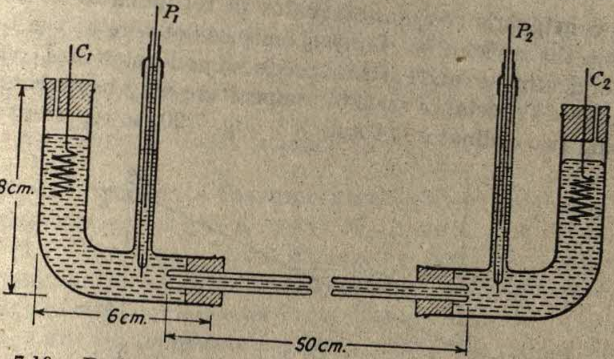

FIG. 7·19.—Resistivity of mercury at room temperature (or 0° C.).

The variation of resistance with temperature.—By measuring the resistance of a wire at different temperatures it has been found that, in general, the resistance increases as the temperature is raised. For most pure metals the curve showing the variation of resistance with temperature measured on the gas scale is almost a perfect parabola, and for platinum CALLENDAR has shown that the variation of resistance with temperature is very accurately represented by the equation

$$R = R_0(1 + \alpha\theta + \beta\theta^2),$$

where θ is the temperature on a centigrade scale with air as the thermometric substance, R being the resistance at θ° C., and R_0 that at 0° C. To determine the constants α and β it is necessary to measure the resistance of a coil of platinum wire at three well-known temperatures, e.g. those of melting ice, steam at atmospheric pressure, and sulphur vapour. These are 0° C., 100° C. and 444·60° C. respectively, if in the last two instances the barometer reads 76 cm. of mercury under standard conditions, and errors not exceeding 0·001 deg. C. are tolerated. The last temperature has been determined accurately with the aid of a compensated gas thermometer.

The above equation may be written

$$\left(\frac{R - R_0}{R_0}\right)\frac{1}{\theta} = \alpha + \beta\theta.$$

Hence α and β may be calculated when the values of $\left(\frac{R - R_0}{R_0}\right)\frac{1}{\theta}$

are known at two different temperatures, 0° C. being excluded because the expression then becomes indeterminate; of course a graphical method could be used by making observations on R at several known temperatures.

On centigrade resistance scales of temperature.—If R_0 and R_{100} are the resistances of any given piece of wire at the temperatures of melting ice and of steam produced under standard conditions, a centigrade resistance scale of temperature may be constructed by drawing two ordinates OA and NB, Fig. 7·20, to represent to scale

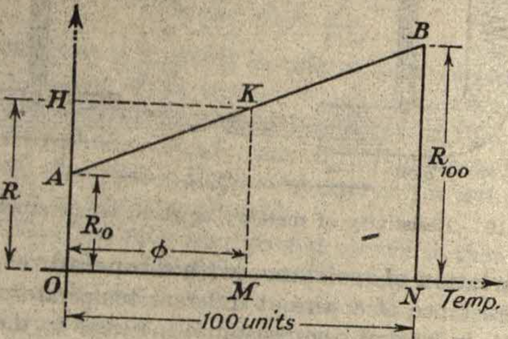

FIG. 7·20.—A centigrade resistance scale of temperature.

R_0 and R_{100}, where the distance ON is 100 arbitrary units. To find the temperature ϕ, on this scale, corresponding to a resistance R, OH is constructed to represent R, HK is drawn parallel to ON to intersect AB in K, and KM is drawn normal to ON to intersect this line in M. Then OM is ϕ units long, so that ϕ becomes known.

For temperatures measured on this scale of temperature, the relationship between R and R_0 is a linear one and may be written

$$R = R_0 + \frac{\phi}{100}(R_{100} - R_0) = R_0\left[1 + \frac{\phi}{100}\cdot\frac{R_{100} - R_0}{R_0}\right]$$
$$= R_0(1 + \kappa\phi),$$

where κ is a constant, known as *the resistance-temperature coefficient as measured on the specified scale of temperature.* Now it so happens that for pure metals κ is very nearly equal to α in the expression $R = R_0(1 + \alpha\theta + \beta\theta^2)$, which means of course that β is a small quantity; it is also found that $\kappa \fallingdotseq \alpha \fallingdotseq \frac{1}{273}$ deg.$^{-1}$ C. For alloys, κ and α, while still remaining nearly equal for any given alloy, vary considerably from one alloy to another and in the case of constantan and manganin tend to zero. It is for this reason that laboratory and standard resistance coils are constructed from these alloys.

To determine κ for iron, or nickel, a length of wire is wrapped on a wooden or mica frame and placed in a test-tube. Thick copper leads enable the coil to be connected to a post-office box. The coil is placed in turn in melting ice and in steam and the resistance determined in each instance. The steam temperature ϕ_s is deduced from the barometric height and κ calculated from the equation

$$\kappa = \frac{R_{\phi_s} - R_0}{R_0\phi_s}.$$

No thermometer is used in this experiment but a calibrated thermometer would be necessary if we wished to make a series of observations of R_θ at different temperatures θ and thereby show that $\frac{R_\theta}{R_0}$ is very nearly a linear function of θ.

Platinum resistance thermometers.—The variation of resistance with temperature as a means of measuring temperature was first used by SIEMENS. His thermometer consisted of a platinum wire wound on a clay cylinder and mounted in an iron tube. The resistance of this thermometer in ice was not constant after it had been used at high temperatures, for the clay attacked the wire and gases from the furnace passed through the iron causing the wire to become brittle. The physicists of his day therefore regarded the method as unpromising and it was not until about 1887, when CALLENDAR wound the wire without straining it on a mica frame, and mounted the whole in a glass tube, that this method of thermometry was developed. To-day it is one of the most reliable means of measuring temperatures from − 40° C. to 1200° C.

A typical platinum resistance thermometer is indicated in Fig. 7·21 (*a*). The fine platinum wire is wound on a mica frame. This wire is joined by intermediate short lengths of thicker platinum wire to thick copper, silver or platinum leads. To compensate for the fact that the leads are at temperatures different from that of the platinum spiral a pair of compensating leads is used. These are identical with the other leads and are joined to a short length of the same fine platinum wire through intermediate pieces of thicker platinum wire. The leads are held in position by an ebonite cap and mica washers and crosses; these also serve to insulate the leads from one another. The difference in resistance between each pair of leads and the wires connecting them depends only on the temperature of the platinum spiral; hence there is no troublesome and uncertain correction for stem exposure when such a thermometer is used. The above difference in resistance is measured on a CALLENDAR-GRIFFITHS bridge, the connexions for which are shown in Fig. 7·21 (*b*). A resistor R_0, about 50 ohms, is placed in each arm AH and HB, i.e. the ratio arms are equal. MN is a manganin wire corresponding to that in the Carey Foster bridge; in using a

Callendar-Griffiths bridge, however, the resistance to be measured is never interchanged with the standard. The thermometer R is placed with the thermometer leads connected to the gap $P_1'P_2'$ and its compensating leads to the gap $C_1'C_2'$; these and a series of coils S of known resistance constitute the fourth arm of the bridge. A 50-ohm resistance coil is placed in series with the battery to prevent a large current from being sent through the thermometer.

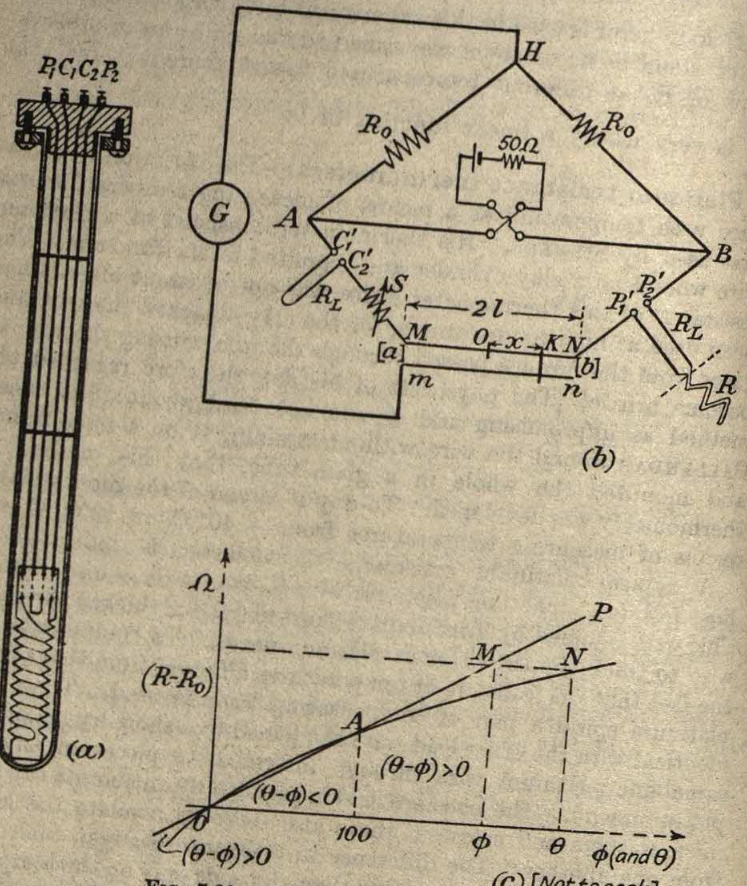

FIG. 7·21.—Platinum thermometry.

To make contact with the bridge wire, a wire *mn* of the same metal is placed alongside it and contact is made by means of a bridge-piece of the same metal. To enable this bridge-piece to be moved a mechanism is provided to lift it clear of the slide-wire; the bridge-piece can then be displaced sidewards without damaging the wires. The bridge is balanced by finding the setting for which there is no change in galvanometer deflexion when the bridge current is reversed. This eliminates the effect of all parasitic e.m.fs. and, at the same time, doubles the sensitivity.

Instead of attempting to obtain an ideal balance, as suggested above, the jockey is set at one of the two mm. divisions nearest to the balance point and the change in the galvanometer deflexion observed when the bridge current is reversed; let it be s_1 scale divisions to the right. The jockey is then set on the second such mm. division and the deflexion s_2 scale divisions to the left noted when the current is reversed. The true balance point is $\left(\frac{s_1}{s_1 + s_2}\right)$ mm. from the first mm. division.

The coils of the bridge are made of manganin on account of its small resistance-temperature coefficient. Unfortunately, however, these coils must be calibrated from time to time since the resistance of a manganin resistor gradually changes. In part this is caused by the varnish coating which absorbs moisture and the accompanying change in volume sets up stresses in the wire. To-day the resistors and bridge-wire are immersed in moisture-free paraffin; this is constantly stirred and only very small variations in its temperature are permitted.

When the bridge is balanced let us assume that K is x cm. from the *centre* O of the wire. Let ρ be the resistance per unit length of the wire. Then if $2l$ is the length of the bridge wire, a and b the end corrections to the wire, and R_L the resistance of the leads and compensating leads

$$S + R_L + \rho(l + x + a) = R + R_L + \rho(l - x + b),$$

since the ratio arms are equal. Hence $R = S + \rho(2x + a - b)$.

To correct for the small excess temperature between the coil of the thermometer and its surroundings whose temperature is to be found, this excess being due to the Joule heating effect, Callendar balanced the bridge using one cell to supply current to it and then balanced it again when two cells were used. The heating effect was thereby quadrupled, so that the increase in resistance of the coil was three times that due to the current from one cell.

In using the bridge it is first necessary to determine $R_{100} - R_0$ where R_0 and R_{100} are the resistances of the thermometer at 0° C. and 100° C. If S_0, S_{100}, x_0 and x_{100} are the values of S and x in these two instances, we have

$$R_{100} - R_0 = S_{100} - S_0 + 2\rho(x_{100} - x_0),$$

and this is independent of the end corrections a and b. Thus while the bridge can never be used to find a precise value for the resistance-temperature coefficient κ, since R_0 cannot be found, yet as a means of measuring temperatures from 0° C. to 630·5° C., the freezing point of antimony, and with slightly less accuracy up to 1000° C., it is unsurpassed except by Smith's difference bridge, cf. p. 194. The effect of not knowing the value of $(a - b)$ is merely to shift the line AB of Fig. 7·20 parallel to itself; its slope, which determines the size of the fundamental interval, $R_{100} - R_0$, remains unaltered.

The factor 2 in the above equation is troublesome when the bridge is used continually, so Callendar selected a bridge wire having a resistance per unit length equal to 0·00500 ohm.cm.$^{-1}$. Then $R = S + 0{\cdot}0100\ (x)$.

The platinum resistance scale of temperature.—The centigrade resistance scale of temperature for platinum is very important, for it was used by Callendar to determine temperatures on the gas scale very accurately. Of course any metal may be used to define a centigrade resistance scale of temperature, but it is only for platinum that the relation between R and θ is parabolic and that α and β, in $R = R_0(1 + \alpha\theta + \beta\theta^2)$, do remain constant.

Since $R = R_0(1 + \kappa\phi)$, if R_0 and R_{100} are known,

$$\phi = \frac{R - R_0}{R_{100} - R_0} \times 100,$$

where ϕ is the temperature of the wire on the platinum resistance scale. $R_{100} - R_0$ is termed the ***fundamental interval*** of the thermometer and corresponds to 100 degrees.

To determine the temperature, θ, on the nitrogen centigrade gas scale corresponding to ϕ, the procedure is as follows. We have

$$\phi = \left(\frac{\frac{R}{R_0} - 1}{\frac{R_{100}}{R_0} - 1}\right) \times 100 = \frac{\alpha\theta + \beta\theta^2}{100\alpha + 10{,}000\beta} \times 100 = \frac{\alpha\theta + \beta\theta^2}{\alpha + 100\beta}.$$

$$\text{Hence}\quad \theta - \phi = \frac{100\beta\theta - \beta\theta^2}{\alpha + 100\beta} = \frac{-\beta\theta(\theta - 100)}{\alpha + 100\beta} = d.\theta.(\theta - 100),$$

where $d = \dfrac{-\beta}{\alpha + 100\beta}$; since β is negative, d is positive. For pure platinum $d = 1{\cdot}493 \times 10^{-4}$ deg.$^{-1}$ C. and in practical thermometry it is convenient to write $d = \delta \times 10^{-4}$ deg. C., where δ is termed the ***difference coefficient.*** Thus

$$\theta - \phi = \delta\left(\frac{\theta}{100}\right)\left(\frac{\theta}{100} - 1\right).$$

To determine a value for the difference coefficient it is usual to measure R_0, R_{100} and $R_{444 \cdot 6}$, where the temperature of sulphur vapour formed under a pressure of one standard atmosphere is taken to be 444·600° C. Then ϕ_s, the sulphur point on the platinum scale, is calculated; it is given by

$$\phi_s = \frac{R_s - R_0}{R_{100} - R_0} \times 100.$$

Hence

$$444{\cdot}60 - \phi_s = \delta\left(\frac{444{\cdot}6}{100}\right)\left(\frac{444{\cdot}6}{100} - 1\right),$$

so that a value for the difference coefficient may be obtained. For pure platinum, $\delta = 1{\cdot}493$ deg. C.

The effect of impurity in the platinum wire, or of residual strain due to imperfect annealing, is to lower the ratio $\frac{R_{100}}{R_0}$ and to increase the value of the difference coefficient. Internationally it is agreed that, for precision work, the above ratio, which is 1·3925 for pure platinum, must not be less than 1·3910, while

$$1{\cdot}488 \text{ deg. C.} < \delta < 1{\cdot}498 \text{ deg. C.}$$

Example.—$R_0 = 20{\cdot}765$ ohm., $R_{100} = 28{\cdot}905$ ohm. and $R_s = 55{\cdot}093$ ohm.

Assuming $\theta_s = 444{\cdot}60°$ C., find a value for the difference coefficient δ and the temperature on the gas scale corresponding to $R = 45{\cdot}284$ ohms.

The fundamental interval is $8{\cdot}140\ \Omega$, so that

$$\phi_s = \frac{34{\cdot}328}{8{\cdot}140} \times 100 = 421{\cdot}73° \text{ C.}$$

$$\therefore\quad 444{\cdot}60 - 421{\cdot}73 = \delta\left(\frac{444{\cdot}60}{100}\right)\left(\frac{444{\cdot}60}{100} - 1\right),$$

i.e.

$$\delta = 1{\cdot}4927 \text{ deg. C.}$$

Similarly,

$$\phi = \frac{24{\cdot}519}{8{\cdot}140} \times 100 = 301{\cdot}28° \text{ C.}$$

$$\therefore\quad \theta - \phi \simeq 1{\cdot}4927\left(\frac{301{\cdot}3}{100}\right)\left(\frac{301{\cdot}3}{100} - 1\right) \simeq 9{\cdot}05 \text{ deg. C.}$$

$$\therefore\quad \theta \simeq 310{\cdot}3° \text{ C.}$$

We now use this value for θ in the difference equation and get

$$\theta - \phi \simeq 1{\cdot}4927\left(\frac{310{\cdot}3}{100}\right)\left(\frac{310{\cdot}3}{100} - 1\right)$$

$$\simeq 9{\cdot}74 \text{ deg. C.}$$

$$\therefore\quad \theta = 310{\cdot}96° \text{ C.}$$

This process could be continued, but, in practice, it is seldom necessary to proceed beyond this stage.

Further remarks on the platinum scale of temperature.—To express the resistance-temperature coefficient κ in terms of α and

β, we have

$$\kappa = \frac{R_{100} - R_0}{R_0 \times 100} = \alpha + 100\beta.$$

But

$$d = \frac{-\beta}{\alpha + 100\beta} = -\frac{\beta}{\kappa}.$$

Hence, for pure platinum, for which $\kappa = 3{\cdot}925 \times 10^{-3}$ deg.$^{-1}$ C.,

$$\beta = -\kappa d = -\kappa\frac{\delta}{(100)^2} = -3{\cdot}925 \times 10^{-3} \times 1{\cdot}493 \times 10^{-4}$$

$$= -5{\cdot}86 \times 10^{-7} \text{ deg.}^{-2}\text{ C.}$$

$$\therefore \alpha = \kappa - 100\beta = 0{\cdot}003984 \text{ deg.}^{-1}\text{ C.}$$

Fig. 7·21 (*c*) shows how $R - R_0$ varies with ϕ and with θ. From it we see that $(\theta - \phi) > 0$ when $\theta > 100°$ C. or $< 0°$ C.; over the range $0 < \theta < 100$, $(\theta - \phi) < 0$. Also

$$\frac{1}{R_0}\left[\frac{d}{d\theta}(R - R_0)\right]_{\theta=0} = \alpha.$$

Theoretically the parabola has its vertex when $\theta = \dfrac{\alpha}{-2\beta}$, i.e. 3600° C., but since platinum melts at 1769° C. and the parabolic formula only holds over a restricted range, this conclusion has no physical significance.

Smith's difference bridge for use with platinum thermometers.—One of the main difficulties in precision determinations of temperature by means of a platinum thermometer is that of making quite sure that the resistance of the leads has been completely eliminated. By using a resistance bridge with equal ratio-arms and a pair of compensating leads, as described on p. 189, Callendar sought to eliminate the resistance of the leads from his measurements. Callendar recommended single well insulated copper leads to connect the thermometer and its compensating leads to the corresponding terminals on his bridge. The four wires necessary were arranged in pairs as in Fig. 7·22, instead of having each pair of wires twisted together with one pair leading from P_1 and P_2 to P_1' and P_2', etc. In this way non-uniformity of the temperature of the air round the leads affected each side of the bridge to the same extent, and at the time when Callendar showed that a platinum thermometer could be an instrument of precision his method of making the measurement of the resistance was a remarkable achievement. By 1912, however, F. E. Smith * had realized that all was not well with the Callendar bridge and showed that the equality between the leads was an assumption scarcely ever realized in work of the highest precision. Smith therefore designed a platinum thermometer which consisted of a coil of non-inductively wound platinum having at each end both a current lead and a potential lead. He also designed a bridge so that for all practical purposes the resistance of the leads was eliminated.

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* *Phil. Mag.*, **24**, 541, 1912.

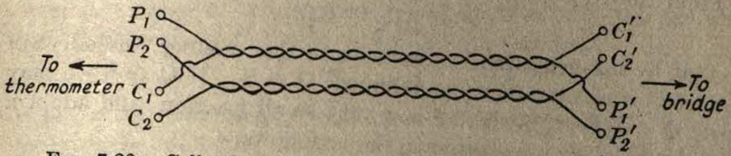

FIG. 7·22.—Callendar's arrangement of the leads connecting the thermometer proper to the bridge.

In principle, the bridge and the method of using it are as follows. Q and S, Fig. 7·23 (*a*), are two *equal* resistances which, together with an adjustable resistance R, comprise three arms of a Wheatstone bridge network. P represents the thermometer with its current leads L_1 and L_4, and potential leads L_2 and L_3. An accumulator and a galvanometer G are arranged as shown to complete the bridge; in the first stage of the operations the current lead L_4 is 'idle'. When the bridge is balanced, the current being reversed in order to eliminate thermal e.m.fs., we have

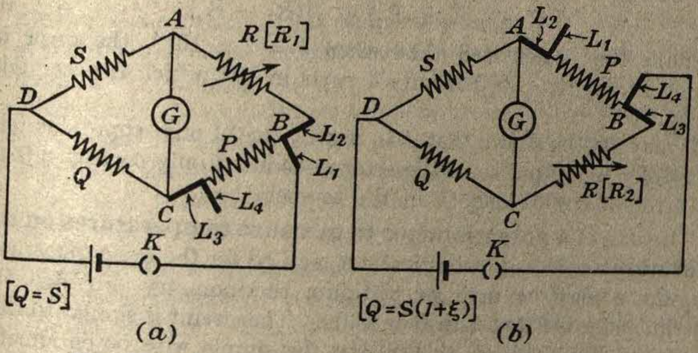

FIG. 7·23.—The principle of Smith's difference bridge.

$$P + L_3 = R_1 + L_2,$$

where R_1 is the value of R when the bridge is balanced under these conditions.

L_2 is then disconnected from R and joined to Q; L_3 is joined to R and the cell to L_4, so that L_1 is now the 'idle' current lead. If R_2 is the value of R at balance in these circumstances, we have

$$P + L_2 = R_2 + L_3.$$

From these two equations we obtain

$$P = \tfrac{1}{2}(R_1 + R_2),$$

so that by balancing the bridge twice in this manner, the effect of the leads is eliminated if, as assumed, $Q = S$. If $Q \neq S$ we may write $Q = S(1 + \xi)$, where ξ is a very small fraction, and adopt a slightly different procedure; it is as follows.

At the same time as L_2 and L_3 are reversed in position and the cell connected to L_4, let the arms containing P and R be interchanged, as shown in Fig. 7·23 (*b*). Then the conditions for balance in the two instances are respectively

$$P + L_3 = \frac{Q}{S}(R_1 + L_2) = (1 + \xi)(R_1 + L_2),$$

and

$$P + L_2 = \frac{S}{Q}(R_2 + L_3) \fallingdotseq (R_2 + L_3)(1 - \xi).$$

From these we have

$$2P = (R_1 + R_2) + \xi[(R_1 + L_2) - (R_2 + L_3)].$$

Suppose now that $L_2 = L_3$. Then

$$P = \tfrac{1}{2}(R_1 + R_2) + \tfrac{1}{2}\xi(R_1 - R_2) \quad . \quad . \quad \text{(i)}$$

Smith then calculated that when $\xi = 2 \times 10^{-4}$, the error in taking $P = \tfrac{1}{2}(R_1 + R_2)$ is only 4 parts in 10^8, which is absolutely negligible.

It may also be shown that if L_2 and L_3 are 0·1 ohm approximately and differ by 10 per cent, the error introduced only corresponds to an error of 0·0001 deg. C. in the temperature.

The use of a potentiometer to measure temperatures on the platinum scale.—A potentiometer method for the measurement of resistance may be used in platinum thermometry provided the thermometer is fitted with four leads. The circuit is similar to that shown in Fig. 7·14, cf. p. 179, but the simple wire potentiometer shown there must be replaced by a precision vernier potentiometer, cf. p. 180, capable of detecting a change in potential difference of 0·1 μV. The correction for lead resistance does not arise for the potential leads carry no current at balance, but at least four readings are necessary to make a single measurement of resistance; one for each resistance and a repeat with the current reversed in order that the effect of parasitic e.m.fs. in the circuit may be eliminated.

If an attempt is made to use a wire potentiometer for the above purpose the results are unreliable on account of the relatively large currents which must be used; such currents raise the temperature of the platinum thermometer considerably above that which is to be measured and the necessary correction is uncertain.

Platinum resistance thermometry at low temperatures.—Over the range of temperature − 40° C. to 600° C. the temperature θ defined by Callendar's well-known formula

$$R = R_0(1 + \alpha\theta + \beta\theta^2) \qquad \text{(i)}$$

is practically indistinguishable from the temperature given by the thermodynamic scale. If the thermometers are constructed of pure platinum, mounted so as to avoid mechanical strain due to temperature changes and properly annealed, a temperature scale can be reproduced with much greater accuracy than is possible with any other type of thermometer.

Below the region of − 40° C., however, temperatures defined by equation (i) depart from the thermodynamic scale by amounts which increase very rapidly below − 100° C., the calculated temperature being about 2 degrees too low at liquid air temperatures.

In 1925 van Dusen proposed the following simple modification of the difference formula, viz.

$$\theta - \phi = \delta\left(\frac{\theta}{100} - 1\right)\left(\frac{\theta}{100}\right) + \xi\left(\frac{\theta}{100} - 1\right)\left(\frac{\theta}{100}\right)^3 \qquad \text{(ii)}$$

where ξ is a constant. For temperatures above 0° C. the last term in the above equation is omitted when it is identical with Callendar's formula for obtaining $\theta - \phi$. The constant ξ is determined from a calibration at liquid oxygen temperature and the term involving ξ is only used for temperatures below 0° C. The transition from one formula to the other is very gradual since the term containing ξ is very small throughout the range $-40°\,\text{C.} < \theta < 100°\,\text{C.}$ The value of ξ is about 0·11 deg. C.

Thermistors.—A thermistor is a resistor with a high negative resistance-temperature coefficient, i.e. its conductance increases with temperature. The material of this resistor is a semi-conductor in the form of a complex metal oxide compound prepared in a special manner to ensure reproducibility and stability of its characteristics. During manufacture it is fired at an elevated temperature so that it may be used without detriment to itself up to 500° C. Experiment shows that the resistance, R, of a thermistor is related to its absolute temperature, T, by the equation

$$R = a \exp\left(\frac{b}{T}\right),$$

where a and b are constants. Thus, the resistance-temperature coefficient at a temperature T, viz. $\frac{1}{R}\frac{dR}{dT}$, is $-\frac{b}{T^2}$.

The thermistor element consists of a very small bead of resistance material about 0·5 mm. in diameter, which is integrally formed on two parallel and very fine platinum wires. The platinum wires are welded to copper coated nickel-iron * leads and the whole assembly is sealed in a glass bulb of small dimensions ; this bulb may be either gas-filled or exhausted.

Since a thermistor is a temperature-sensitive device, changes in the ambient temperature will give rise to changes in the resistance of the thermistor element. These may be compensated by the use of a similar element in the opposite arm of the bridge which is used to measure the resistance.

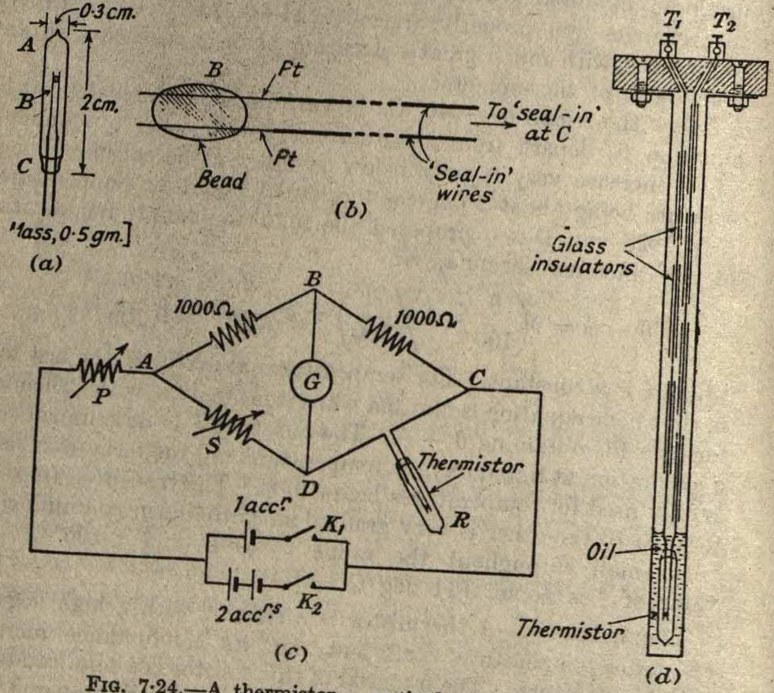

FIG. 7·24.—A thermistor ; method of standardizing.

A simple form of thermistor is shown in Fig. 7·24 (*a*). A is a glass envelope approximately 2 cm. × 0·3 cm. B is the metal oxide bead fused on the platinum leads ; the method of mounting is shown in greater detail in Fig. 7·24 (*b*).

In order to standardize a thermistor for use over the range 0 – 100° C., it is placed in the arm CD of the Wheatstone bridge network shown in Fig. 7·24 (*c*), the values of the other resistances being selected to match that of the resistance being measured. [If a compensating thermistor is used in the arm AD then the ratio arms must be equal.] P is a resistor of such value that the wattage in the thermistor R does not exceed the rated value even when two cells are used in conjunction with the bridge. The bridge is first balanced with one cell only in use. Let R_1 be the measured value of R; when the bridge is balanced with two cells supplying the current to it let R_2 be the measured value of R. Then, since the resistance of a thermistor decreases with rise in temperature,

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* Known to the valve industry as 'cunife' [German, *Kupfermanteldraht*].

$$R_1 - R_2 = 3(\Delta R),$$

where ΔR is the decrease in the resistance of R on account of the Joule heating effect in the thermistor when one cell supplies current to the bridge. Then $R_1 + \Delta R = R$, the resistance of the thermistor in the absence of the Joule heating effect.

In this way the resistance of the thermistor may be determined at 0° C., the transition point of sodium sulphate (32·38° C.), the melting point of naphthalene (79·8° C.) and at the steam temperature. Thus the constants a and b in the formula $R = a \exp\left(\frac{b}{T}\right)$ may be found.

To protect the thermistor from mechanical shocks while the standardization is carried out, it is enclosed in a brass case as shown in Fig. 7·24 (*d*). Thin glass tubes insulate the 'leads-in' while a thick oil serves to make good thermal contact between the thermistor and its surroundings. This method of mounting increases the thermal capacity of the thermometer but this is of little consequence when steady temperatures are being measured.

Note on a submarine cable imperfectly insulated.—Let Fig. 7·25 (*a*) represent a short but finite length Δx of a wire surrounded by an insulator in the form of a coaxial cylindrical sheath whose outer surface is at zero potential. [The cable of which the element considered is a short length lies, for example, in sea-water.] If χ is the resistivity of the insulator, the resistance of an element defined by the cylindrical surfaces r and $r + \delta r$, and of length Δx is $\chi \frac{\delta r}{2\pi r . \Delta x}$. Hence ΔR, the resistance for the length Δx, is

$$\Delta R = \frac{\chi}{2\pi\Delta x} \ln\left(\frac{b}{a}\right),$$

where a and b are the inner and outer radii of a cross-section and this is the insulation resistance of the length Δx of the cable.

If we imagine the whole length l of the cable divided into n sections, which are essentially in parallel, and let $(\Delta R)_k$ be the resistance of the kth section, then the insulation resistance R of the cable is given by

$$\frac{1}{\mathrm{R}} = \sum_{k=1}^{n} \frac{1}{(\Delta \mathrm{R})_k} = \sum_{k=1}^{n} \frac{2\pi(\Delta x)_k}{\chi \ln\left(\dfrac{b}{a}\right)},$$

if l is not so large that the potential at all points along the copper core cannot be considered constant. At once, we have

$$\mathrm{R} = \frac{\chi}{2\pi l} \ln\left(\frac{b}{a}\right),$$

It will be noticed that the insulation resistance decreases as the length of the cable increases.

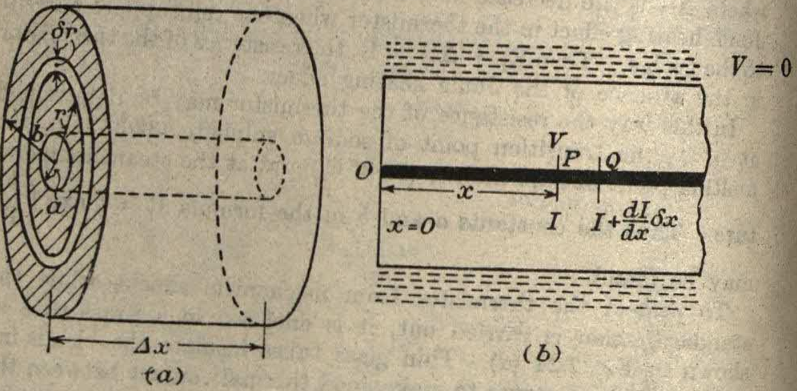


FIG. 7·25.—Resistance of a submarine cable.

For a long submarine cable, but without capacitance and self-inductance, the distribution of potential along it may be found in the following way. Let V be the potential at a point in the wire at a distance x from the transmitting station—cf. Fig. 7·25 (b). If we write

$$\xi = \frac{\chi}{2\pi} \ln\left(\frac{b}{a}\right),$$

the insulation resistance of a short length δx of the cable is $\dfrac{\xi}{\delta x}$.

Then if I is the current in the cable at the section defined by x, at the section $x + \delta x$ it will be $\mathrm{I} + \dfrac{d\mathrm{I}}{dx}\delta x$. Hence the current leak from the cable between the sections considered is $-\dfrac{d\mathrm{I}}{dx}\delta x$. This leak is a current of electricity flowing across a resistance $\dfrac{\xi}{\delta x}$, the potential drop across the insulator at the section considered having a mean value V. Hence by Ohm's law,

$$V = -\left(\frac{dI}{dx}\delta x\right)\left(\frac{\xi}{\delta x}\right),$$

or

$$\frac{dI}{dx} = -\frac{V}{\xi} = -VG \quad . \quad . \quad . \quad \text{(i)}$$

where $G = \frac{1}{\xi}$ is the conductance per unit length of the insulator.

If R is the resistance per unit length of the core, the fall in potential across the element δx is $-\frac{dV}{dx}.\delta x = IR.\delta x$.

$$\therefore -\frac{dV}{dx} = IR \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Eliminating I from (i) and (ii) we obtain

$$\frac{d}{dx}\left(\frac{1}{R}\frac{dV}{dx}\right) = VG.$$

If R is constant for all points along the cable, we have $\frac{d^2V}{dx^2} = RGV$ and if G is constant, a solution of this equation is

$$V = A_1 \cosh \sqrt{RG}.x + B_1 \sinh \sqrt{RG}.x,$$

where A_1 and B_1 are integration constants to be determined in any particular problem from the 'end-conditions'.

Alternatively, we may write

$$V = A_2 \exp(\sqrt{RG}.x) + B_2 \exp(-\sqrt{RG}.x),$$

where A_2 and B_2 are integration constants.

Similarly, by eliminating V from (i) and (ii), we have

$$\frac{d^2I}{dx^2} = -G\frac{dV}{dx} = RGI.$$

$$\therefore I = A \exp(\sqrt{RG}.x) + B \exp(-\sqrt{RG}.x),$$

where A and B are integration constants. If the line is very long, $\exp(\sqrt{RG}.x) \to \infty$, so that $A \to 0$ since the current must be finite. Under such conditions

$$I = B \exp(-\sqrt{RG}.x).$$

The mechanism of metallic conduction.—Many attempts have been made to develop in detail the idea of an electric current as a process of diffusion, but before the introduction of the electron theory towards the end of the nineteenth century these attempts could hardly be described as very successful. The difficulty arises in the absence of any sign of transport of matter or of any chemical change accompanying the transport of electricity through metals. Every circuit must consist of two or more portions composed of different materials which may contain no element in common and we must assume either that the particles which convey the current can pass from one material to the other or that they cannot. Either assumption leads to enormous difficulties. If the particles cannot cross the boundary between the different metallic conductors they must remain piled up at the boundaries, even if they are identical with the atoms of the material through which they move and this piling up must alter the distribution of mass in the conductor, while if they are composed of some substance different from the rest of the material, some sort of chemical change may be expected to take place. The most careful experiments on pure metals and on alloys, however, have failed to reveal any change in any property of a metallic conductor induced by the passage of an electric current through it. On the other hand, if we suppose that the particles can pass freely from one material to another, new difficulties appear. It is known that the atoms and groups of atoms of different elements differ markedly in their properties, and chemical or spectroscopic analysis is available to detect traces of foreign atoms should these enter another metal. Since the properties of the metals forming a non-uniform circuit are unchanged by the passage of the current, if this latter is due to the transport of particles then these entities must be of a nature common to all elements. Previous to the discovery of the electron no such particles of a fundamental nature were known to exist.

The discovery of the electron and the consequence of a so-called 'electron theory' removed in one stroke many of the difficulties inherent to the earlier theories. According to this theory there are in every metal a very large number of electrons freely movable in the spaces between the atoms, and it is the diffusion of these electrons through the metal under the action of an external electric field that is the essence of an electric current. If there is no external field the velocities of these free electrons will be distributed equally in all directions; there will be no tendency for them to move in one direction in preference to another, but if the metal is subject to an external electric field the electrons are subject to a force in a definite direction by virtue of their charge so that those having component velocities in the direction of the field will have these components decreased while those with components in the direction opposite to the field will be increased. Thus there is a transport of electricity against the field and it is this transportation which constitutes an electric current. [The direction of this electron current is in a sense opposite to that of the conventional electric current since the electrons are negatively charged.] The velocity of drift is kept in check, however, by the continual encounters of the electrons with the atoms of the metal and with each other when additional forces arise tending to deflect the electrons from their forward motion. If it were not for these collisions with the atoms in the metal the momentum of the electrons would increase indefinitely under the action of the impressed forces but this growth is continually arrested by the collisions that occur.

Let us suppose that there are N electrons per unit length of a conductor and that at any given moment these have an average forward velocity u through the material of the conductor, i.e. u is considered positive when it is in the direction of the field. If m is the mass of each electron, the total momentum of the moving charges in unit length will be Nmu. The rate at which this total momentum is checked by collisions will be proportional to N and to u, and may be taken as γNu, where γ is a constant. The rate at which the momentum is increased by the electric field, E, is NeE, where e is the charge on an electron, so that the motion of the particles is determined by the equation

$$\frac{d}{dt}(Nmu) = NeE - \gamma Nu \quad . \quad . \quad . \quad (i)$$

In unit time the number of electrons passing any fixed section of the conductor is Nu, so that the total charge passing per unit time is Neu. If i is the current in the conductor,

$$i = Neu \quad . \quad . \quad . \quad . \quad (ii)$$

Equation (i) may therefore be written

$$\frac{di}{dt} = \frac{Ne^2}{m}\left(E - \frac{\gamma}{Ne^2}i\right) \quad . \quad . \quad . \quad (iii)$$

This equation shows that if a steady electric field of strength E is applied to the free electrons in a wire the current will not increase indefinitely but will remain stationary when it has reached a value given by

$$i = \frac{Ne^2}{\gamma}E \quad . \quad . \quad . \quad . \quad (iv)$$

If V is the potential at any section of the wire and s is a coordinate measured along the wire, $E = -\frac{dV}{ds}$, so that

$$-\frac{dV}{ds} = \frac{\gamma}{Ne^2}i \quad . \quad . \quad . \quad . \quad (v)$$

Integrating between any two fixed points P and Q of the conductor the distances of P and Q from the origin of coordinates being s_1 and s_2

respectively, $[s_1 < s_2]$, we have

$$V_P - V_Q = i\int_P^Q \frac{\gamma}{Ne^2}ds = \frac{\gamma i}{Ne^2}(s_2 - s_1),$$

which is ir, if r is the resistance of the length PQ of the wire.

It is in this way that the simple electron theory attempts to account for Ohm's law. It will be noticed that according to this hypothesis Ohm's law is only true for steady currents. The equations we have obtained indicate that the resistance per unit length of a conductor is $\frac{\gamma}{Ne^2}$. Thus, generally speaking, materials in which there is a copious supply of free electrons should have high electrical conductivities, i.e. their resistivities should be low, and conversely.

To estimate a value for γ let us suppose, as a rough approximation, that in a conductor each free electron moves without restraint for a certain time t between two consecutive collisions with atoms. If we assume the momentum to be destroyed completely at each collision, so that after a collision an electron has zero velocity, the average velocity acquired in time t will be given by $u = \frac{1}{2}\frac{Ee}{m}t$, so that the average forward momentum of an electron at any instant will be $\frac{1}{2}Eet = mu$.

$$\therefore\ i = Neu = \frac{1}{2}\frac{Ne^2}{m}tE = \frac{Ne^2}{\gamma}E,$$

by the above. Hence γ is of the order of magnitude $\frac{2m}{t}$ and $r = \frac{2m}{Ne^2t}$. To find χ, the resistivity of the material of the conductor, let A be the cross-sectional area of the wire. Then, since we are dealing with unit length of wire,

$$r = \frac{\chi}{A} = \frac{2m}{Ne^2t}.$$

$$\therefore\ \chi = \frac{2mA}{Ne^2t} = \frac{2m}{ne^2t},$$

where n is number of free electrons per unit volume of the metal.

The simplification of network analysis.—Even in relatively simple networks, the use of Kirchhoff's laws and of Maxwell's cyclic currents is often very laborious and time consuming. To simplify the calculations many theorems have been stated; some of these will now be discussed and their applications illustrated by examples.

The principle of superposition.—In calculating the currents in the various branches of a network containing several sources of current, it is found that a solution is most readily obtained by determining the currents in the branches resulting from the presence of one source at a time; the final solution for the current in any one branch is given by the algebraic sum of the individual currents in that branch. This is known as the ***principle of superposition;*** when it is used, however, all omitted sources must be replaced by resistors, each with a resistance equal to the internal resistance of the omitted source.

Example.—Apply the principle of superposition to find the currents in the various branches of the network shown in Fig. 7·26 (*a*); the internal resistance of each battery may be taken as 1 ohm.

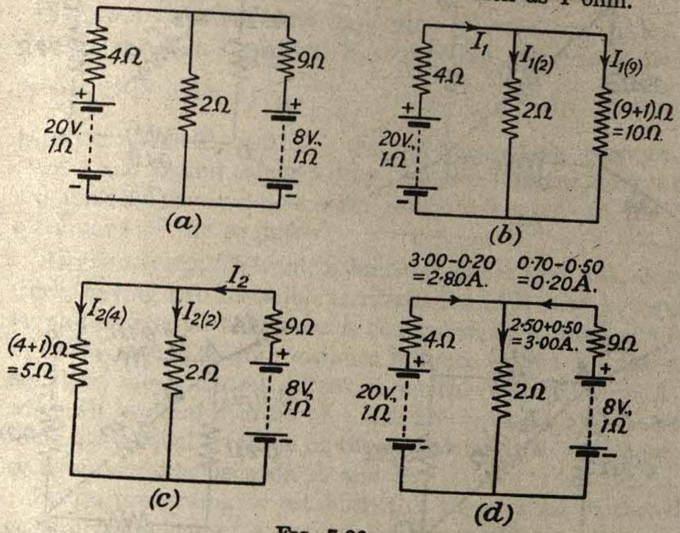


FIG. 7·26.

First imagine the 8V. battery removed and replaced by a 1 Ω resistor as shown in Fig. 7·26 (*b*). With the notation indicated, we have

$$I_1 = \frac{20}{(4+1) + \left(\frac{2 \times 10}{12}\right)} = \frac{240}{80} = 3{\cdot}00\text{A.}$$

$$\therefore\ I_{1(2)} = \frac{10}{12} \times 3 = 2{\cdot}50\text{A.},$$

and

$$I_{1(9)} = \frac{2}{12} \times 3 = 0{\cdot}50\text{A.}$$

When the 20V. battery is replaced by a 1 Ω resistor, the circuit is as shown in Fig. 7·20 (*c*). With the notation shown,

$$I_2 = \frac{8}{10 + \frac{10}{7}} = 0{\cdot}70\text{A.},$$

$$\therefore\ I_{2(2)} = 0{\cdot}7 \times \frac{5}{7} = 0{\cdot}50\text{A, and } I_{2(4)} = 0{\cdot}20\text{A.}$$

These currents are superimposed in Fig. 7·26 (*d*); it shows that the currents from the 20V. and 8V. batteries are 2·80A and 0·20A respectively, the current through the 2 Ω resistor being 3·00A; the directions of these currents are indicated.

Example.—Apply the principle of superposition to obtain a value for the current in the branch AC of the network shown in Fig. 7·27 (*a*).

First consider the 2A current leaving from B and the 2A component of the 10A current which enters at A; in future such a distribution of current will be referred to as the 2A load. From Fig. 7·27 (*b*) we have, by applying Kirchhoff's second law to the loop ABC,

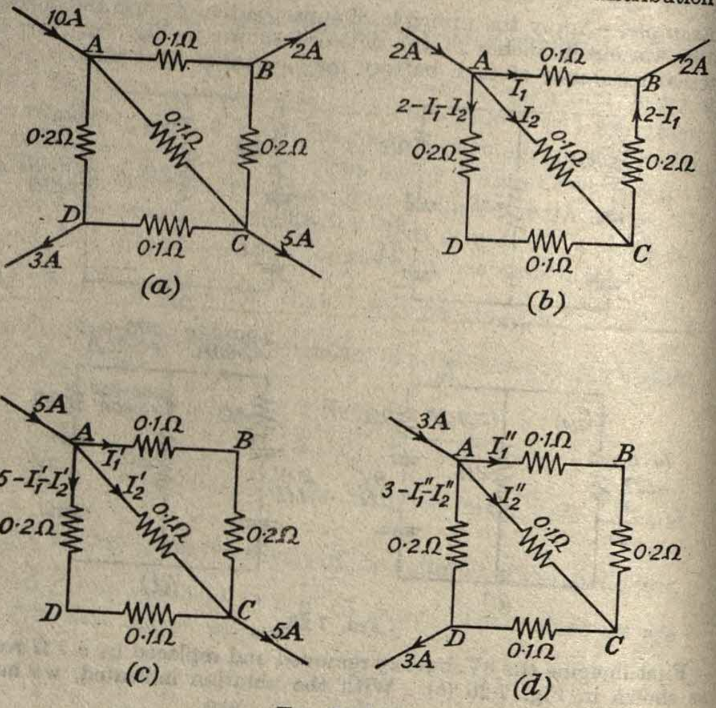


FIG. 7·27.

$$0{\cdot}1\,I_1 - 0{\cdot}2(2 - I_1) - 0{\cdot}1\,I_2 = 0.$$

Similarly, for the loop ACD,

$$0{\cdot}3(2 - I_1 - I_2) = 0{\cdot}1\,I_2.$$

These equations become

$$-\,0{\cdot}4 + 0{\cdot}3\,I_1 = 0{\cdot}1\,I_2,$$

and

$$0{\cdot}6 - 0{\cdot}3\,I_1 = 0{\cdot}4\,I_2,$$

whence, by addition,

$$I_2 = 0{\cdot}40\text{A}.$$

Now consider the 5A load acting alone. With the notation indicated in Fig. 7·27 (*c*), we have for the loop ABC,

$$0{\cdot}3\,I_1' = 0{\cdot}1\,I_2',$$

and for the loop ACD,

$$0{\cdot}3(5 - I_1' - I_2') = 0{\cdot}1\, I_2'.$$

From these equations we find $I_1' = 1{\cdot}0$A and $I_2' = 3{\cdot}0$A. Similarly for the 3A load, cf. Fig. 7·27 (*d*), we find

$$0{\cdot}3\, I_1'' = 0{\cdot}1\, I_2'',$$

and $$0{\cdot}2(3 - I_1'' - I_2'') = 0{\cdot}1\, I_2'' + 0{\cdot}1(I_1'' + I_2'').$$

$$\therefore I_1'' = 0{\cdot}4\text{A, i.e. } I_2'' = 1{\cdot}2\text{A}.$$

Hence, using the principle of superposition, the total current in the branch AC, when all loads are operative, is given by

$$I_{AC} = I_2 + I_2' + I_2'' = 4{\cdot}6\text{A}.$$

Thévenin's theorem.—According to this theorem, the current flowing through a resistor R_L connected across any two points A and B is given by $\dfrac{V_{AB}}{R + R_L}$, where V_{AB} is the drop in voltage between A and B, and R is the resistance of the network as seen from A and B, and calculated by assuming that the current supplies are replaced by resistors with resistances equal to the internal resistances of the sources.

Alternatively, Thévenin's theorem states that any circuit network, having two accessible terminals A and B, may be replaced so far as its external behaviour is concerned, by a single e.m.f. acting in series with a single resistance between A and B; the e.m.f. is equal to the potential difference which exists between A and B when all internal sources of e.m.f. are replaced by resistors with resistances equal to those of the sources and the resistance is that then measurable between A and B.

Before proceeding to establish this theorem let us consider the general behaviour of a potential divider. This, as shown in Fig. 7·28 (*a*), consists of two resistors R_1 and R_2 in series with a cell of e.m.f. V and negligible internal resistance. The potential difference across R_1 is $\dfrac{R_1V}{R_1 + R_2}$ and this is called the open circuit voltage (o.c.v.) between the two terminals A and B which are connected to the ends of R_1; this o.c.v. will be denoted by V_0. Now let a load, resistance R_L, be placed across AB, i.e. R_L is in parallel with R_1; it is desired to examine how V_L, the voltage across the load, varies with I_L, the current through it.

If I is the current from the cell, we have

$$\begin{aligned} V - V_L &= \text{voltage available to send current through } R_2 \\ &= R_2 I. \end{aligned}$$

Considering the drop in voltage across R_1, we have

$$(I - I_L)R_1 = V_L.$$

Eliminating I from these two equations, we find

$$\frac{V - V_L}{R_2} - I_L = \frac{V_L}{R_1},$$

i.e.

$$\frac{V}{R_2} - I_L = V_L\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

$$\therefore\ V_L = \frac{VR_1}{R_1 + R_2} - \frac{R_1R_2}{R_1 + R_2} I_L$$

$$= V_0 - \frac{R_1R_2}{R_1 + R_2} I_L.$$

If now we consider the circuit shown in Fig. 7·28 (*b*), it is seen at once that this expression also represents the voltage across the load in this instance. Hence, if we have two boxes containing the alternative circuits shown in Fig. 7·28 (*c*) and (*d*), there is no ***external*** experiment which can be devised to distinguish one box from the other.

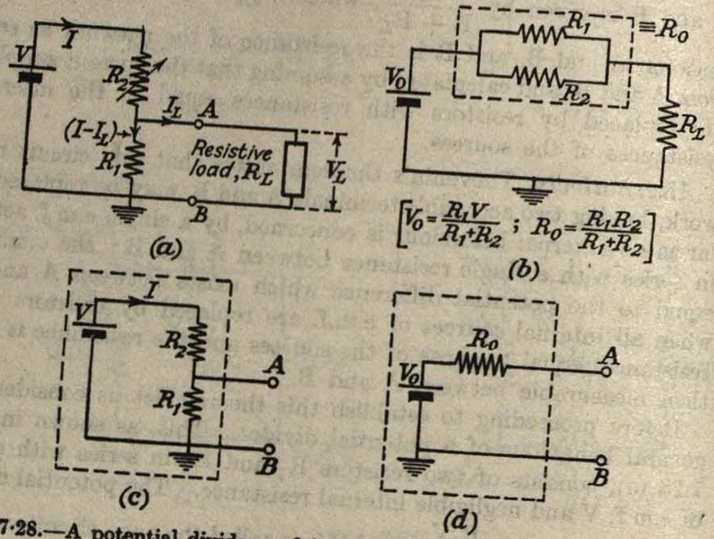


FIG. 7·28.—A potential divider and its equivalent circuit; an illustration of Thévenin's theorem.

If the points A and B are short-circuited, the currents from the cells in (*c*) and (*d*) are $\frac{V}{R_2}$ and $\frac{V_0}{R_0}$; these are equal since

$$V_0 = \frac{R_1V}{R_1 + R_2}, \quad \text{and} \quad R_0 = \frac{R_1R_2}{R_1 + R_2}.$$

This example is an illustration of Thévenin's theorem, for according to it, since the resistance, as seen from AB, consists of R_1 in parallel with R_2 (the internal resistance of the cell is again assumed to be zero),

$$I_L = \frac{V_0}{\dfrac{R_1R_2}{R_1 + R_2} + R_L} = \frac{VR_1}{R_1R_2 + R_L(R_1 + R_2)}.$$

But from Fig. 7·28 (*a*),

$$\frac{V}{R_2 + \dfrac{R_1R_L}{R_1 + R_2}} = I,$$

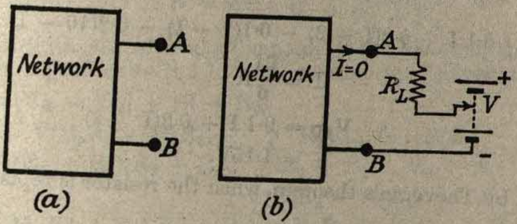
and

$$R_1(I - I_L) = R_LI_L.$$

Eliminating I from these two equations we find, as before,

$$I_L = \frac{VR_1}{R_1R_2 + R_L(R_1 + R_2)}.$$

Proof of Thévenin's theorem.—Fig. 7·29 (*a*) represents a linear network containing both resistors and voltaic cells, and having two accessible terminals A and B. Fig. 7·29 (*b*) shows the same network but with an external load, R_L, and an adjustable voltage supply arranged in series between A and B. Let this voltage supply have a value V when no current enters or leaves the network, i.e. $I = 0$. Thus V is the open circuit voltage between A and B, i.e. $V = V_{AB}$.



(*a*) (*b*)

FIG. 7·29.—Proof of Thévenin's theorem.

Now by the principle of superposition the zero current in the external circuit may be regarded as the algebraic sum of (*a*) the current, I_{in}, from V when all current sources in the network are replaced by appropriate resistors and (*b*) the current, I_{out}, from all the sources within the network when V is replaced by an appropriate resistor. For (*a*) we have

$$I_{in} = \frac{V}{R + R_L} = \frac{V_{AB}}{R + R_L},$$

where R is the resistance of the network as seen from A and B. But $I_{in} - I_{out} = 0$, so that

$$I_{out} = \frac{V_{AB}}{R + R_L}.$$

Example.—Use Thévenin's theorem to solve the example on p. 206. When the resistor AC is removed from the network shown in Fig. 7·30 (*a*), the resistance of the network, as seen from A and C, is given by

$$\frac{1}{R} = \frac{1}{0{\cdot}3} + \frac{1}{0{\cdot}3}, \quad \text{i.e.} \quad R = 0{\cdot}15\ \Omega.$$

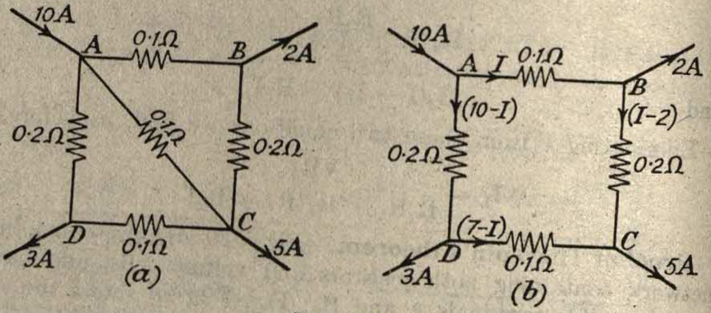


FIG. 7·30.

If I, cf. Fig. 7·30 (*b*), is the current in AB, for the loop ABCD, we have

$$0{\cdot}1\,I + 0{\cdot}2(I - 2) - 0{\cdot}1(7 - I) - 0{\cdot}2(10 - I) = 0,$$

i.e.

$$I = \frac{31}{6}\text{A}.$$

$$\therefore \quad V_{AC} = 0{\cdot}1\,I + 0{\cdot}2(I - 2)$$
$$= 1{\cdot}15\text{V}.$$

Hence, by Thévenin's theorem, when the resistor is replaced between A and C

$$I_{AC} = \frac{1{\cdot}15}{0{\cdot}1 + 0{\cdot}15} = 4{\cdot}60\ \text{A},$$

and so agrees with the solution obtained by the other method.

Example.—Fig. 7·31 (*a*) represents a Wheatstone bridge for the measurement of resistance. The source of current is a cell of e.m.f. V and negligible internal resistance. The potential of the point B is taken to be zero. Show that if the galvanometer is removed ($R_g = \infty$),

$$V_H - V_K = V(R_2R_3 - R_1R_4) \div [(R_1 + R_2)(R_3 + R_4)].$$

Also, if the cell is shorted, i.e. A and B are at the same potential, show that the resistance between H and K is given by

$$R = \frac{R_1R_2}{R_1 + R_2} + \frac{R_3R_4}{R_3 + R_4}.$$

Hence, using Thévenin's theorem, determine the galvanometer current, I_g, when the circuit is restored to its initial form.

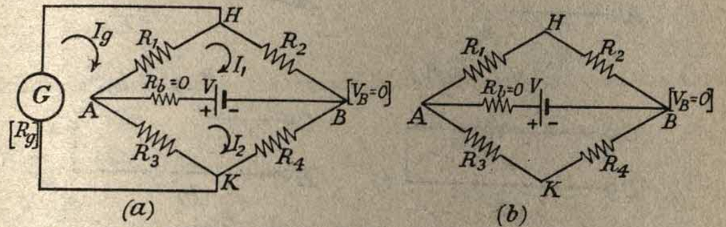


FIG. 7·31.—A Wheatstone bridge network with a cell of negligible internal resistance.

Fig. 7·31 (*b*) shows the bridge when the galvanometer is removed. We have

$$V_H = \frac{R_2 V}{R_1 + R_2}, \quad \text{and} \quad V_K = \frac{R_4 V}{R_3 + R_4}.$$

Hence

$$V_H - V_K = V\left[\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4}\right]$$
$$= V . \frac{R_2R_3 - R_1R_4}{(R_1 + R_2)(R_3 + R_4)}.$$

When, in addition, the cell is removed, we have in effect two resistances R_1 and R_2 in parallel arranged in series with R_3 and R_4 in parallel. Hence R, the effective resistance of the combination, is given by

$$R = \frac{R_1R_2}{R_1 + R_2} + \frac{R_3R_4}{R_3 + R_4}.$$

When both the cell and galvanometer are restored, the galvanometer may be regarded as the 'load', so that by Thévenin's theorem

$$I_g = (V_H - V_K) \div (R + R_g)$$
$$= \frac{V(R_2R_3 - R_1R_4)}{(R_1 + R_2)(R_3 + R_4)} \div \left(\frac{R_1R_2}{R_1 + R_2} + \frac{R_3R_4}{R_3 + R_4} + R_g\right)$$
$$= \frac{V(R_2R_3 - R_1R_4)}{R_g(R_1 + R_2)(R_3 + R_4) + R_1R_2(R_3 + R_4) + R_3R_4(R_1 + R_2)}.$$

To verify this expression, let Maxwell's cyclic currents be as shown in Fig. 7·31 (*a*). Then

$$R_1(I_g - I_1) + R_3(I_g - I_2) + R_gI_g = 0,$$
$$R_1(I_1 - I_g) + R_2I_1 + 0 = V,$$

and

$$R_4I_2 + R_3(I_2 - I_g) + 0 = -V.$$

These equations give for I_g the expression obtained above.

Delta-star and star-delta transformations.—The solution of certain problems concerning networks may be made less arduous by using a so-called ***delta-star transformation.*** Consider, therefore, the two networks shown in Fig. 7·32 (*a*) and (*b*); they will be equivalent electrically if the resistance between any two terminals of the star network is the same as the resistance viewed from the corresponding terminals of the delta system.

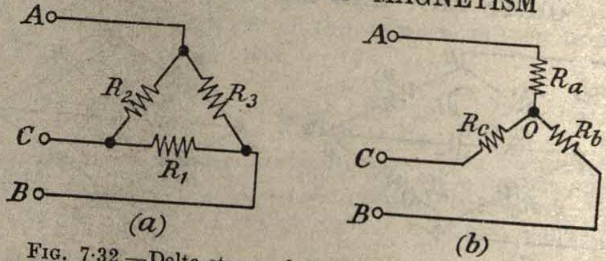


FIG. 7·32.—Delta-star and star-delta transformations.

Thus, between terminals A and B, we have

$$R_a + R_b = R_3(R_1 + R_2) \div (R_1 + R_2 + R_3), \qquad (i)$$

since in the delta system the resistances R_1 and R_2 form a series combination in parallel with R_3.

Similarly,

$$R_b + R_c = R_1(R_2 + R_3) \div (R_1 + R_2 + R_3), \qquad (ii)$$

and

$$R_c + R_a = R_2(R_3 + R_1) \div (R_1 + R_2 + R_3) \qquad (iii)$$

Adding equations (i) and (iii) and subtracting (ii), we find

$$R_a = R_2R_3 \div (R_1 + R_2 + R_3). \qquad (iv)$$

Similarly,

$$R_b = R_3R_1 \div (R_1 + R_2 + R_3), \qquad (v)$$

and

$$R_c = R_1R_2 \div (R_1 + R_2 + R_3). \qquad (vi)$$

Equations (iv), (v) and (vi) express the resistances of the equivalent star components in terms of those of the delta network.

To express the values of R_1, R_2 and R_3 in terms of the components of the star network, we may proceed as follows. From equations (iv)–(vi), we have

$$\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} = (R_1 + R_2 + R_3)\left[\frac{1}{R_2R_3} + \frac{1}{R_3R_1} + \frac{1}{R_1R_2}\right]$$
$$= \frac{(R_1 + R_2 + R_3)^2}{R_1R_2R_3}.$$

Hence

$$R_bR_c\left[\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}\right]$$
$$= \frac{R_3R_1}{(R_1 + R_2 + R_3)} \cdot \frac{R_1R_2}{(R_1 + R_2 + R_3)} \cdot \frac{(R_1 + R_2 + R_3)^2}{R_1R_2R_3}$$
$$= R_1.$$

Similar expressions for R_2 and R_3 are easily written down.

Example.—Find the equivalent resistance of the network shown in Fig. 7·33 (*a*) between points A and D by using a delta-star transformation on the loop ABC.

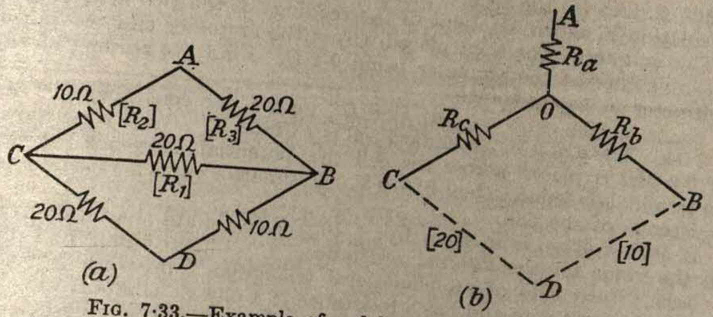


FIG. 7·33.—Example of a delta-star transformation.

When the delta ABC of Fig. 7·33 (a) is transformed into the equivalent star formation, the whole network assumes the form shown in Fig. 7·33 (b) where

$$R_a = \frac{10 \times 20}{20 + 20 + 10} = 4{\cdot}00\ \Omega,$$

$$R_b = \frac{20 \times 20}{20 + 20 + 10} = 8{\cdot}00\ \Omega,$$

and

$$R_c = \frac{20 \times 10}{20 + 20 + 10} = 4{\cdot}00\ \Omega.$$

Hence the resistance of the branch OCD is 24 Ω, while that of OBD is 18 Ω.

∴ Equivalent resistance between A and D, say R_{AD}, is given by

$$R_{AD} = 4{\cdot}00 + \frac{24 \times 18}{24 + 18} = 14{\cdot}3\ \Omega.$$

EXAMPLES VII

7·01. State Kirchhoff's laws for an electrical network fed from a z.f. supply.

A point D is situated within a triangle ABC and a network is formed by placing a cell in the arm AB. The resistance in this arm, including that of the cell is zero. The other arms contain resistances as follows: AC 1 ohm, BC 2 ohms, AD 2 ohms, BD 3 ohms, and CD 1 ohm. What resistance does the network offer to the cell? [1·87 ohm.]

7·02. A simple slide-wire potentiometer 100 cm. long and of resistance 10 ohms is connected to a storage cell of e.m.f. 2 V. and negligible internal resistance. The e.m.f. of a Leclanché cell is measured in the usual way and balance is obtained at a reading of 75 cm. on the potentiometer wire. If the internal resistance of the Leclanché cell is 5 ohms and the resistance of the galvanometer is 15 ohms, estimate the current that will flow through the galvanometer when the slider is moved to the 75·1 cm. mark on the potentiometer wire. [$9{\cdot}14 \times 10^{-5}$ amp.]

7·03. To measure a resistance by the voltmeter-ammeter method an ammeter was placed in series with a resistance R and a voltmeter was placed in parallel with R. The current measured was 1·2 A. and the voltmeter reading was 15·0 V. If the resistance of the voltmeter is 500 Ω. find (*a*) the true value of R, and (*b*) the percentage error in calculating R as the quotient of the readings of the two instruments.

The voltmeter was then placed across the ammeter and resistance, the resistance of the ammeter being 0·1 Ω. Find the reading of the voltmeter in this instance.

[(*a*) 12·8 Ω., (*b*) 2·4 per cent low, 15·2 V.]

7·04. The e.m.f. of a standard cell is 1·01862 V. When a resistance of 0·5 MΩ is placed across the cell the potential difference between its terminals is found to be 1·01810 V. Estimate a value for the internal resistance of the cell.

If each voltage is in error by ± 10^{-5} V. estimate the uncertainty in the value for the internal resistance. [246 ohm., 10 ohm.]

7·05. Each side of an equilateral triangular framework is made from a uniform wire and has a resistance of 4 ohms. Uniform wires of resistance 3 ohms join each apex to the mid-point of the opposite side and these 'cross-wires' are soldered together at the 'centre of gravity' of the triangle. If a current of 1 A. enters the framework at one apex and leaves at another, calculate the currents in each part of the framework. Hence show that the equivalent resistance is 1·45 ohm.

7·06. A network of wire is in the form of a rectangle, length na and width a. Parallel wires are arranged so that the network consists of n squares, all joints being perfect. A steady current enters the network at one extremity of a shorter side of the rectangle and leaves at the other. If R_n is the equivalent resistance of the network and r the resistance of a length a of the wire, prove that

$$\frac{1}{R_n} = \frac{1}{R_{n-1} + 2r} + \frac{1}{r}.$$

Verify that the resistances of networks consisting of one, two, or three squares are as 1 : 0·978 : 0·976.

7·07. P, Q, R, and S are the resistances in the four arms AB, BC, AD, and DC respectively of a Wheatstone bridge. A z.f. current I amp. enters at A and leaves at C. If G is the resistance of the galvanometer and i_g is the current from B to D through the galvanometer, prove that

$$i_g = \frac{RQ - PS}{G(P + Q + R + S) + (P + R)(Q + S)} \cdot I.$$

If the bridge is *nearly* balanced and the current I is supplied from a cell of e.m.f. V and negligible resistance, the effective resistance of the bridge may be taken as that of a resistance (P + Q) in parallel with (R + S). Show that

$$I = \frac{V(P + Q + R + S)}{(P + Q)(R + S)}.$$

[N.B.—The above problem is to indicate the method to be adopted in solving the next two problems.]

7·08. In a Wheatstone bridge which is nearly balanced the resistances of the arms are 100·01, 10, 1000 and 100 ohms. The cell used has an e.m.f. of 2 V. and negligible internal resistance. The galvanometer has a resistance of 100 ohms and sensitivity 10 cm. deflexion per μA. Calculate the galvanometer deflexion in the two possible arrangements of battery and galvanometer and indicate the direction of the current through the galvanometer. Comment briefly on your results. [0·83 cm. in each instance. Second arrangement gives lower heating effect in coil to be measured.]

7·09. State Kirchhoff's laws for electrical circuits.

A resistance is measured as 200 ohms with a post office box, using 1000 ohms and 100 ohms in the ratio arms and 2000 ohms in the variable arm. If a 2-volt accumulator is used with the ratio arms in series between its terminals and the resistance in the variable arm is changed from 2000 ohms to 2001 ohms, calculate the current through the galvanometer, of which the resistance is 100 ohms. (G)

[0·22 μA.]

7·10. Of the four arms P, Q, R, S, of a Wheatstone network, P and Q are ratio arms of resistances 1000 and 10 ohms respectively; S is a resistance coil of nominal value 15 ohms, and when R is made equal to 1500 ohms the balance appears to be exact. The cell (e.m.f. 2 V., internal resistance negligible) is connected to the junctions of P, R and Q, S. If the galvanometer, which is of resistance 100 ohms, can just detect a current of 10^{-7} ampere., between what limits does the resistance S lie? You may assume that the other resistances are accurate. (G)

[15 ± 0·01 ohm.]

7·11. R_1, R_2 and R_3 are respectively the resistances of the resistors placed in the arms AB, BC, and AD of a Wheatstone bridge. The arm CD contains a galvanometer of resistance R_g. The points A and C are connected to a cell of e.m.f. V, the total resistance of this branch being R_0. The points B and D are connected to a key. Prove that, if $R_g = \frac{R_2R_3}{R_1}$, the current through the galvanometer is independent of whether the key in BD is making contact or not, and that this current is

$$\frac{V(R_1 + R_2)}{R_0(R_1 + R_2 + R_3 + R_g) + (R_1 + R_2)(R_3 + R_g)}.$$

7·12. A leaky condenser behaves as if its terminals were connected through a very high resistance—this is termed the insulation resistance of the condenser. When such a condenser, of capacitance 1 μF., is connected between the quadrants of an electrometer and given a charge the deflexion is observed to fall from 210 mm. to 200 mm. in two minutes. On substituting between the quadrants a cell of e.m.f. 1·08 volts, a deflexion of 108 mm. is observed. From these data calculate values for (*a*) the charge lost by the condenser in two minutes; (*b*) the mean potential difference across the condenser; (*c*) the mean current flow; and (*d*) the insulation resistance of the condenser.

[10^{-7} coulomb., 2·05 V., $0{\cdot}83 \times 10^{-9}$ A., $2{\cdot}46 \times 10^{9}$ ohm.]

7·13. Two dynamos, A and B, each having an internal resistance of 0·1 ohm, feed the opposite ends of two conductors of uniform section, one conductor being connected to the positive poles of both dynamos and the other conductor to the negative poles. Each conductor is 900 metres long and has a resistance of 1·667 μΩ.metre.$^{-1}$. At a distance of 300 metres from the machine A a current of 100 A. is taken by a motor C connected to the conductors, while at a point situated 600 metres from A another motor D is similarly supplied with a current of 150 A. If the e.m.f. developed by A is 240 V. while that developed by B is 230 V., find the current given by each dynamo and the power dissipated in each motor. [174 A., 76 A.; 22·2 kW., 33·2 kW.]

7·14. ABCD is a four-arm network. The arms AB and AD contain adjustable resistances R_1 and R_2 respectively. Cells of e.m.fs. E_1 and E_2 and internal resistances r_1 and r_2 are placed in the arms BC and CD respectively. The points A and C are connected to a galvanometer.

By means of a diagram show how the cells must be arranged so that when R_1 and R_2 are adjusted no current shall flow through the galvanometer. What condition is then fulfilled? Show also that if balances are obtained for pairs of values (R_1, R_2), (R_1', R_2') of the variable resistances, then

$$\frac{E_1}{E_2} = \frac{R_1 - R_1'}{R_2 - R_2'}.$$

7·15. Describe and explain how a simple potentiometer may be used to compare two resistances. Hence show that by comparing the potential difference across a resistor AB with those across the two parts AC and CB into which the resistor may be divided, it is possible to determine the end-correction at the 'zero-end' of a potentiometer wire.

A telegraph wire connects two stations A and B, and at an intermediate point there is a leakage to earth through an unknown resistance. When A and B are each maintained simultaneously at the same potential relative to earth, the currents at A and B are in the ratio $\alpha : \beta$. Prove that the position of the fault divides AB in the ratio $\beta : \alpha$.

7·16. A simple series circuit ABCDA is made up of a battery AB of e.m.f. 12 volt and resistance 10 ohm, a coil BC of 40 ohm, a variable resistance CD of P ohm, and a battery DA of e.m.f. 8 volt and resistance 5 ohm. Show how the batteries must be connected so that a value for P may be found for which a galvanometer connected between A and C shows no deflexion. Calculate the value of P required for this, and the differences of potential then existing across each of the batteries.

[28·33 Ω., $V_{AB} = 9{\cdot}6$ V., $V_{AD} = 6{\cdot}8$ V.]

7·17. A current is sent through a resistance of 0·10 ohm in series with two accumulators arranged in opposition. The current is varied and the following readings are observed.

Potential difference across 0·1 ohm coil (mV.)	Potential difference across both accumulators (mV.)
1·9	4·4
2·2	5·0
3·3	5·4
4·8	6·9
7·0	8·0
11·8	11·8

Show that the internal resistance of the pair of accumulators is 0·074 ohm and find a value for the difference between their e.m.fs.

[$3{\cdot}1 \times 10^{-3}$ V.]

7·18. Obtain an expression connecting the resistance-temperature coefficient of a metal with temperature referred to 0° C., with the coefficient referred to 20° C.

A coil of copper wire at 20° C. has a resistance of 100 ohm. If the coefficient of increase in resistance with temperature for copper is $4{\cdot}26 \times 10^{-3}$ deg.$^{-1}$ C., obtain a value for the resistance of the coil at the temperature of steam when the corrected barometric height is 74·2 cm. of mercury.

[$\kappa = \kappa_0(1 + 20\kappa_0)^{-1}$, 131·1 ohm.]

7·19. A coil of iron wire, resistance 10 Ω, is connected in series with a coil of copper wire, resistance 20 Ω; this copper coil is shunted by a coil of iron wire. If the temperature rises everywhere from 0° C. to 50° C. the potential difference across the 10 Ω coil increases by 5 per cent. If the resistance-temperature coefficients for iron and for copper may be taken as 0·0060 deg.$^{-1}$ C. and 0·0040 deg.$^{-1}$ C., respectively, obtain a value for the resistance at 0° C. of the iron shunt. [Assume p.d. across system to be constant.] [332 ohm.]

7·20. A battery, e.m.f. V and internal resistance B, is connected in series with a high resistance and a sensitive galvanometer. The total resistance of the circuit excluding that of the battery is R and the galvanometer deflexion is θ_1. When a coil of resistance S is placed in parallel with the battery, the galvanometer deflexion is θ_2. Show that

$$B = \frac{SR(\theta_1 - \theta_2)}{(R + S)\theta_2 - S\theta_1}.$$

7·21. Two resistors A and B, made from different materials, have resistance-temperature coefficients of 0·0032 deg.$^{-1}$ C. and 0·00048 deg.$^{-1}$ C., respectively. If A has a resistance of 10·00 Ω at 0° C., calculate the resistance of B at the same temperature if it is such that when joined in series with A the resistance-temperature coefficient for the combination shall be 0·0020 deg.$^{-1}$ C. [7·90 Ω.]

7·22. Define *temperature on the platinum scale*. A platinum thermometer has a resistance of 10·091 ohms at 0° C., and 14·091 ohms at 100° C. At 444·60° C. its resistance is 27·060 ohms. If the variation of its resistance with temperature is represented by $R = R_0(1 + \alpha\theta + \beta\theta^2)$, determine the difference coefficient and the constants α and β. If the resistance of the above thermometer is 22·091 ohms at a certain temperature, what is that temperature (*a*) on the platinum scale, (*b*) on the gas scale?

[$\alpha = 3{\cdot}926 \times 10^{-3}$ deg.$^{-1}$ C., $\beta = -5{\cdot}85 \times 10^{-7}$ deg.$^{-2}$ C.
$\phi = 300{\cdot}00°$ C., 309·76° C.]

7·23. Give an account of *one* method for comparing small resistances. When a very small current passes through a certain conductor its resistance is R_1 and its temperature may be taken as ϕ° C., viz. that of its surroundings. When a large current is passed the resistance is R_2 and the temperature of the surrounding air is θ° C. If κ is the resistance-temperature coefficient referred to 0° C., show that the temperature of the wire exceeds that of its surroundings by

$$\frac{R_2}{R_1}\left(\phi + \frac{1}{\kappa}\right) - \left(\theta + \frac{1}{\kappa}\right).$$

7·24. AB is a uniform telegraph wire, resistance R ohms. At some unknown point C in the wire there is a fault, i.e. C is connected to earth through an unknown resistance r. Two measurements of the resistance to earth are made at the end A; the first, R_1, with the end B insulated and the second, R_2, with B earthed. If x is the resistance of the portion AC of the telegraph wire, $R_1 = x + r$, and $R_2 = x + \dfrac{r(R - x)}{r + R - x}$. Eliminating r from these equations, prove that

$$x = R_2 - \sqrt{(R_1 - R_2)(R - R_2)}.$$

7·25. In the network shown in Fig. Ex. 7·25 determine the current flowing from each battery and the potentials at the points A and B. Hence determine the ultimate charge on the positive plate of each condenser. It may be assumed that the condenser 'star-point' N, is initially without charge.

[0·1 A., $V_A = 7$ V., $V_B = 26$ V., 132 μC., 124 μC., 256 μC.]

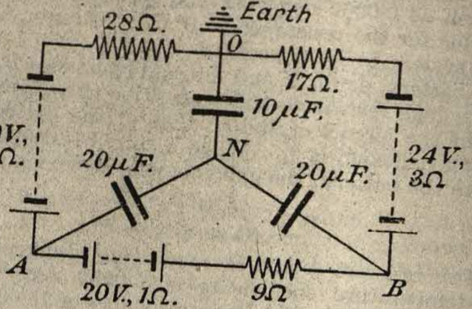


FIG. Ex. 7·25.

7·26. A cable with a copper core of diameter $2a$ is covered with a uniform layer of insulating material with an overall diameter $2b$. If l is the length of the cable and χ the resistivity of the insulation, show that the insulation resistance of the cable is

$$R = \frac{\chi}{2\pi l}\ln\left(\frac{b}{a}\right).$$

If $2a = 1{\cdot}5$ cm., $2b = 3{\cdot}0$ cm. and $R = 350$ MΩ for a cable of length 2000 metres, determine a value for χ. If the insulation resistance is to be raised to 500 MΩ by an additional uniform layer of insulation whose resistivity is $7{\cdot}2 \times 10^8$ MΩ.cm., what is the thickness of this layer?

[$6{\cdot}35 \times 10^{14}$ ohm.cm., 0·45 cm.]

7·27. A thick copper tube, radii r_1 and r_2 ($r_1 < r_2$), and length l is divided diametrically into two equal portions. One portion is used as a resistor, the current entering and leaving at the plane ends. If χ is the resistivity of copper, show that the resistance, R, of the resistor is given by

$$R = \frac{\pi\chi}{l}\,\frac{1}{\ln\left(\frac{r_2}{r_1}\right)}.$$

If $\chi = 1{\cdot}73 \times 10^{-6}$ ohm.cm., $r_1 = 4{\cdot}0$ cm., $r_2 = 6{\cdot}0$ cm. and $l = 4{\cdot}0$ cm., obtain a value for R.

[3·36 μΩ.]

7·28. The arms AB, BC, CD and DA of a Wheatstone bridge contain coils of resistance 2·0, 3·0, 2·0 and 3·0 ohms respectively. The point A is at a potential of 2·0 volts while C is at zero potential. If the points B and D are joined by a galvanometer of resistance 10 ohms, determine the magnitude and direction of the galvanometer current.

[$0{\cdot}032_3$ A. from B to D.]

7·29. Use Thévenin's theorem to determine the current I through the 10 Ω resistor in the circuit shown in Fig. Ex. 7·29. Verify your solution by using Maxwell's cyclic currents. If the connexions to the 6 V. cell are reversed, show that the current is from B to A and of magnitude 0·07A.

[0·21 A.]

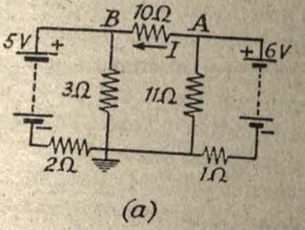


FIG. EX. 7·29.

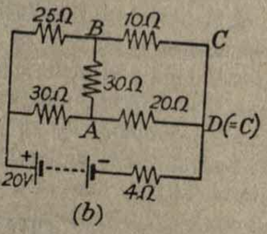


FIG. EX. 7·30.

7·30. By using a delta-star transformation on the loop ABCD of the circuit shown in Fig. Ex. 7·30, determine a value for the current from the battery. Check your solution by an independent method.

[0·80 A.]

7·31. A potentiometer wire has a resistance R_0 ohms and a steady potential difference V_0 volts is maintained across it. It is used to measure a voltage V, the total resistance of the galvanometer and source of V being R ohms. When the sliding contact is at a distance δl from the balance position l, show that the galvanometer current I_g is given approximately by

$$I_g = \pm \frac{\delta l . L V_0}{RL^2 + LlR_0 - R_0 l^2} \text{ amp.,}$$

where L is the total length of the potentiometer wire.

In a certain instance L = 100 cm. and the other circuit parameters are such that

$$|I_g| = \frac{20\,\delta l}{20{,}000 - 100l - l^2} \text{ amp.}$$

Plot a graph to show how $|I_g|$ varies with l for a given value of δl. If $\frac{|I_g|}{\delta l}$ expresses the sensitivity, and $|I_g| \div \left(\frac{\delta l}{l}\right)$ expresses the accuracy, use your graph to show that the best sensitivity is obtained when the balance point is near either end of the wire, and best accuracy combined with best sensitivity when the balance point is such that $l \rightarrow 100$ cm.

CHAPTER VIII

ELECTROMAGNETIC INDUCTION

Faraday's discovery of electromagnetic induction.—In 1831 FARADAY discovered that induced currents were set up in a closed circuit whenever a current in a neighbouring circuit was made or broken, i.e. when there was a change in the flux of magnetic induction through the closed circuit. For several years previous to this Faraday had failed to detect the presence of these currents, a fact due to the low sensitivity of the galvanometer he used. From his published account of this work it appears that his first successful attempt was carried out on the following lines. About 200 feet of copper wire were coiled round a large block of wood; a second long length of similar wire was then interposed as a helix between the turns of the first coil, twine serving as an insulator. One helix was connected to a galvanometer and the other to a battery. When the battery circuit, the so-called ***primary circuit***, was closed, 'there was a sudden and very slight effect (deflexion) at the galvanometer', i.e. there was an induced current of a transient nature produced in the galvanometer circuit, the so-called ***secondary circuit***. There was also a similar effect, but in a contrary sense when the primary current was broken. Faraday was very careful to emphasize the fact that the current in the secondary circuit is a transient one and that no current exists there when the current in the primary is fully established.

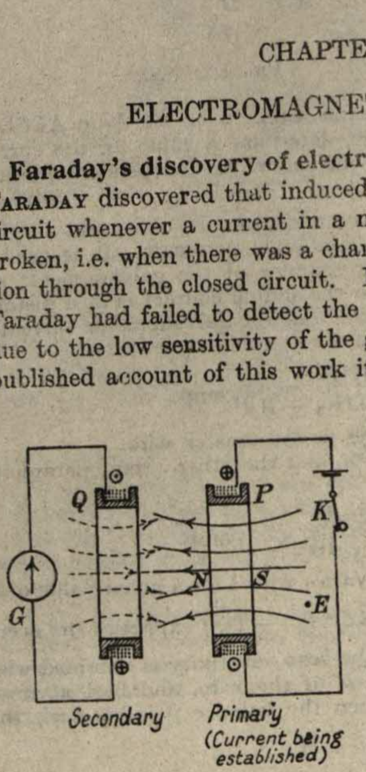
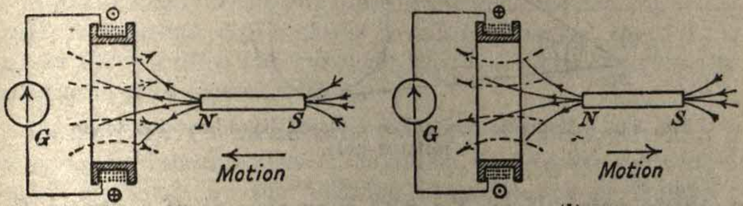


FIG. 8·01.—Faraday's discovery of electromagnetic induction.

The above results may be verified in the laboratory in the following manner. P, Fig. 8·01, is the primary coil connected to a battery and a key K. Q is the secondary coil connected to a ballistic galvanometer G. It will be found that when K is closed there is an induced current in the secondary circuit; also when the primary current is broken. Both these currents are of short duration but opposite in direction. It will also be noticed that there is no current in the secondary circuit when that in P is fully established. If, however, the current in P is increased there is a transient current in Q; it is in the same direction as that established when the primary is first closed. If the primary current is reduced, an induced current in the opposite direction is temporarily established in Q.

Let us suppose that the current in P is such that to an observer at E it appears to flow in a clockwise direction. Then the lines of magnetic induction are as indicated. Then the current in Q, when that in P is being established, is such that it appears to flow in an anti-clockwise direction to an observer at E; the lines of magnetic induction are shown by the dotted curves. The induced current is such that it tends to maintain constant the number of lines of magnetic induction threading the primary coil; this is a general principle applicable to all induced currents.



(*a*) (*b*)

FIG. 8·02.—Induced currents produced by the motion of a bar magnet near to a closed coil.

Faraday obtained similar results when a bar magnet was moved in the neighbourhood of a closed coil. Fig. 8·02 (*a*) shows the direction of the lines of magnetic induction due to the magnet NS and the lines of magnetic induction due to the induced current when NS approaches the coil. The direction of the induced current is such that the number of lines of induction threading the coil tends to remain constant. Fig. 8·02 (*b*) shows the direction of the induced current and its associated lines of magnetic induction when NS is being drawn away from the coil.

Magnetic flux.—If $\vec{B}$ is the magnetic induction at every point in an element of area δS, then $\vec{B}\cdot\hat{n}\,\delta S$, where $\hat{n}$ is unit vector along the outward-drawn normal, is called the flux of magnetic induction across δS (it is often called the magnetic flux). The unit of magnetic flux is the ***maxwell.***

A unit tube of magnetic induction.—If we consider any small tube of magnetic induction and apply Gauss' theorem to it, we can show, as for a tube of electric induction, that the flux of magnetic induction across any normal section of the tube is constant, i.e. B δS is a constant for a tube. If B δS is unity we have a ***unit tube of magnetic induction.***

Magnetic flux across a closed boundary due to an isolated

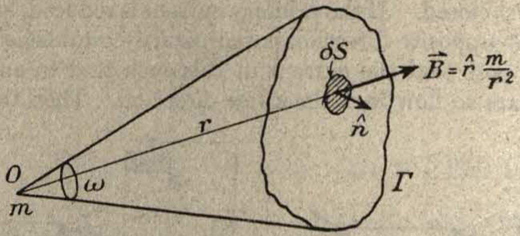


FIG. 8·03.—Magnetic flux across a closed boundary due to an isolated pole.

magnetic pole.—If δS, Fig. 8·03, is an element of a surface Γ, m a magnetic pole at O, distant r from δS, then the flux across δS is

$$\frac{m}{r^2}\hat{r}\cdot\hat{n}\,\delta S = \frac{m}{r^2}\cos(\hat{r}, \hat{n})\delta S$$

$$= m\,\delta\omega,$$

where $\delta\omega$ is the solid angle δS subtends at O.

The total flux across the surface is

$$\int m\,d\omega = m\int d\omega = m\omega = \Phi \text{ (say)}.$$

If the surface is closed and completely surrounds m, $\Phi = 4\pi m$. Hence the total flux from unit pole is 4π, and this is often called, somewhat erroneously, the 'number of lines of force' from a unit pole; it may be shown to be equal to the number of unit tubes. For consider a unit pole at the centre of a sphere of radius r. At its surface $B = \frac{1}{r^2}$, and if δS is the cross-section of a unit tube where it intersects the surface of the sphere, $B.\delta S = 1$.

$$\therefore \text{ Number of unit tubes from unit pole} = \frac{4\pi r^2}{\delta S} = \frac{4\pi}{B.\delta S} = 4\pi.$$

Magnetic linkages.—The flux through a boundary has already been defined as $\int \vec{B}\cdot\hat{n}\,dS = \Phi$. When a coil consists of N turns, wound closely together, the flux through each turn is still Φ, but the ***effective flux*** is $N\int \vec{B}\cdot\hat{n}\,dS = N\Phi$, or Ψ. It is called the number of ***linkages*** associated with the coil and is expressed in ***maxwell-turns.***

Lenz's law.—The facts stated previously with regard to the production of induced currents were summarized by LENZ in a law bearing his name. As modified by MAXWELL, it may be stated as follows: ***The e.m.f. induced in a circuit, when a magnet moves in its neighbourhood or the current in an adjacent circuit varies, tends to produce a current, the magnetic field due to which opposes any change in the value of the magnetic flux linked with that circuit.***

Lenz's law may also be expressed as follows: ***When the magnetic flux linked with a closed circuit is changing, there is set up in the circuit a current which tends to keep constant the magnetic flux linked with it.***

Fleming's right-hand rule.—When a conductor moves in a magnetic field the directions of the motion, the field, and the induced e.m.f. are given by the following statement due to FLEMING: ***If the thumb and first two fingers of the right hand are spread out so that they point in three directions at right angles to one another, the first finger giving the direction of the magnetic field, the thumb indicating the direction of the motion of the conductor, then the second finger indicates the direction of the induced e.m.f.***—cf. Fig. 8·04 (*a*).

[Strictly speaking, this rule is only applicable when the magnetic field is normal to the plane in which the displacement occurs. When it is not, the first finger must point in the direction of the component of the field normal to the plane in which the conductor moves.]

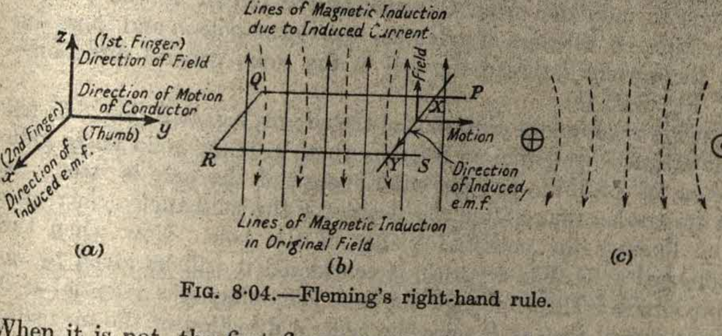

z
(1st. Finger)
Direction of Field
Direction of Motion of Conductor
(Thumb)
y
(2nd Finger)
x
Direction of Induced e.m.f.
(a)
Lines of Magnetic Induction due to Induced Current
Field
Q
P
X
Motion
R
Y
S
Direction of Induced e.m.f
Lines of Magnetic Induction in Original Field
(b)
⊕
⊙
(c)

FIG. 8·04.—Fleming's right-hand rule.

Fleming's right-hand rule deduced from Lenz's law.—Let PQRS, Fig. 8·04 (*b*), be a system of long wires, connected as indicated, and lying in a plane normal to the lines of magnetic induction in the medium. Let XY be a conductor bridging the arms of the above system. Suppose XY moves to the right. Then there is a tendency for the flux of magnetic induction through the closed circuit XQRY to increase. The induced current, by Lenz's law, will be such that the magnetic flux due to it tends to prevent the above increase, i.e. the lines of induction will be downwards [dotted in the diagram]. The current in XY must therefore flow from X to Y—cf. Fig. 8·04 (*b*) and (*c*). This direction coincides with that expressed by Fleming's right-hand rule.

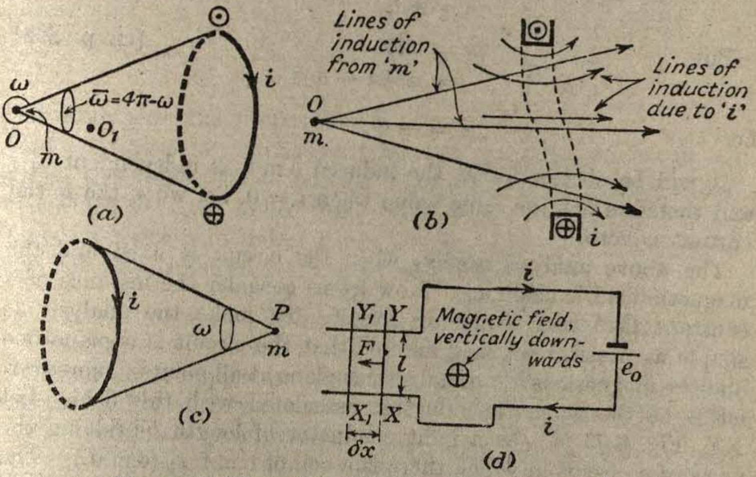
Faraday's law of electromagnetic induction.—FARADAY, by 1831, had established experimentally the conditions under which electromagnetic induction occurs. In 1845 NEUMANN expressed the results of Faraday's work by a formula giving the magnitude of the induced e.m.f. If the positive direction of flux and of e.m.f. are assumed to be related like the directions of translation and rotation of a right-handed screw, the formula giving the induced e.m.f. in magnitude and direction is

$$e = -\frac{d\Psi}{dt} = -\dot{\Psi}.$$

Deduction of Faraday's law from the principle of the conservation of energy.—HELMHOLTZ, and a little later KELVIN, showed that Faraday's law was a direct consequence of the principle of the conservation of energy. Their work appeared about 1850. Let us suppose that a single magnetic pole, of strength m, lies at a point O on the *south* side of a closed circuit of one turn in which there is a current i (e.m.u.)—cf. Fig. 8·05 (*a*). Then the flux of magnetic induction from O traverses the circuit in the positive direction, i.e. the lines of induction due to the pole m and those due to the current i thread the coil in the same general direction—cf. Fig. 8·05 (*b*).

When m lies on the *north* side of the circuit, e.g. at P, Fig. 8·05 (*c*), the magnetic potential at P is $i\omega$, where ω is the solid angle indicated, cf. p. 127. The potential energy of the pole is $mi\omega$. As m approaches the coil from the position P the solid angle ω increases and likewise the potential energy of the pole. Now when the potential at O, Fig. 8·05 (*a*), is considered it will be given by $i\omega$ provided ω is the solid angle shown. [The magnetic potential at O must exceed that at P since the general direction of the magnetic field is from left to right.] These facts are also consistent with the view that at a very distant point on the north side of the circuit the magnetic potential is zero, while at a very distant point on the south side it is $4\pi i$; the fact that these two potentials differ by $4\pi i$ although the points are each at infinity is in accord with Ampère's circuital relation—cf. p. 142. In this connexion it must be emphasized that for currents the magnetic potential at a point is multivalued, since it depends upon the number of times the circuit is 'threaded' by the path along which a small pole is moved. On the other hand, with a magnetic shell the potential is single valued and has a negative value at each point on the south side of the shell—cf. p. 125.

Now let the forces acting on m and due to the current in the circuit shown in Fig. 8·05 (a) cause the pole to move to O_1, i.e. to a point nearer the circuit so that the magnetic flux Φ increases.



ω
O
m
ϖ=4π-ω
•O₁
i
(a)
Lines of induction from 'm'
O
m.
Lines of induction due to 'i'
i
(b)
i
ω
P
m
(c)
i
Y₁ Y
F
l
Magnetic field, vertically downwards
e₀
X₁ X
δx
i
(d)

FIG. 8·05.—Deduction of Faraday's law, $\epsilon = -\dot{\Psi}$.

Let $\bar{\omega}$ be the solid angle defined by the equation

$$4\pi - \omega = \bar{\omega},$$

i.e. $\bar{\omega}$ is the solid angle which the circuit actually subtends at O. The angle $\bar{\omega}$ is selected so that when O becomes O_1, $\bar{\omega}$ becomes $\bar{\omega} + \delta\bar{\omega}$. Thus the potential energy of the pole at O is

$$mi\omega = mi(4\pi - \bar{\omega}).$$

As the pole moves from O to O_1 its increase in potential energy will be equal to the work done on it, i.e. the work done is

$$mi[\{4\pi - (\bar{\omega} + \delta\bar{\omega})\} - (4\pi - \bar{\omega})]$$

$$= -\,mi\,\delta\bar{\omega} = mi\,\delta\omega.$$

If this movement takes place in time δt, the rate at which work is done by the circuit on the pole is $mi\dot{\omega}$. As the pole moves, let the current in the circuit be maintained constant by increasing the e.m.f. of the generator from E to $(E + \epsilon)$. The additional rate of supply of energy from the generator is εi. By the principle of the conservation of energy we have, therefore,

$$mi\dot{\omega} + \varepsilon i = 0,$$

or

$$\varepsilon = - m\dot{\omega}.$$

The induced e.m.f., e, which is equal and opposite to ε, is given by

$$e = m\dot{\omega}.$$

But

$$\begin{aligned} \Phi &= m\bar{\omega} \qquad \text{[cf. p. 222]} \\ &= m(4\pi - \omega), \end{aligned}$$

i.e.

$$e = - \dot{\Phi}.$$

It will be observed that the induced e.m.f. is independent of i and therefore has the same value when $i = 0$, i.e. when the initial current is zero.

The above analysis applies when the circuit is fixed and the magnetic field is changing. Now let us consider the instance of a constant field and a moving circuit. To make the analysis as simple as possible we shall assume that the circuit is a plane one and the magnetic field normal to this plane at all points; moreover, let B be the magnetic induction associated with this field. Let XY, Fig. 8·05 (d), be a light conductor of length l bridging the arms of a circuit in which there is a cell of e.m.f. e_0 (e.m.u.). The force, F, on the wire is liB acting in the direction shown. Suppose that this force causes the wire to move to X_1Y_1, a distance δx in time δt. The work done by the cell in effecting this displacement is

$$\mathrm{F}.\delta x = li\mathrm{B}.\delta x.$$

Now in time δt, the energy supplied by the cell is $e_0 i.\delta t$, and if r is the resistance (e.m.u.) of the circuit the heat generated therein is $i^2 r.\delta t$ (in work units). By the principle of the conservation of energy

$$e_0 i.\delta t = i^2 r.\delta t + li\mathrm{B}.\delta x,$$

so that

$$i = \frac{e_0 - l\mathrm{B}.\dfrac{dx}{dt}}{r} = \frac{e_0 - lu\mathrm{B}}{r},$$

where u is the velocity with which XY is moving.

In other words, in addition to the electromotive force of the cell there is an induced e.m.f. equal to $- lu$B, which is minus the rate at which the flux through the circuit is increasing, i.e.

$$e = - \frac{d\Phi}{dt}.$$

If the moving conductor is not part of a circuit with finite resistance and it moves across a magnetic field as above, there will be a potential difference $-lu\mathrm{B}$ between its ends. To establish this, let r be the resistance of the moving conductor and suppose its ends are joined to a high resistance r_1. The current in the circuit will be $-\dfrac{d\Phi}{dt} \div (r + r_1)$, so that the potential difference across r_1 is

$$-\frac{d\Phi}{dt}.\frac{r_1}{r_1+r} = -\frac{d\Phi}{dt}, \qquad \text{if } r_1 \rightarrow \infty.$$

This establishes the proposition.

Here it must be emphasized that the equation $e = -\dot{\Phi}$ expresses the value of the induced e.m.f. in a closed linear circuit with which Φ is the linked flux. If there are N such circuits arranged in series with linked fluxes $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, then the total induced e.m.f. will be given by

$$e = -\left[\frac{d\Phi_1}{dt} + \frac{d\Phi_2}{dt} + \ldots + \frac{d\Phi_n}{dt}\right].$$

This equation applies to a coil of N turns since no turn in that coil can occupy exactly the same position as another. If $\Phi_1 = \Phi_2 = \ldots = \Phi_n = \Phi$ (say), then

$$e = -\mathrm{N}\frac{d\Phi}{dt},$$

and this will apply if the turns in the coil are equal and closely wound. Sometimes one writes $\Psi = \mathrm{N}\Phi$, where Ψ is the ***effective flux*** or ***number of linkages*** associated with the coil. Then

$$e = -\frac{d\Psi}{dt},$$

but when this is done it must be remembered that it is only changes in Φ which can give rise to an induced electromotive force.

The equation $e = -\dfrac{d\Phi}{dt}$.—The 'proofs' just given that $e = -\dot{\Phi}$ can only be regarded as plausible indications that this is a quantitative statement of Faraday's law of electromagnetic induction for the free energy associated with the magnetic field has not been considered and it may be shown that it is only by assuming this energy to remain constant that the proofs are valid. Accordingly they must be regarded as pointers to a rigorous proof and we have to consider the law as a basic assumption. Its successful application to electromagnetic theory and in measuring resistance in absolute units, cf. p. 697, are the real justifications for the use of this equation in physics and electrical engineering.

The quantity of electricity transformed when the effective flux linked with a circuit changes.—It has just been shown that if the magnetic flux linked with a closely wound coil of N equal turns is changing then

$$e = -\mathrm{N}\frac{d\Phi}{dt}.$$

If r is the resistance of the circuit of which the coil is part, the induced current will be given by

$$i = \frac{e}{r} = -\frac{\mathrm{N}}{r}\frac{d\Phi}{dt}.$$

Hence δq, the quantity of electricity set in motion in time δt, is given by

$$\delta q = i\,\delta t = -\frac{\mathrm{N}}{r}\frac{d\Phi}{dt}\delta t = -\frac{\mathrm{N}}{r}\delta\Phi.$$

$$\therefore q = -\frac{\mathrm{N}}{r}\int_{\Phi}^{\Phi_2} d\Phi = \frac{\mathrm{N}}{r}[\Phi_1 - \Phi_2].$$

These expressions are only valid if absolute units are used. If R is the resistance in ohms, then in the e.m. system $(r) = 10^9\ (\mathrm{R})$.

The expression for q given above shows that the total quantity of electricity induced is independent of the rate at which the effective flux changes, but inversely proportional to the resistance of the circuit. It must be remembered, however, that if q is to be measured with a ballistic galvanometer, the change in linkages must be completed before the suspended magnet (or coil) of the galvanometer has moved from its zero position.

Eddy currents.—Currents are not only induced in closed wire circuits when the number of lines of magnetic induction threading them varies but also in any conducting material placed in a varying magnetic field. These are termed ***eddy*** currents. These currents are frequently the source of much trouble in metal apparatus placed in a varying magnetic field. They may cause the metal to become very hot. This may be avoided to a great extent by building up the apparatus from flat metal strips insulated from one another so that the currents are reduced in magnitude.

In recent years advantage has been taken of these eddy currents to melt metals. The specimen is placed in a magnetic field which may alternate at from two kilocycles per second to ten megacycles per second. The field of lower frequency is produced mechanically while the latter is obtained with the aid of a thermionic valve. Not only does the melting take place rapidly, but alloys hitherto unobtainable may be prepared by placing the constituent metals in a high vacuum. Under such conditions the metals do not oxidize and an alloy may be formed.

The damping of a perfectly conducting wheel rotating in its own plane in a transverse magnetic field, when a circuit of resistance R ohms connects axle and rim.—It will be assumed that initially the wheel is rotating freely in a plane normal to the magnetic field with induction B. If I is the moment of inertia of the wheel about its axis of rotation, its kinetic energy, W, when making n revolutions per second is given by

$$W = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\pi a^2 \rho d . \frac{a^2}{2} . 4\pi^2 n^2,$$

where a is the radius of the wheel, d its thickness and ρ the density of its material. Hence

$$W = \pi^3 a^4 \rho\, d\, n^2.$$

If e is the magnitude of the induced e.m.f., then

$$e = \pi a^2 n B. \qquad \text{(e.m.u.)}$$

Now the rate of loss of energy of the wheel is equal to the rate at which energy is dissipated in the resistance, r e.m.u. or R ohms, i.e.

$$-\frac{dW}{dt} = \frac{e^2}{r} = \frac{\pi^2 a^4 n^2 B^2}{r}$$

$$= \frac{\pi^2 a^4 B^2}{r} . \frac{W}{\pi^3 a^4 \rho d} = \frac{B^2}{r(\pi \rho d)} . W.$$

If W_0 is the initial value of W, and if W is the energy at time t.

$$\ln \frac{W_0}{W} = \frac{B^2}{r(\pi \rho d)} t = \frac{B^2}{10^9 R \pi \rho d} . t.$$

$$\therefore \text{Half-period decay time} = \frac{(R \times 10^9)\pi \rho d}{B^2} \ln 2.$$

Also, if n_0 is the initial speed of the wheel, the current will be halved when $n = \frac{1}{2}n_0$, i.e. at time τ (say). Thus the energy will be $0{\cdot}25\, W_0$, when

$$\tau = \frac{10^9 R \pi \rho d}{B^2} \ln 4.$$

It should be noted that the total energy dissipated in the resistance before the wheel comes to rest is equal to the initial kinetic energy of the wheel.

Example.—A thin wire and spherical brass bob. mass m, constitute a pendulum of length a which swings under gravity in a plane normal to a uniform magnetic field with induction B. If the semi-vertical angle through which the pendulum swings is θ, obtain an expression for the maximum potential difference between the extremities of the pendulum.

Since the potential energy of the bob in the position 'θ' exceeds its potential energy in the position of static equilibrium by

$$mga(1 - \cos \theta) = 2mga \sin^2 \tfrac{1}{2}\theta,$$

the velocity u_0 of the bob as it swings through the above position is given by

$$\tfrac{1}{2}mu_0^2 = 2mga\sin^2\tfrac{1}{2}\theta,$$

i.e.

$$u_0 = 2\sqrt{ga}\sin\tfrac{1}{2}\theta.$$

Now the maximum velocity at points along the wire varies uniformly from 0 to u_0; let u be the maximum velocity at a point P at distance x from the fixed end of the pendulum and v the potential (e.m.u.) at P, that at the fixed end of the pendulum being zero. Then at a point Q in the wire, where PQ $= \delta x$, the potential will be $v + \delta v$, where

$$\delta v = \mathrm{B}u\,\delta x,$$

and $u = u_0\left(\dfrac{x}{a}\right)$. Hence v_0, the maximum potential difference between the extremities of the pendulum, is given by

$$v_0 = \int_0^a u_0\mathrm{B}\left(\frac{x}{a}\right)dx = u_0\mathrm{B}\left(\frac{a^2}{2a}\right) = \frac{1}{2}\,u_0\mathrm{B}a$$

$$= a\mathrm{B}\sqrt{ga}\sin\tfrac{1}{2}\theta \text{ e.m.u.}$$

In practical units this becomes

$$\mathrm{V}_0 = \frac{a\mathrm{B}\sqrt{ga}\sin\frac{1}{2}\theta}{10^8} \text{ volt.}$$

Example.—Calculate the force on one 'half' of a current carrying solenoid.

It will be assumed that the solenoid is so long that, with the usual notation, $\mathrm{H} = 4\pi ni$. If the permeability of the core on which the solenoid is wound is constant and equal to μ (whose value will generally not differ much from unity), the flux is $4\pi\mu ni(\pi a^2)$, where a is the radius of cross-section. Since the flux from a unit pole is 4π, the solenoid may be replaced by a magnet with pole-strength $\pi\mu nia^2$. When the solenoid is imagined to be severed into two portions, poles of strength $\pm\,\pi\mu nia^2$ may be assumed to appear at the 'junction'. Now the field due to one half of the solenoid is $2\pi ni$, so that the force on the other half is

$$(2\pi ni)\pi\mu nia^2 = 2\pi^2\mu n^2a^2i^2.$$

To obtain this formula by another method, let the solenoid be replaced by its equivalent magnetic shell. For a length δz of the solenoid, the effective current is $ni\,\delta z$, so that the intensity of magnetization in the shell is $\mathrm{J} = \mu ni$. At the junction between the two portions the surface density of the magnetism will be $\pm\,\mu ni$. The stress is $\dfrac{2\pi\sigma^2}{\mu} = 2\pi\mu n^2i^2$. Hence the force is

$$2\pi\mu n^2i^2 \times \pi a^2 = 2\pi^2\mu n^2a^2i^2,$$

as before.

The potential energy of a current-carrying coil lying in a magnetic field.—Let i be the current in the coil which is situated in a medium with permeability μ. Then the equivalent magnetic shell will have a strength ϕ such that $\phi = \mu i$, cf. p. 131. Now the potential energy of this shell, cf. p. 125, is

$$-\frac{\phi}{\mu} \times \text{flux of magnetic induction across the positive side of the shell}$$

$$= -i \times \text{flux of } \vec{B} = -i\int \vec{B}\cdot\hat{n}\,dS,$$

where $\hat{n}$ is a unit vector normal to an element δS of the equivalent shell. This equation has been derived since, in the sequel, it will be used on many occasions.

Mutual inductance.—Let A_1 and A_2 be two rigid circuits, in a medium of magnetic permeability μ, carrying currents i_1 and i_2 respectively. The strengths of the equivalent magnetic shells are μi_1 and μi_2. The mutual potential energy may therefore be written as

$$-\mu i_1 . \frac{\Psi_2}{\mu} \quad \text{or} \quad -\mu i_2 . \frac{\Psi_1}{\mu}, \qquad [\text{since } W = -\phi \times \text{flux of magnetic field strength}].$$

i.e. as

$$-i_1\Psi_2, \quad \text{or} \quad -i_2\Psi_1,$$

where Ψ_2 is the effective flux of magnetic induction through A_1 due to i_2, and Ψ_1 is the effective flux through A_2 due to i_1.

Now provided ferromagnetic materials are absent, Ψ_2 is directly proportional to i_2, so that we may write $\Psi_2 = m_2 i_2$, where m_2 is a constant depending only on the geometry of the system. Similarly $\Psi_1 = m_1 i_1$, and hence the mutual potential energy is

$$-m_1 i_1 i_2 \quad \text{or} \quad -m_2 i_1 i_2.$$

Thus $m_1 = m_2$, i.e. the flux of magnetic induction through A_1, due to unit current in A_2 is equal to the flux of magnetic induction through A_2 due to unit current in A_1. If we call $m = m_1$ (or m_2), the constant m is known as the ***coefficient of mutual induction*** or ***the mutual inductance of the two circuits.***

Since $\Psi = mi$, where i is the current in one coil and Ψ the flux through the other, and $e = -\dot{\Psi}$, we have $e = -m\frac{di}{dt}$. Thus m is numerically equal to the e.m.f. induced in the secondary when the current in the primary is changing at the rate of one e.m.u. of current per second.

In practical units, we may write $E = -M\frac{dI}{dt}$, so that in practical units the mutual inductance, M, of two coils is numerically equal to the e.m.f. (in volts) induced in one coil when the current in the other is changing at the rate of one ampere per second.

Units of mutual inductance.—(*a*) Two coils have unit mutual inductance if the e.m.f. induced in one of them is one e.m.u. of potential difference when the current in the other is changing at the rate of one e.m.u. of current per second.

(*b*) The practical unit of mutual inductance is the ***henry*** and two coils have a mutual inductance of one henry if the e.m.f. induced in one of them is one volt, when the current in the other is changing at the rate of one ampere per second.

Self-inductance.—When a current i flows in a circuit there is a definite number of linkages associated with that circuit. This number will be proportional to the current, since the magnetic induction is everywhere proportional to the current if ferromagnetic materials are absent and the frequency of the current is low (no skin-effect). Accordingly we may write for such a circuit

$$\Psi = li,$$

where l is a constant known as the self inductance of the circuit.

Since $e = -\frac{d\Psi}{dt} = -l\frac{di}{dt}$, we may say that the self inductance of a circuit is numerically equal to the e.m.f. induced in that circuit and expressed in e.m.u., when the current is changing at a rate of one e.m.u. per second.

In practical units we may write

$$\mathrm{E} = -\mathrm{L}\frac{d\mathrm{I}}{dt},$$

where L is the self-inductance in henries, and a circuit has a self-inductance of one henry if the induced e.m.f. is one volt when the current is changing at the rate of one ampere per second.

Self inductance of a solenoid.—(*a*) Let us calculate the self-inductance of a long uniformly wound solenoid. The solenoid must be narrow compared with its length λ, so that any want of uniformity in the magnetic field inside the solenoid may be neglected. The magnetic induction in such a solenoid, if the permeability of the core on which it is wound is μ, which for all non-ferromagnetic materials may be assumed constant and equal to unity, is equal to $4\pi\mu ni$, where i is the current in e.m.u. and n is the number of turns per cm. length of the solenoid. If a is the radius of the coil, the magnetic flux across any section and hence through each turn of the coil is $4\pi\mu ni.\pi a^2$. The effective flux through the coil is therefore $4\pi^2\mu na^2i.n\lambda = 4\pi^2\mu n^2a^2\lambda i$. The self inductance, which is the number of linkages per unit current, is therefore given by

$$l = 4\pi^2\mu n^2a^2\lambda \text{ e.m.u. [cm.].}$$

In practical units, cf. p. 238,

$$\mathrm{L} = 4\pi^2\mu n^2a^2\lambda \times 10^{-9}\ \mathrm{H.} = [4\pi^2\mu n^2a^2\lambda \times 10^{-3}]\,\mu\mathrm{H.}$$

(*b*) Let us now calculate the self inductance of a short and narrow solenoid on the assumption that the magnetic induction is constant over each cross-section of the solenoid normal to its axis; of course, in general, the induction will be different for the various cross-sections. Let O, Fig. 8·06, be the centre of the solenoid and P a point on its axis, where OP $= x$. If $2a$ is the diameter of the solenoid and 2λ its length, the magnetic induction at P will be given by $B = \mu H$, where μ is the permeability (assumed constant) of the non-ferromagnetic core on which the solenoid is wound and, cf. p. 141,

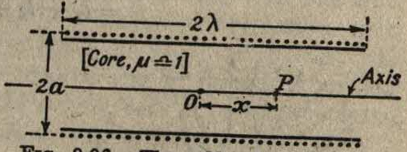


FIG. 8·06.—The self inductance of a short and narrow solenoid.

$$H = 2\pi ni\left[\frac{\lambda + x}{\{(\lambda + x)^2 + a^2\}^{1\cdot 5}} + \frac{\lambda - x}{\{(\lambda - x)^2 + a^2\}^{1\cdot 5}}\right].$$

Since this is assumed constant over the cross-section at P, the flux of induction across this section is given by

$$\Phi = 2\pi^2\mu a^2 ni\,[.\ .\ .],$$

i.e. the number of linkages associated with a length δx of the solenoid is $\Phi n\,\delta x$. Hence, for the whole solenoid, the number of linkages is given by

$$\begin{aligned}\Psi &= 2\int_0^\lambda \Phi n\,dx = 4\pi^2\mu a^2 n^2 i\int_0^\lambda\Big[.\ .\ .\Big]dx\\ &= 4\pi^2\mu a^2 n^2 i\left[\Big\{(\lambda + x)^2 + a^2\Big\}^{0\cdot 5} - \Big\{(\lambda - x)^2 + a^2\Big\}^{0\cdot 5}\right]_0^\lambda\\ &= 4\pi^2\mu a^2 n^2 i\,[(4\lambda^2 + a^2)^{0\cdot 5} - a].\end{aligned}$$

$$\therefore \text{Self inductance} = l = \frac{\Psi}{i} = 4\pi^2\mu a^2 n^2[(4\lambda^2 + a^2)^{0\cdot 5} - a] \text{ e.m.u.}$$

In practical units this becomes

$$L = 4\pi^2\mu a^2 n^2\,[(4\lambda^2 + a^2)^{0\cdot 5} - a] \times 10^{-9} \text{ henry},$$

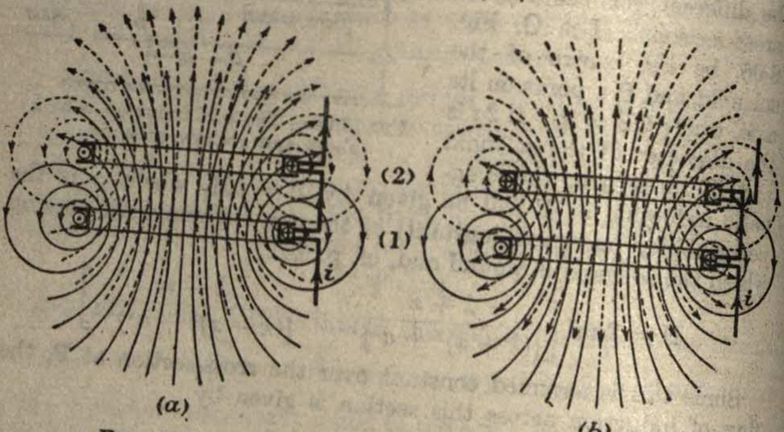
and μ may be taken as unity.

Positive and negative mutual inductances.—If two coils (1) and (2), of self-inductances l_1 and l_2 respectively, are connected in series the effective flux through the coil (2), say, may or may not be in the same sense as the flux through (2) due to the current in it. In the former case the mutual inductance, M, is said to be ***positive***; in the other it is ***negative.***

When m is positive, cf. Fig. 8·07 (*a*), and the two coils carry a current i, the number of linkages associated with coil (1) is

$l_1 i + |m| i$; for coil (2) it is $l_2 i + |m| i$. Thus the total number of linkages for both coils is $l_1 i + 2|m| i + l_2 i$, i.e. the effective self inductance, which is the flux per unit current, is given by

$$l' = l_1 + l_2 + 2|m|.$$



(a) (b)

FIG. 8·07.—Positive and negative mutual inductances.

In Fig. 8·07 (b) the mutual and self-produced fluxes are in opposition at any instant, i.e. m is negative, and we have

$$l'' = l_1 + l_2 - 2|m|.$$

Thus if l' and l'' are measured, we have a means of finding $|m|$, for $l' - l'' = 4|m|$.

Similarly, if the inductances are expressed in henries,

$$L' - L'' = 4|M|.$$

The N.P.L. primary standard of mutual inductance.—In a primary standard of mutual inductance, as the value is calculated from the dimensions, the design must be such as to allow of accurate measurement of the important dimensions. Also the nature of the materials and the construction must be such as to ensure permanency. There is always considerable uncertainty in determining the dimensions of many-layered coils, and modern experience has shown that a single-layered coil wound with bare wire in a screw thread on a cylinder of fused quartz gives the nearest approach to ideal conditions. If, however, both primary and secondary circuits are single-layer coils, it is not easy to obtain a value of the mutual inductance large enough for practical purposes. In Campbell's type of fixed mutual inductance the primary only is single layered, while the secondary is a many-layered coil of many turns, but the geometrical design of the system is such that the dimensions of the secondary coil do not need to be determined with high accuracy, as the accuracy of the calculated value for the mutual inductance is made to depend much more on the measurement of the primary. This consists of two equal single-layer coils BC and GH, Fig. 8·08, arranged in series and wound on the same cylinder of fused quartz, the gap between them being twice the axial length JK of one of the coils. To make these coils the copper wire is wound in uniform grooves which are 'lapped' and the wire itself has a cross-sectional area which is remarkably constant. The secondary consists of a channel-wound coil Q coaxial with the primary coils and placed midway between them; the former on which the secondary is wound consists of a glass disc, circular in section, and mounted coaxially between two slightly larger glass discs; the surfaces of these discs are optically flat. The central winding of the secondary lies in a region throughout which the magnetic field due to the current in the primary is zero for all values of the current. For this standard at 20° C.,

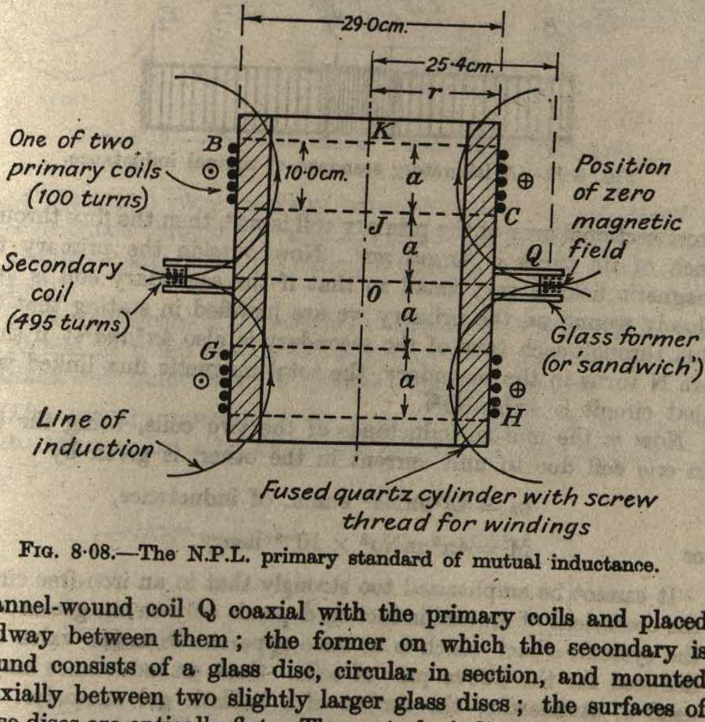

29·0cm.
25·4cm.
r
K
B
One of two primary coils (100 turns)
10·0cm.
a
J
C
Q
Position of zero magnetic field
Secondary coil (495 turns)
O
a
a
a
G
H
Glass former (or 'sandwich')
Line of induction
Fused quartz cylinder with screw thread for windings

FIG. 8·08.—The N.P.L. primary standard of mutual inductance.

$$M = (10{,}006{\cdot}52 \pm 0{\cdot}03)\mu\text{H}.$$

[The original Campbell standard was set up in 1907 but in 1962 it was replaced by the one just described; the temperature effect is almost negligible and the uncertainty in the estimated error has been reduced three times. These improvements are due to RAYNER.]

Other 'standards' of mutual inductance.—A practical standard of mutual inductance, cf. Fig. 8·09, is constructed by winding a coil of many turns of wire (the secondary S_1S_2) on the middle portion of a long uniformly wound solenoid (the primary P_1P_2). Let this solenoid have n turns per unit length, and suppose that the current through it is i. If μ is the permeability of the core on which the primary is wound and for non-ferromagnetic materials μ may be considered constant with a numerical value equal to unity, the magnetic induction inside the solenoid is $4\pi\mu ni$ and it is uniform over any cross-section near the middle of the coil. If the cross-sectional area of the primary coil is πa^2, then the flux through each of its turns is $4\pi\mu ni . \pi a^2$. Now outside the primary the magnetic field is very small so that if the secondary coil is very closely wound on the primary we are justified in stating that the flux through each turn of the secondary is also $4\pi^2\mu na^2i$; if there are N turns in the secondary, the total magnetic flux linked with that circuit is $4\pi^2\mu n\mathrm{N}a^2i$.

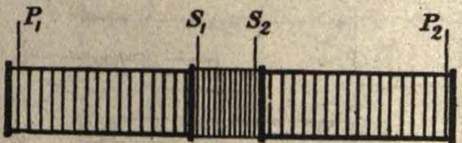


FIG. 8·09.—A laboratory standard of mutual inductance.

Now m the mutual inductance of the two coils, being the flux in one coil due to unit current in the other, is given by

$$m = 4\pi^2\mu n\mathrm{N}a^2 \text{ e.m.u. of inductance,}$$

or

$$\mathrm{M} = 4\pi^2\mu n\mathrm{N}a^2 \times 10^{-9} \text{ henry.}$$

It cannot be emphasized too strongly that in an iron-free circuit the coefficient of mutual induction depends only on the geometry of the system, a fact which is at once apparent from the value of m deduced for the above pair of coils. Such an inductance may be used to standardize the throws of a ballistic galvanometer.

Neglecting the dimensions of μ, cf. p. 688, those of m are $[m] = \text{cm.}^{-1}\text{cm.}^2 = \text{cm.}$, and thus we see that the e.m.u. of inductance is 1 cm.

It must be pointed out that the above mutual inductance is not even a second-class standard on account of the departures which occur in practice from the assumptions made in deriving the formula.

The Campbell mutual inductometer.—A standard but variable mutual inductance is known as a *mutual inductometer.* In its simplest form it is sufficient to arrange that one of two circuits can be moved relatively to the other and that its position may be read off on a scale which is graduated, after comparison with a standard of mutual inductance, to give the corresponding mutual inductance in millihenries.

The general arrangement of the coils is shown in Fig. 8·10 (*a*) and (*b*). The primary circuit consists of two coils C_1 and C_2 which are coaxial and arranged in series. S is the secondary coil which moves in a horizontal plane midway between C_1 and C_2 and about an eccentric axis through O; this axis is parallel to that of C_1 and C_2. A pointer P attached to S indicates the mutual inductance of the system.

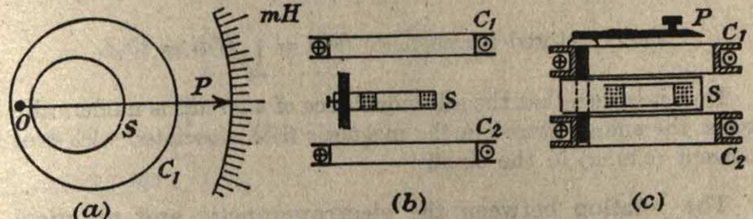


FIG. 8·10.—A variable mutual inductometer (Campbell).

The arrangement of the coils is shown more clearly in Fig. 8·10 (*c*). It should be noted that if all three coils are connected in series a variable self inductance, of limited range, is obtained.

The energy possessed by a coil on account of its self inductance.—Let e be the steady potential difference applied to an inductive coil which may be regarded as a pure resistance r in series with a pure inductance l, cf. Fig. 8·11 (*a*). This circuit may be replaced by that shown in Fig. 8·11 (*b*) in which it is imagined that a cell of e.m.f. $e - l\frac{di}{dt}$ is applied to a pure resistance r.

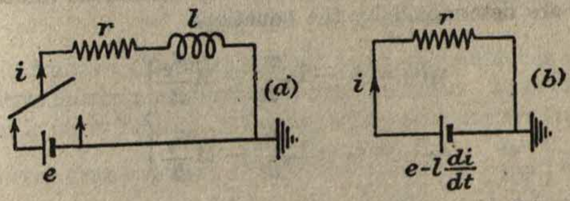


FIG. 8·11.—Energy possessed by a coil on account of its self inductance.

Then, by Kirchhoff's second law,

$$e - l\frac{di}{dt} = ri, \quad \text{or} \quad e = ri + l\frac{di}{dt}.$$

Multiplying throughout by $i\,\delta t$, we have

$$ei\,\delta t = ri^2\,\delta t + li\,\delta i,$$

i.e. the energy supplied from the cell = heat developed (ergs)
+ energy stored in the magnetic field.

If the current reaches a steady value i_0, we have

$$\text{Energy stored in magnetic field} = \int_0^{i_0} li\,di = \tfrac{1}{2}li_0^2.$$

Thus it is seen that the self inductance of a circuit is numerically twice the energy stored in the magnetic field associated with unit current (e.m.u.) in the circuit.

The relation between the electromagnetic and practical units of inductance.—Consider the energy associated with a coil of self inductance, (l) e.m.u. or (L) henries, where (l) and (L) are numbers. When the current is 1 e.m.u. (10 amps) the energy is $\tfrac{1}{2}(l)(1^2)$ erg. or $\tfrac{1}{2}(\mathrm{L})(10^2)$ joule $= \tfrac{1}{2}(\mathrm{L})(10^9)$ erg.

$$\therefore\ (\mathrm{L}) = (l) \times 10^{-9}.$$

Similarly, $(\mathrm{M}) = (m) \times 10^{-9}$.

To prove that $\mathrm{M}^2 < \mathrm{L}_1\mathrm{L}_2$.—Let e_1 and e_2 be the instantaneous values of the e.m.fs. in two circuits; i_1 and i_2 the instantaneous currents (practical units). It will be assumed that the circuits are fixed so that L_1, L_2, and M are constants and it must be remembered that M is considered algebraically. The instantaneous values of the currents are determined by the equations

$$\left.\begin{aligned} i_1\mathrm{R}_1 &= e_1 - \mathrm{L}_1\frac{di_1}{dt} - \mathrm{M}\frac{di_2}{dt}, \\ \text{and}\qquad i_2\mathrm{R}_2 &= e_2 - \mathrm{L}_2\frac{di_2}{dt} - \mathrm{M}\frac{di_1}{dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(i)}$$

The rate at which energy is being stored in the medium is given by

$$\begin{aligned} \frac{d\mathrm{W}}{dt} &= e_1i_1 - i_1^2\mathrm{R}_1 + e_2i_2 - i_2^2\mathrm{R}_2 \\ &= i_1\mathrm{L}_1\frac{di_1}{dt} + i_1\mathrm{M}\frac{di_2}{dt} + i_2\mathrm{L}_2\frac{di_2}{dt} + i_2\mathrm{M}\frac{di_1}{dt} \quad . \quad . \quad . \quad \text{(ii)} \end{aligned}$$

$$\therefore\ \mathrm{W} = \int d\mathrm{W} = \tfrac{1}{2}\mathrm{L}_1\Big[i_1^2\Big]_{t=t} + \mathrm{M}\Big[i_1i_2\Big]_{t=t} + \tfrac{1}{2}\mathrm{L}_2\Big[i_2^2\Big]_{t=t},$$

where the square brackets denote that the quantity enclosed is the value at time t, and there is no integration constant since $\mathrm{W} \rightarrow 0$

when i_1 and i_2 are each zero. Now W is always positive and if, for brevity, we call $\Big[i_1\Big]_{t=t} = \mathrm{I}_1$, etc., we may write

$$2\frac{\mathrm{W}}{\mathrm{L}_1} = \mathrm{I}_1{}^2 + 2 . \frac{\mathrm{M}}{\mathrm{L}_1}\mathrm{I}_1\mathrm{I}_2 + \left[\left(\frac{\mathrm{M}}{\mathrm{L}_1}\mathrm{I}_2\right)^2\right] + \frac{\mathrm{L}_2}{\mathrm{L}_1}\mathrm{I}_2{}^2 - \left[\left(\frac{\mathrm{M}}{\mathrm{L}_1}\mathrm{I}_2\right)^2\right]$$

$$= \left(\mathrm{I}_1 + \frac{\mathrm{M}}{\mathrm{L}_1}\mathrm{I}_2\right)^2 + \left(\frac{\mathrm{L}_2}{\mathrm{L}_1} - \frac{\mathrm{M}^2}{\mathrm{L}_1{}^2}\right)\mathrm{I}_2{}^2.$$

Let $\mathrm{I}_1 = -\frac{\mathrm{M}}{\mathrm{L}_1}\mathrm{I}_2$. Then since W is positive for all values of I_1 and I_2, $\mathrm{M}^2 < \mathrm{L}_1\mathrm{L}_2$.

The coefficient of coupling.—Consider two coils, having respectively N_1 and N_2 turns, which are so close together that the effective flux in one coil is completely linked with the other. If Φ is the flux in each turn of (1) when the current is i_1, the linkages for coil (1) number $\mathrm{N}_1\Phi$. The self inductance of this coil is therefore $l_1 = \frac{\mathrm{N}_1\Phi}{i_1}$. The mutual inductance is $\frac{\mathrm{N}_2\Phi}{i_1} = m = \frac{\mathrm{N}_2}{\mathrm{N}_1}l_1$.

Similarly, if i_2 is the current in (2) and Φ' the flux, $l_2 = \frac{\mathrm{N}_2\Phi'}{i_2}$, and $m = \frac{\mathrm{N}_1\Phi'}{i_2} = \frac{\mathrm{N}_1}{\mathrm{N}_2}l_2$.

$$\therefore\ m^2 = l_1 l_2, \quad \text{or} \quad \mathrm{M} = \sqrt{\mathrm{L}_1\mathrm{L}_2}.$$

As stated above, this relationship is only true when the whole of the effective flux from one coil links with the other. In practice this condition is never fulfilled, although it is approximately true if the coils are very close together. The ratio $\frac{\mathrm{M}}{\sqrt{\mathrm{L}_1\mathrm{L}_2}}\left(\text{or } \frac{m}{\sqrt{l_1 l_2}}\right)$ is known as the *coefficient of coupling* and is of importance in radio work. When it tends to unity, the two coils are said to be 'close-coupled'; when it is considerably less than unity the coils are 'loose-coupled'.

To prove $\mathrm{M}_{12} = \mathrm{M}_{21}$.—Energy considerations enable us to establish the fact that $\mathrm{M}_{12} = \mathrm{M}_{21}$, where M_{12} is defined by the statement that the e.m.f. induced in (1) is $-\mathrm{M}_{12}\frac{di_2}{dt}$ when the current in (2) is varying. A similar definition applies to M_{21}. Consider two neighbouring circuits defined by the parameters i_1, L_1, R_1 and i_2, L_2, R_2 respectively.

If e_1 and e_2 are the instantaneous values of the applied voltages, we have

$$\left.\begin{aligned} i_1R_1 &= e_1 - L_1\frac{di_1}{dt} - M_{12}\frac{di_2}{dt} \\ \text{and}\qquad i_2R_2 &= e_2 - L_2\frac{di_2}{dt} - M_{21}\frac{di_1}{dt}. \end{aligned}\right\} \quad . \quad . \quad \text{(i)}$$

$\therefore$ Energy given to surrounding medium in time δt is

$$(i_1L_1 + i_2M_{21})\delta i_1 + (i_2L_2 + i_1M_{12})\delta i_2 = \delta W \quad . \quad \text{(ii)}$$

Since the state of the medium at any instant is determined by i_1 and i_2 alone, δW must be a complete differential, i.e.

$$\frac{\partial}{\partial i_2}(i_1L_1 + i_2M_{21}) = \frac{\partial}{\partial i_1}(i_2L_2 + i_1M_{12}).$$

$$\therefore M_{21} = M_{12} \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

This proof is valid even when ferromagnetic materials are present in the neighbourhood of the circuits.

The energy due to currents in two rigid coils which are not fixed.—If M is the mutual inductance, the e.m.f. induced in the first circuit due to the changing current in the second is $\frac{d}{dt}(Mi_2)$. Equations (i) in the previous paragraph then become

$$i_1R_1 = e_1 - L_1\frac{di_1}{dt} - \frac{d}{dt}(Mi_2) \quad . \quad . \quad . \quad \text{(iv)}$$

and

$$i_2R_2 = e_2 - L_2\frac{di_2}{dt} - \frac{d}{dt}(Mi_1) \quad . \quad . \quad . \quad \text{(v)}$$

Multiplying (iv) by $i_1\,\delta t$ and (v) by $i_2\,\delta t$, we find

$$\begin{aligned} \delta W &= L_1i_1\,\delta i_1 + L_2i_2\,\delta i_2 + i_1\,\delta(Mi_2) + i_2\,\delta(Mi_1) \\ &= L_1i_1\,\delta i_1 + L_2i_2\,\delta i_2 + \delta(Mi_1i_2). \end{aligned}$$

Hence $W = \frac{1}{2}L_1i_1^2 + Mi_1i_2 + \frac{1}{2}L_2i_2^2 \quad . \quad . \quad . \quad . \quad \text{(vi)}$

This equation gives the energy stored in the field up to the time when the currents are i_1 and i_2 respectively, and M is the value of the mutual inductance at that time.

If M is fixed, the expression for W remains unaltered.

The measurement of the magnetic induction associated with a strong magnetic field.—Let N and S, Fig. 8·12, be the pole-pieces of a large electromagnet. In order to measure the magnetic induction B in the region between the pole-pieces a small coil, known as a search-coil, is placed in the gap with its plane normal to the field. This coil, with N_1 turns each of cross-sectional area A_1, is connected to a ballistic galvanometer, G, an adjustable resistor R_2 and the secondary of a mutual inductance M. The resistor R_2 is to limit, if necessary, the throws of the galvanometer and if G is a moving coil instrument, the total resistance of the secondary circuit must exceed a certain value if the galvanometer is to remain ballistic, cf. p. 339. When the search-coil is rapidly withdrawn from the field there is a change in the number of linkages associated with the coil and the quantity of electricity produced may be measured by G, provided this instrument is standardized. To do this a known current I is quickly reversed in the primary of the mutual inductance M and the corresponding throw of the galvanometer coil (or magnet) noted.

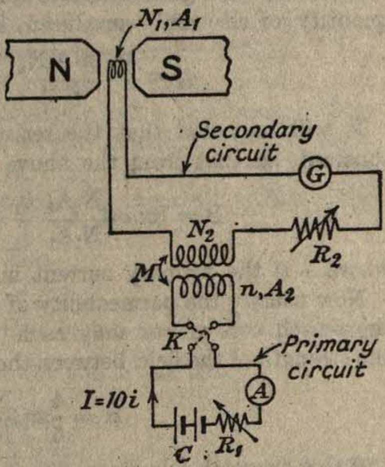


FIG. 8·12.—Measurement of a strong magnetic field.

If a value for M is to be calculated, then the mutual inductor must consist of a long solenoid, with n turns per unit length and a cross-sectional area A_2 and upon this there is wound the secondary, a coil with a large number of turns N_2. Since the solenoid is necessarily finite in length, the secondary should be wound as close as possible to it in order that the flux due to the field outside the primary shall be negligible.

When the search coil is rapidly withdrawn from between N and S, the change in the effective flux associated with the coil is N_1A_1B and if r (e.m.u.) is the resistance of the secondary circuit, the quantity of electricity, q_1, produced is

$$q_1 = \frac{N_1A_1B}{r} = \kappa\theta_1\left(1 + \frac{\lambda}{2}\right),$$

if θ_1 is the throw of the galvanometer, κ a constant and λ the logarithmic decrement per half cycle.

In order to determine κ, the current in the primary of the inductor is reversed and the throw θ_2 observed. If i (e.m.u.) is the current in the primary, the magnetic field due to it is given by $H = 4\pi ni$. Hence the flux through each turn in the primary is $4\pi\mu niA_2$, where μ is the permeability of the core on which the primary is wound and A_2 its cross-sectional area. The number of linkages associated with the secondary is therefore $4\pi\mu niA_2N_2$ and when the current is reversed the change in effective flux is $8\pi\mu niA_2N_2$. Hence q_2, the quantity of electricity produced, is given by

$$q_2 = \frac{8\pi\mu niA_2N_2}{r} = \kappa\theta_2\left(1 + \frac{\lambda}{2}\right).$$

It will be noted that the resistance of the secondary is kept constant, so that from the above two equations, we have

$$B = 8\pi\mu ni\frac{N_2A_2}{N_1A_1}\cdot\frac{\theta_1}{\theta_2} = \frac{4}{5}\pi\mu nI\frac{N_2A_2}{N_1A_1}\cdot\frac{\theta_1}{\theta_2},$$

where I is the primary current in amperes.

Now usually the permeability of the core and of the air in which the search coil moves may each be taken as unity. Hence H, the strength of the field between the poles of the magnet, is given by

$$H = \frac{4}{5}\pi nI\frac{N_2A_2}{N_1A_1}\cdot\frac{\theta_1}{\theta_2},$$

and this expression is dimensionally correct, cf. p. 691. It should be noted that the expressions for B and H do not involve the logarithmic decrement λ.

The motion of a metal ring falling through a horizontal radial magnetic field.—Fig. 8·13 is a section of an electromagnet, M, in which the pole faces are separated by a narrow cylindrical gap.

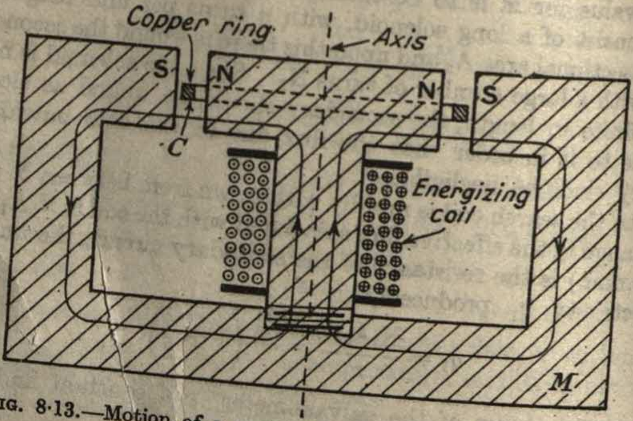


FIG. 8·13.—Motion of a metal ring in a radial magnetic field.

This gap is crossed by lines of magnetic induction which spread out radially from the central core. C is a metal ring placed over the gap and allowed to fall from rest. As the ring falls an e.m.f. will be induced in it. The reaction between the current thereby caused and the radial magnetic field will be a force tending to prevent the motion of the ring, i.e. the ring will fall more slowly than it would if the motion took place in the earth's gravitational field alone. To examine this problem quantitatively, we proceed as follows.

Let B be the magnetic induction in the annular air-gap, A the cross-sectional area of the ring (in a plane parallel to its axis), s its perimeter, χ the resistivity (in ohm.cm.) and ρ the density of its material. At time t let the velocity downward of the ring be u. Consider an element of the ring of length δs. The e.m.f. in this element is $B(\delta s)u$, so that for the whole ring the e.m.f. is Bsu. The current induced is given

$$i = \frac{Bsu}{r},$$

where r is the resistance of the ring (e.m.u.). If A is small, the resistance may be taken as $\frac{\chi s}{A} \times 10^9$ e.m.u., so that

$$i = \frac{BuA}{\chi \times 10^9}.$$

The force on the ring acts upwards and is given by

$$siB = \frac{B^2uAs}{10^9\chi}.$$

Hence if m is the mass of the ring, its equation of motion is

$$m\frac{du}{dt} = mg - \frac{B^2uAs}{10^9\chi},$$

where g is the intensity of gravity. Since $m = \rho As$, we have

$$\frac{du}{dt} + \frac{B^2}{10^9\chi\rho}u = g,$$

which, on integration, gives

$$u = \frac{10^9g\rho\chi}{B^2}\left[1 - \exp\left(-\frac{B^2}{10^9\chi\rho}t\right)\right],$$

since at $t = 0$, $u = 0$.

If $u = \dot{z}$, we have

$$z = \frac{10^9g\rho\chi}{B^2}\left[t - \frac{10^9\chi\rho}{B^2}\left\{1 - \exp\left(-\frac{B^2}{10^9\chi\rho}t\right)\right\}\right],$$

for the distance traversed in time t.

The limiting velocity is $\frac{10^9g\rho\chi}{B^2}$; for copper $\rho = 8{\cdot}9$ gm.cm.$^{-3}$, $\chi = 1{\cdot}8 \times 10^{-6}$ ohm.cm., so that if $B = 10^4$ gauss

$$\Big[u\Big]_{t\to\infty} = \frac{980 \times 8{\cdot}9 \times 1{\cdot}8 \times 10^{-6} \times 10^9}{10^8}$$
$$= 0{\cdot}16 \text{ cm.sec.}^{-1}.$$

On account of this slow motion the ring appears as if it were levitating in the annular air gap.

The self inductance of a coaxial line.—This system of conductors consists of a pair of coaxial and very long cylinders carrying equal currents in opposite directions; the inner conductor is a thin-walled cylinder of external diameter $2a$, while the outer conductor is also thin walled and with internal diameter $2b$. The direction of the current in each conductor is shown symbolically in Fig. 8·14. The magnetic field at all points such as P_0, outside the system and at a distance r_0 from its central axis, is constant in magnitude, and using Ampère's circuital relation is given by H_0, where

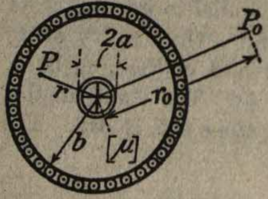


FIG. 8·14.—The self inductance of a coaxial line.

$$2\pi r_0 H_0 = 4\pi(i - i) = 0,$$

i.e.

$$H_0 = 0.$$

If μ is the permeability of the medium between the conductors, the magnetic induction B at a point P in this region, and at a distance r from the above axis, is given by

$$B = \frac{2\mu i}{r},$$

for the field strength is $\frac{2i}{r}$, cf. p. 132. Since the field, and therefore the induction, inside the thin walled inner conductor is everywhere zero, the flux per unit length of the system is given by

$$\Phi = \int_a^b \frac{2\mu i}{r} dr,$$

since B is uniform over an element of area of length unity (normal to the plane of the diagram) and width δr.

$$\therefore \Phi = 2\mu i \ln \frac{b}{a},$$

and this is also the number of linkages associated with unit length of the system.

$\therefore$ Self inductance per unit length $= \bar{l} = 2\mu \ln \frac{b}{a}$; in practical units this becomes

$$L = 2\mu \ln \frac{b}{a} \times 10^{-9} \text{ henry.cm.}^{-1}$$

Now let us assume that the medium between the cylinders is air so that its permeability μ_a may be taken as unity, while the inner cylinder is a solid core with constant permeability μ. Then in the region between the cylinders,

$$\bar{\Phi}_1 = 2\mu_a i \ln \frac{b}{a},$$

where $\bar{\Phi}_1$ is the flux per unit length in this region. To determine $\bar{\Phi}_2$, the flux per unit length in the wire itself, let i_d be the current density in the wire. Then, by the circuital relation,

$$4\pi(i_d . \pi r^2) = 2\pi r \mathrm{H},$$

where H is the field strength at a point distant r from the axis of the wire, $(r < a)$.

$$\therefore\ \mathrm{H} = 2\pi r i_d, \quad \text{or} \quad \mathrm{B} = 2\pi\mu r i_d$$

$$\therefore\ \bar{\Phi}_2 = \int_0^a \mathrm{B}\, dr,$$

since B is normal to the radius of cross-section of the wire. Hence

$$\bar{\Phi}_2 = \pi\mu a^2 i_d = \mu i,$$

where i is the current in the wire. The self inductance per unit length of the core itself is therefore μ, and hence, with a notation which is self explanatory,

$$\bar{l} = \bar{l}_1 + \bar{l}_2 = 2\mu_a \ln \frac{b}{a} + \mu.$$

It must be noted that the values of the inductances thus obtained are valid only for direct or low-frequency alternating currents, for it has been assumed that the current density in the wire is constant. When the frequency is very high the current is confined to the surface of the wire so that $\bar{\Phi}_2 = 0$. In such instances the self inductance per unit length will be $2\mu_a \ln \frac{b}{a}$.

The self inductance of a pair of parallel wires in air.—Suppose we have two parallel conductors, each a long wire of diameter $2a$; let s be the distance between their axes and it is assumed that $\frac{a}{s} \to 0$. Unit length of the system is shown in Fig. 8·15 and it is assumed that Q acts as a 'return path' for the current in P. The flux of magnetic induction across this portion of the loop and due to the current in P is given by

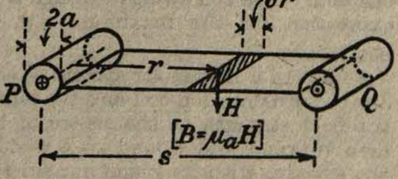


FIG. 8·15.—The self inductance of a pair of parallel wires in air.

$$\bar{\Phi} = \int_a^s \frac{2\mu_a i}{r} dr,$$

where μ_a is the permeability of air, for at all points in the element of area shown,

$$B = \mu_a H = \mu_a . \frac{2i}{r},$$

and a is so small that the flux within it may be neglected.

$$\therefore \ \bar{\Phi} = 2\mu_a i \ln \left(\frac{s}{a}\right).$$

Since the current in Q gives rise to an equal flux across the loop, and the total flux outside the system is zero,

$$l = 4\mu_a \ln \left(\frac{s}{a}\right),$$

or

$$L = 4\mu_a \ln \left(\frac{s}{a}\right) \times 10^{-9} \text{ henry.cm.}^{-1}$$

The measurement of self and mutual inductance.—In 1891 **Rayleigh** described one of the first successful methods for measuring the self inductance of a coil, and although it is known as a d.c. method, yet in it use is really made of the transient currents which exist before the current reaches its final steady value; it was a modification of an earlier method due to **Maxwell**. Some years later **Carey Foster** used a d.c. method for measuring mutual inductance. These methods are now only of historical importance, for in each instance the accuracy attainable has been surpassed many times by using alternating currents, some of the methods used will be described in Chapter XVII.

EXAMPLES VIII

8·01. A thin uniform rod, mass m and length $2a$, swings under gravity in a vertical plane normal to H_0 about a horizontal axis through one end. The rod swings through a semi-vertical angle θ. Obtain an expression for the maximum p.d. between the extremities of the pendulum.

[$2a\mu_a H_0 \sqrt{3ga} \sin \frac{1}{2}\theta$ e.m.u.]

8·02. In a Helmholtz system of coils the radius is 100 cm. If there are 10 turns in each coil and the current is 10 A., calculate a value for the field strength at the centre of the system. If a coil of effective area 10^4 cm.^{2} and wound on a wooden core is placed at the centre of the system, with its plane normal to the lines of induction, calculate a value for the mutual inductance of the system.

[89·9 oersted., $8{\cdot}99 \times 10^{-4}$ H.]

8·03. Define self inductance and mutual inductance. Show that the latter may be positive or negative. Obtain the relation between the c.g.s. absolute and the practical units of inductance. What is the self inductance of a solenoid of length 30 cm. consisting of 150 turns of thin wire? The cross-sectional area of the core on which the solenoid is wound is 10 cm.^{2} and its permeability is 200. [18·8 mH.]

8·04. A circular metal disc, diameter 48 cm., rotates on its axis

50 times in a second. If the plane of the disc is normal to a magnetic field of strength 70 oersteds calculate the current when copper brushes touch the axis and periphery of the disc and the resistance of the external circuit is 22 ohm.

[2·88 mA.]

8·05. Calculate a value for the mutual inductance of two coaxial coils of wire, the inner one consisting of 400 turns of wire wound on a wooden core 80 cm. long and 3 cm. in diameter, while the outer one consists of 1100 turns wound closely round the middle of the inner coil. Would you regard such an inductance as an absolute standard of mutual inductance? Give reasons for your answer.

[488 μH.]

8·06. Describe methods for measuring a magnetic field and discuss their relative merits. A copper disc, 50 cm. in diameter, rotates at a constant speed of 2400 revolutions per minute in a uniform magnetic field normal to the plane of the disc. The field is due to a current of 5 amperes in a long solenoid having 10 turns per cm. Obtain a value, in volts, for the potential difference between the axle and the periphery of the disc. If this potential difference is balanced against that between the ends of a small resistance connected in series with the solenoid, what is the value of this resistance?

[$4·94 \times 10^{-2}$ volt., $0·988 \times 10^{-2}$ ohm.]

8·07. Deduce a formula for the field inside a long uniformly wound solenoid. State carefully the law used as the starting point in your proof.

A long solenoid of 5 turns per cm. and 3·0 cm. diameter is divided into two equal parts connected in series. When the current in the solenoid is 2·0 A. what is the force of attraction each part exerts on the other? N.B.—The position of each half is not altered when the coil is divided.

[44·5 dyne.]

8·08. A small magnet of moment M can move along the axis of a circular coil of radius a. Show that when the magnet is at a distance z from the coil the flux of magnetic induction across the coil is

$$\Phi = 2\pi Ma^2(a^2 + z^2)^{-1·5}.$$

If the coil has resistance r and its self inductance is negligible, show that when the magnet is moving with velocity u along the above axis, the current in the coil has a magnitude

$$\frac{6\pi Ma^2zu}{r(a^2 + z^2)^{2·5}}.$$

8·09. A simple alternator consists of a plane rectangular coil of N turns, each of area A, rotating with constant angular velocity ω in a uniform magnetic field with induction B. The terminals of the coil are connected together, the total resistance of the circuit so formed being r. Neglecting the effects of self inductance, obtain expressions for:—

(*a*) The peak value of the e.m.f. induced in the coil,
(*b*) The peak value of the current in the coil,
(*c*) The mean couple on the coil,
and (*d*) The rate at which work must be done to maintain the coil in rotation.

[(*a*) $NAB\omega$ e.m.u., (*b*) $\dfrac{NAB\omega}{r}$ e.m.u., (*c*) $\dfrac{1}{2}\dfrac{N^2A^2B^2\omega}{r}$ dyne.cm.,
(*d*) $\dfrac{1}{2}\dfrac{N^2A^2B^2\omega^2}{r}$ erg.sec.$^{-1}$.]

8·10. A coil of wire in the form of a square of side a and with N turns is rotated with constant angular velocity ω about an axis parallel to one side and passing through the centre of the coil. If there is a magnetic field with induction B normal to the axis prove that the mean rate at which work must be done to maintain the rotation is $\dfrac{\frac{1}{2}N^2a^4B^2\omega^2r}{r^2+\omega^2l^2}$, where r is the resistance and l the self inductance of the coil, each in e.m.u.

8·11. Two similar coaxial flat circular coils, of mean radius 30 cm., whose planes are separated by a distance equal to their common radius, are each of 30 ampere-turns, the current being in the same sense in each coil. A circular metal disc, of radius 1·5 cm., coaxial with the coils and with its centre midway between them, rotates about the common axis of the system. Find the rate of rotation necessary to establish a potential difference of 10 microvolts between the rim and the centre of the disc.

Establish any formulae used in the calculation and comment on any assumptions which are made. (S)

[157 rev.sec.$^{-1}$.]

CHAPTER IX

ELECTROTHERMAL EFFECTS AND THERMOELECTRICITY

ELECTROTHERMAL EFFECTS

The heating effect of a current.—Let A and B be two cross-sections in a wire at potentials V_1 and V_2 volts respectively ($V_1 > V_2$). Suppose that a charge δQ coulomb passes from A to B. At A the potential energy associated with the charge is $V_1 . \delta Q$; at B it is $V_2 . \delta Q$. The loss in potential energy associated with this charge as it passes from A to B is $(V_1 - V_2)\delta Q$. If a current of I amperes is passing along the wire, $\delta Q = I . \delta t$, if δt is the time taken to transfer the charge from A to B. The energy lost appears as heat in the conductor and the rate at which energy is dissipated, the so-called ***power,*** is VI watts, if $V_1 - V_2 = V$, the drop in potential between A and B. The energy dissipated in t seconds is given by $W = VIt$ joules.

If v_1, v_2 and i are the potentials and current in e.m.u. we may show in a similar way that the rate of working is vi erg.sec.$^{-1}$.

Now since the actual amount of energy evolved per second must be independent of the units in which it is expressed, we have

$$VI \text{ watt.} = vi \text{ erg.}$$

But $I = 10i$, and 1 watt. $= 10^7$ erg.sec.$^{-1}$.

$$\therefore V(10i)10^7 = vi, \quad \text{or} \quad 10^8 V = v,$$

i.e. 1 e.m.u. of potential difference is equal to 10^{-8} volt.

If R is the resistance of the wire between the points A and B, we have $V = IR$, so that

$$W = VIt = I^2Rt, \quad \text{or} \quad \frac{V^2}{R}.t.$$

Now although these alternative expressions for W, which measures the so-called ***Joule heating effect*** in the wire, are theoretically identical, the use in calorimetric work of those which involve the resistance R is much to be deprecated.

Callendar always emphasized that the value of R under the conditions applicable to a calorimetric experiment was never known unless both V and I were measured. The reason for this is that the temperature of the wire is always several degrees in excess of that of the calorimetric fluid; this excess is unknown, so that even if the resistance of the wire has been measured at a standard or known temperature and an accurate value for the resistance-temperature coefficient for the material of the wire is available, it is nevertheless impossible to calculate a reliable value for the resistance of the wire when it is used in a calorimetric experiment. The effect is small if manganin wire is used but the argument is greatly clarified by the following experiment.

A nickel wire, s.w.g. No. 34 and about 40 cm. long, was used as a heater. At room temperature its resistance was about 0·85 ohm. Assuming a value for J, the mechanical equivalent of heat, it was hoped to use the above wire in a simple calorimetric experiment to determine the specific heat of paraffin oil. When a high resistance voltmeter was placed across the wire, the current through it being about 2 amp., the fraction $\frac{V}{I}$ differed considerably from the resistance of the wire as measured by means of a post-office box, even when the liquid was thoroughly stirred. When the stirring ceased, the voltmeter reading decreased by as much as 10 per cent; the resistance of the wire had increased considerably so that the current through it was reduced. Hence, in calorimetric experiments, where the rate of supply of heat is controlled electrically, it is most undesirable to use any other expression than V*It* joules for the energy dissipated in the calorimeter.

Experimental determination of the thermal capacity, at room temperature, of a calorimeter and its contents.—As shown in Fig. 9·01 (*a*), the apparatus consists essentially of the calorimeter containing a thin insulating oil; the calorimeter C is fitted with an ebonite lid which carries the heating coil H, the whole being surrounded by a large double-walled copper vessel D, the space between its walls being filled with water. For preference, the calorimeter should be suspended by three fine strings. The temperature of the water in the outer jacket is recorded by the thermometer T_2 and is the 'temperature of the enclosure', or 'room temperature'. The current is supplied from a battery as shown in Fig. 9·01 (*b*) and is kept constant by means of the sliding resistance R; its magnitude is measured by an ammeter A. The potential difference across the heating coil is measured by a high resistance voltmeter V. To obtain a reliable result in any calorimetric experiment of this nature, the energy must be supplied at a rate of 20–30 watts and, for really accurate work, a potentiometer should be used to measure both the current and the potential difference involved.

Now whenever a voltmeter and an ammeter have to be used simultaneously to measure power (or determine the characteristic curve of a diode, etc.) it is well to consider the circuit arrangements.

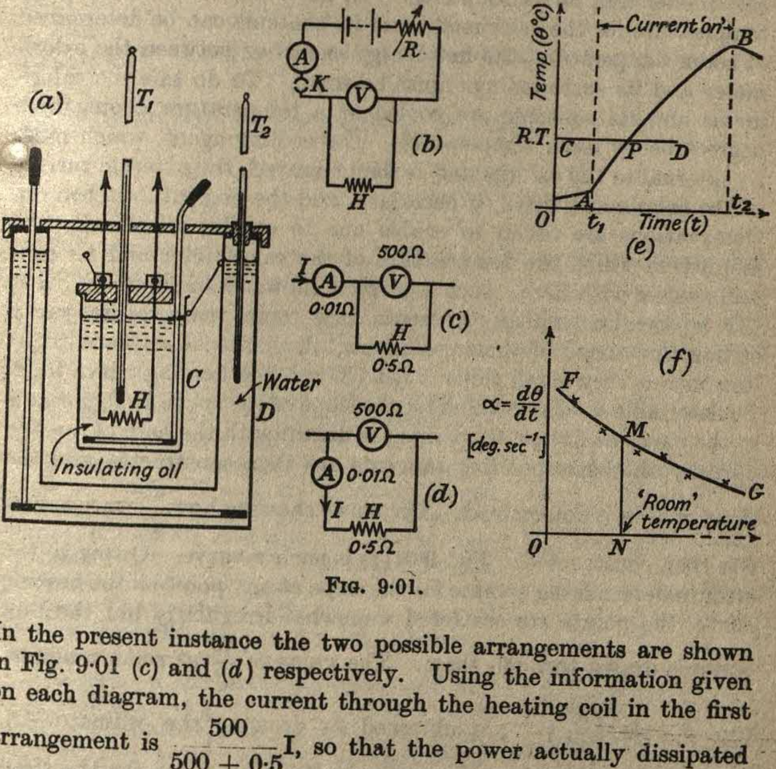


FIG. 9·01.

In the present instance the two possible arrangements are shown in Fig. 9·01 (*c*) and (*d*) respectively. Using the information given on each diagram, the current through the heating coil in the first arrangement is $\frac{500}{500 + 0\cdot5}$I, so that the power actually dissipated in the heater H is

$$V\left\{\left(\frac{500}{500 + 0\cdot5}\right)I\right\},$$

which differs from VI, the power determined from the instrument readings only, by 0·1 per cent.

In the second arrangement the potential difference across the heater is $\left(\frac{0\cdot5}{0\cdot5 + 0\cdot01}\right)$V, so that the power in the coil is, in fact,

$$\frac{0\cdot5}{0\cdot51}.VI,$$

the error now being 2 per cent ; the first arrangement must therefore be used.

To determine the thermal capacity of the calorimeter and its contents it is necessary to determine the rate at which the temperature of the calorimeter and its contents is changing at a given instant. Now, in general, the observed rate of rise in temperature will be affected by the rate at which energy is being exchanged between the calorimeter and its surroundings. If, however, the rate of rise in temperature for the calorimeter and its contents can be determined at room temperature, the net energy exchange between the calorimeter and its surroundings must be small. To do this the calorimeter and its contents are cooled to a temperature about three degrees below room temperature. The cooling agent, which must be external to the calorimeter, is then removed, the external surface of the calorimeter dried, if necessary, and the current switched on. Observations are taken to enable one to exhibit graphically the manner in which the temperature of the calorimeter and its contents varies with time ; such a graph is shown in Fig. 9·01 (*e*). Let CD be the line which represents how room temperature varies during the course of the experiment ; it will be almost a straight line with a very small slope. Let CD cut the heating curve in P. To determine a reliable value for the slope of the curve at P, normals to the curve at five or six points are drawn with the help of a plane mirror ; the corresponding tangents are then constructed and the slope of each is determined. The curve showing how $\frac{d\theta}{dt}$ varies with θ is then constructed. Fig. 9·01(*f*) is such a curve. Owing to the error in determining a value for the slope at any point on the heating curve, the points are scattered somewhat irregularly but the best curve is drawn through them. The value of $\frac{d\theta}{dt}$ at room temperature θ_r, viz. $\left(\frac{d\theta}{dt}\right)_{\theta=\theta_r}$, is obtained by drawing the ordinate MN, where ON $= \theta_r$, to cut the curve in M. The value for the slope at P thus obtained is very reliable since it has been found by evaluating several slopes in the manner indicated.

If C joule.deg.$^{-1}$ C. is the thermal capacity of the calorimeter and its contents,

$$C\left(\frac{d\theta}{dt}\right)_{\theta=\theta_r} = VI,$$

so that a value for C may be found. If a value for the specific heat of the material of the calorimeter is available, then that of the liquid at room temperature may be calculated provided the mass of liquid in the calorimeter is known.

[Since both V and I are known, a value for the resistance of the heating coil under the conditions in which it is used is at once available; a correction for the power dissipated in the voltmeter is easily made.]

Maximum power to be obtained from a given battery.—Let E be the e.m.f. of a battery with internal resistance B. If a resistance R is connected to the battery the current through R is $\frac{E}{(B+R)}$. Hence the energy dissipated per second in R is $\frac{E^2R}{(B+R)^2} = P$, say. For this to be a maximum or a minimum $\frac{dP}{dR}$ must be zero. Differentiating, we have,

$$\frac{dP}{dR} = \frac{E^2(B+R)^2 - E^2R[2(B+R)]}{(B+R)^4}.$$

This is zero when $R = B$. A second differentiation shows that P is a maximum when $R = B$.

Experiment.—To make use of the theorem which has just been established and obtain useful data about an accumulator, the circuit shown in Fig. 9·02 (*a*) may be used. The source of power is an accumulator E with a 1 ohm coil in series with it; let B be the internal resistance of the accumulator plus that of the coil. Then this combination may be regarded as a cell with an e.m.f. E and an effective internal resistance B; it is arranged in series with a known adjustable resistance R, having components 0·1, 0·2, 0·2, 0·5, 1, 2, 2 and 5 ohms, and an ammeter A, whose resistance is negligible. The potential difference, V, across R is measured with the aid of a high-resistance voltmeter; the current is I.

If a series of corresponding values of V and I is obtained, the graph shown in Fig. 9·02 (*b*) may be constructed and used to find the following facts; the graph is linear and must be extrapolated to cut the axes in points H and K.

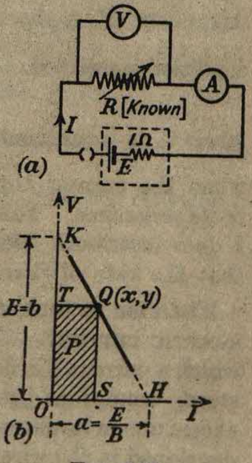


FIG. 9·02.

(*a*) When $R \to \infty$, $I \to 0$ and the reading of V tends to equal the e.m.f. of the cell. Thus OK measures this e.m.f.

(*b*) The maximum current which the cell can supply occurs when $R = 0$, i.e. when the voltmeter reading is zero. This maximum current is therefore given by OH.

(*c*) The equation to HK is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where $a = \frac{E}{B}$ and $b = E$.

Thus
$$y = \mathrm{E} - \mathrm{B}x,$$
so that the slope of HK is $-\mathrm{B}$.

(*d*) If Q is a point (x, y) on the line HK, the power developed in R is given by
$$\mathrm{P} = xy = x(\mathrm{E} - \mathrm{B}x),$$
and this is easily shown to be a maximum when $x = \frac{1}{2}\frac{\mathrm{E}}{\mathrm{B}}$, i.e. when Q is the mid-point of HK. Its coordinates are $\left(\frac{1}{2}\frac{\mathrm{E}}{\mathrm{B}}, \frac{1}{2}\mathrm{E}\right)$, so that the maximum power is $\frac{1}{4}\frac{\mathrm{E}^2}{\mathrm{B}}$, as required by the above theorem; the slope of the line joining O to the mid-point of HK is B.

The principle of least heat.—Let us consider two resistors R_1 and R_2 in parallel, the current through the combination being I. If I_1 is the current through R_1, $(\mathrm{I} - \mathrm{I}_1)$ is the current through R_2.

∴ Rate at which heat is produced in the two resistors
$$= \frac{\mathrm{I}_1{}^2\mathrm{R}_1 + (\mathrm{I} - \mathrm{I}_1)^2\mathrm{R}_2}{\mathrm{J}}\,\text{cal.sec.}^{-1} = \mathrm{P}\ \text{(say)}.$$
$$\therefore \frac{d\mathrm{P}}{d\mathrm{I}_1} = \frac{1}{\mathrm{J}}[2\mathrm{I}_1\mathrm{R}_1 + 2(\mathrm{I} - \mathrm{I}_1)(-\mathrm{R}_2)].$$

This is zero if $\mathrm{I}_1(\mathrm{R}_1 + \mathrm{R}_2) = \mathrm{IR}_2$, and since $\frac{d^2\mathrm{P}}{d\mathrm{I}_1{}^2}$ is positive for this value of I_1, the rate of production is a minimum when this condition is satisfied. It may be written $\mathrm{I}_1 = \frac{\mathrm{I}}{\mathrm{R}_1}\cdot\frac{\mathrm{R}_1\mathrm{R}_2}{\mathrm{R}_1 + \mathrm{R}_2} = \mathrm{A}\cdot\frac{\mathrm{I}}{\mathrm{R}_1}$ where A is a constant; similarly, $\mathrm{I}_2 = \mathrm{A}\cdot\frac{\mathrm{I}}{\mathrm{R}_2}$. Thus P is a minimum when the current through each resistor is inversely proportional to its resistance. This, however, is the manner in which the current I does distribute itself and hence the division of current is such that the rate of production of heat is a minimum.

The rise in temperature of a wire due to the passage of an electric current.—Consider a wire of length l, radius a, through which a current I is flowing. If R is the resistance of the wire when steady conditions have been attained, i.e. when ϕ, its temperature excess above its surroundings, is constant, the rate at which heat is developed in the wire is $\mathrm{I}^2\mathrm{RJ}^{-1}$. If Newton's law of cooling applies $\mathrm{I}^2\mathrm{RJ}^{-1} = \varepsilon . 2\pi al . \phi$, where ε is the surface emissivity of the wire. But $\mathrm{R} = \chi\frac{l}{\pi a^2}$, where χ is the resistivity of the material of the wire at the excess temperature ϕ.
$$\therefore \mathrm{I} = \pi\left[\frac{2\varepsilon\mathrm{J}\phi}{\chi}a^3\right]^{\frac{1}{2}}.$$

Thus the elevation of temperature is independent of the length of the wire, but for a constant current $\frac{\phi}{\chi}$ is inversely proportional to the cube of the radius.

Instead of assuming Newton's law of cooling it may be supposed that the wire behaves as a full radiator (or black body) when the rate of loss of energy in erg.sec.$^{-1}$ is given by

$$I^2R \times 10^7 = 2\pi al\sigma(T^4 - T_0^4),$$

where σ is Stefan's constant, viz. $5{\cdot}74 \times 10^{-5}$ erg.cm.$^{-2}$ sec.$^{-1}$ deg.$^{-4}$ K., T and T_0 the absolute temperatures of the wire and its surroundings. If the wire is a filament in a lamp $T \gg T_0$, so that T_0^4 may be neglected in comparison with T^4. Since $R = \frac{\chi l}{\pi a^2}$,

$$I^2 = 2\pi^2 a^3 \frac{\sigma}{\chi} T^4 \times 10^{-7}.$$

For tungsten $\frac{\chi}{T}$ is almost constant, so that for tungsten wires of constant radius,

$$I^2 \propto T^3, \quad \text{or} \quad T \propto I^{\frac{2}{3}}.$$

Example.—An electric fire radiates at aT^4 watts, and its heater element has a resistance $15 + bT^2$, where T is the absolute temperature of the element and a and b are constants. With a potential difference of 100 V. across the fire the current is 4 A. In order that the fire may lose heat at a rate P watts, and assuming that the loss of energy is by radiation only, show that the voltage across the fire must be adjusted to a value $\sqrt{15P + \frac{1}{2}P^{1\cdot5}}$. Prove, also, that the current is $\sqrt{\frac{P}{15 + \frac{1}{2}\sqrt{P}}}$.

When V is the applied potential difference, let R be the resistance. Then

$$\frac{V^2}{R} = P = aT^4, \quad \text{and} \quad R = 15 + bT^2.$$

If T_1 is the temperature of the heater when V = 100 V. and I = 4 amp., we have

$$400 = aT_1^4,$$

and

$$25 = 15 + bT_1^2, \quad \text{i.e.} \quad 10 = bT_1^2.$$

Eliminating T_1 from these equations, we get,

$$2 = \frac{\sqrt{a}}{b}.$$

Now
$$P = \frac{V^2}{15 + bT^2} = aT^4. \quad \therefore T^2 = \sqrt{\frac{P}{a}}.$$

$$\therefore P = \frac{V^2}{15 + bT^2} = \frac{V^2}{15 + \frac{1}{2}\sqrt{P}}.$$

$$\therefore V = \sqrt{15P + \frac{1}{2}P^{1\cdot5}}.$$

Hence $$I = \frac{V}{R} = \frac{\sqrt{15P + \frac{1}{2}P^{1.5}}}{15 + \frac{1}{2}\sqrt{P}} = \sqrt{\frac{P}{15 + \frac{1}{2}\sqrt{P}}}.$$

Example.—The rate of radiation of heat by an electric fire is proportional to the fourth power of the absolute temperature T of the element, the resistance of which increases with increase of temperature. A given small diminution in mains voltage reduces the steady rate of radiation of heat by the fire. Show that the reduction of heat radiated is less than it would be if R were independent of T, in the ratio

$$1 : 1 + \frac{1}{4}\frac{d(\ln R)}{d(\ln T)},$$

approximately.

Let $P = \text{power} = aT^4$, where a is a constant; also let $R = f(T)$.

Then $$P = \frac{V^2}{R} = aT^4, \quad \text{i.e. } V^2 = RaT^4 = af(T)T^4.$$

$$\therefore\ 2V\delta V = [4aT^3f(T) + aT^4f'(T)]\delta T.$$

$$\therefore\ \frac{dP}{dV} = \frac{dP}{dT}\cdot\frac{dT}{dV} = \frac{8aT^3V}{4aT^3f(T) + aT^4f'(T)}.$$

(*a*) If $R = f(T) = \text{constant}$, let $(\delta P)_1$ be the change in power. Then

$$(\delta P)_1 = \frac{8aT^3V\ \delta V}{4aT^3f(T)} = \frac{2V}{R}\delta V,$$

which could have been written down at once, using $P = \frac{V^2}{R} = \alpha V^2$, where α is a constant.

(*b*) If $f(T)$ is not constant, let the change in power be $(\delta P)_2$. Then

$$(\delta P)_2 = \frac{4aT^3.2V}{4aT^3f(T) + aT^4f'(T)}\delta V.$$

$$\therefore\ \frac{(\delta P)_2}{(\delta P)_1} = \frac{2V\ \delta V}{f(T) + \frac{1}{4}Tf'(T)}\cdot\frac{1}{2\frac{V}{R}.\delta V}$$

$$= \frac{R}{f(T) + \frac{1}{4}Tf'(T)} = \frac{1}{1 + \frac{1}{4}\cdot\frac{T}{R}\frac{dR}{dT}} = \frac{1}{1 + \frac{1}{4}\cdot\frac{d(\ln R)}{d(\ln T)}}.$$

Experimental investigation of the power dissipated in a 6V. lamp.—For this purpose the lamp is placed in the arm CD of the network shown in Fig. 9·03. The arm AB contains a known adjustable resistor R_1 with a maximum resistance of 2000 ohms; a voltmeter with a range 0–6V. is placed in BC, while AD contains a coil of wire of about 1 ohm resistance in series with a suitable ammeter A. Current to the bridge is supplied by means of a 12V. battery H; this current is controlled by a variable resistor R_0. G is a pointer galvanometer and the bridge is balanced by varying R_1. The 1 ohm resistor in series with the ammeter is to raise the resistance in the arm AD so that R_1 may have a reasonably large value.

When the bridge is balanced the voltage, V, across the lamp is given by the reading of the voltmeter, while the current through it is that through the ammeter, viz. I. Hence the power dissipated in the lamp is $VI = P$, say, while its resistance is $R = \frac{V}{I}$. If it is assumed that $P = aR^n$, then the constants a and n may be found after a series of corresponding values of P and R has been obtained. To proceed, we write

$$\log P = \log a + n \log R,$$

or

$$y = A + nx,$$

where $A = \log a$ and $x = \log R$. This is the equation to a straight line and usually a large-scale graph is required if accurate values for the constants are to be obtained. It is then almost impossible to indicate the origin of coordinates so that a value for A cannot be determined at once. To overcome this difficulty two widely separated points (x_1, y_1) and (x_2, y_2) on the straight line are chosen and the equation to it is found by using

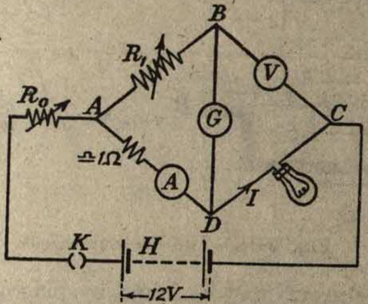


FIG. 9·03.—Experimental determination of the power dissipated in an electric lamp.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This may be arranged as

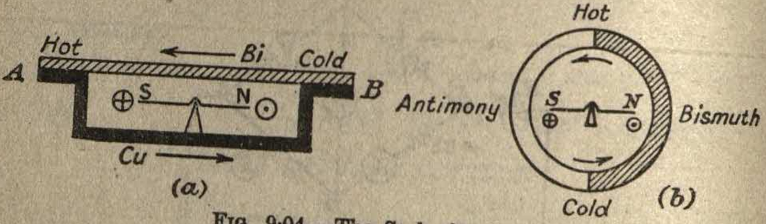
$$y = A + nx,$$

so that A and n are at once known.

THERMOELECTRICITY—FUNDAMENTAL FACTS

The Seebeck effect.—In 1821 SEEBECK discovered that an electric current could be produced by thermal means alone. He showed that a current flows in a circuit consisting of two wires of different materials as long as there is a difference in temperature between the two junctions. Such currents are termed thermoelectric currents and the electromotive force producing them is known as a thermoelectric force.

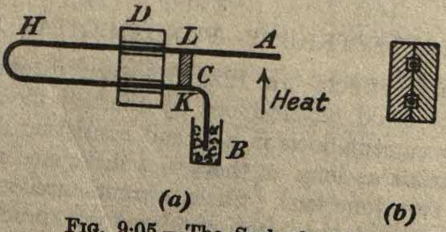
One of the first pieces of apparatus whereby Seebeck discovered how to produce thermoelectric currents is shown in Fig. 9·04 (*a*). The ends of a copper bar were bent, as indicated, and soldered to a bar of bismuth. The plane of this combination of metals was placed in the magnetic meridian, and a small compass needle was supported near to the middle of the circuit. When the junction A was heated the pole N of the magnet was deflected towards the east (this movement is indicated in the diagram by the ⊙ sign), i.e. an electric current had been produced and it flowed from the copper to the bismuth across the cold junction. If the circuit consists of rods of antimony and bismuth, bent for convenience to form a circle, the direction of the current is from the antimony to the bismuth through the cold junction [cf. Fig. 9·04 (*b*)]. The direction of the current may be found by placing the plane of the coil in the meridian and observing the motion of a magnetic needle at its centre. The experiment may be shown to a large audience by insulating the two metals at the 'cold end' by means of a strip of ebonite; the insulated ends are then connected to a sensitive galvanometer which readily detects any current produced by a difference of temperature between the 'junctions'. In each instance the energy necessary to maintain the current is derived from the surroundings, i.e. on the whole, there is an absorption of heat from them.



(*a*) (*b*)

FIG. 9·04.—The Seebeck effect.

Experiment.—AHB is a rod of copper 1 cm. in diameter and bent as indicated in Fig. 9·05 (*a*). It is short-circuited by C, a thick piece of constantan [Cu 60 per cent, Ni 40 per cent]. A large iron block is cut in halves, grooved, and fitted round the rod as shown in Fig. 9·05 (*b*). AB is insulated from the iron blocks by paper. When the end A of the rod is heated while B is kept in ice it is almost impossible to separate the two pieces of iron. This is because the iron has



(*a*) (*b*)

FIG. 9·05.—The Seebeck effect.

become magnetized by the large current in the circuit HKL. This current is produced by the small thermoelectric force in the same circuit which appears when the two junctions K and L are at different temperatures. The current is large since the resistance of the circuit is very small if the junctions K and L have been well made—they should be silver-soldered.

The Peltier effect.—An effect which is the converse of that just described was discovered in 1834 by PELTIER. He noticed that when a current passed across the junction between two dissimilar metals there was either an evolution or an absorption of heat, i.e. the junction was either heated or cooled. For any two metals the condition whether the junction shall be heated or cooled is determined by the direction of the current. This effect is entirely apart from the Joule effect which is irreversible, i.e. it does not depend on the direction of the current in the conductor, whereas the Peltier effect is reversible. In the following experiment the former effect is made negligibly small by using thick rods. B, Fig. 9·06, is a rod of bismuth soldered at each end to a rod of antimony, A. If a current is passed in the direction indicated there is an evolution of heat where the current passes from the antimony to the bismuth. At the other junction heat is absorbed. The effects are reversed when the current is reversed. These effects are clearly shown if coils of thin copper wire are wound round the junctions of the metals. Copper is chosen because its resistance-temperature coefficient is large. The resistances of these coils are measured when a current is passed along the rods. An increase in resistance of the coil at the junction where the current passes from antimony to bismuth shows that heat is developed at this junction. The other junction is cooled for the resistance of the coil round it decreases. When the direction of the current is reversed, contrary effects are obtained, indicating thereby that the effect under investigation is reversible.

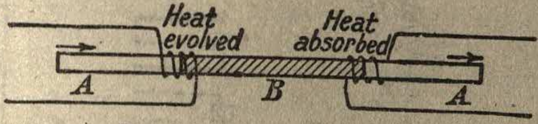


FIG. 9·06.—The Peltier effect.

The existence of the Peltier effect may also be demonstrated in the following way. Let T, Fig. 9·07, be a thermopile connected through a key, K_1, to a sensitive galvanometer, G, and a resistance R_1. Further, let us suppose that when heat radiation falls upon the right-hand face of the thermopile the current in the circuit marked (i) is clockwise. Let B be an accumulator, in series with a reversing key K_0, an ammeter A, preferably of the moving-iron type, an adjustable resistance R_2 and key K_2, so that a current of about 0·1 ampere may be passed for a period of about 10 seconds through the thermopile T [circuit (ii)]. By arranging a resistance $R_0(> 10^4\,\Omega)$ in parallel with A and G, i.e. across PQ, a small fraction of the current I_2 may be made to pass through G and hence, by using the reversing key K_0, through T in the same direction as did the thermoelectric current produced when heat radiations fell on the right-hand face of T. When K_2 is opened and K_1 closed immediately afterwards it will be found that G is deflected ; the direction of this deflexion will be such as to indicate the presence of an anticlockwise current in circuit (i). Thus the junctions of the thermopile on its r.h.s. must have been cooled with respect to those on the l.h.s. If the current from B through T is reversed, the current through G when K_1 is closed will also be reversed.

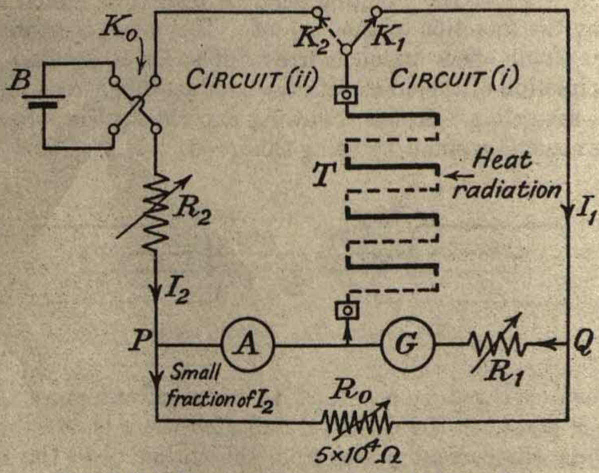


FIG. 9·07.—Demonstration of the Peltier effect.

Both the Seebeck and the Peltier effects may be explained if we assume that there is an electromotive force at the junction of two metals, acting in the case of an antimony-bismuth junction from the bismuth to the antimony, i.e. bismuth is electropositive with respect to antimony. If a circuit consists of two metals only and the temperature is everywhere the same, then the electromotive force at one junction is equal and opposite to that at the other and the total electromotive force in the circuit is zero. If, however, the temperatures of the junctions are different, the opposing electromotive forces are not equal and the difference between them causes a current to flow.

On the other hand, when a current is passed from antimony to bismuth work is done in overcoming the electromotive force at the junction; this appears as heat and the junction is heated. If the current is reversed the junction is cooled. Hence, on these assumptions, the existence of the Peltier effect finds a ready explanation.

In a general way the existence of an e.m.f. at the junction of two metals can be accounted for from our knowledge of the structure of conducting substances. Modern theory suggests that all such bodies contain large numbers of free electrons, i.e. electrons free to move in the interstices between the constituent particles of the conductor. These electrons behave very much like the molecules of a gas so that they are often referred to as an 'electronic gas'. The density of this gas at any fixed temperature of the metal is assumed to vary in different metals, so that when two metals are placed in contact the electrons diffuse from one to the other. This diffusion establishes an electromotive force at the junction which increases in value until it is sufficient to prevent a differential diffusion of the electrons from the one metal to the other. The equilibrium condition finally established is an example of a 'dynamic' equilibrium as distinct from a 'static' equilibrium, for there is no reason to suppose that when equilibrium has been attained the motion of the electrons ceases.

The Thomson effect.—From theoretical considerations KELVIN, when he was SIR WM. THOMSON, proved that if the only seat of potential difference in a thermocouple was at the junctions the total e.m.f. in the circuit should be proportional to the temperature difference between the junctions. Experiment shows that this is not even approximately true so that Kelvin assumed that in any homogeneous wire there must be an e.m.f. whenever there is a temperature gradient in the wire. To test the validity of this conclusion, Kelvin sent a heavy current through a homogeneous bar. The ends of this bar were kept at a constant temperature, but the central portion was heated. Then, with the aid of a sensitive differential thermometer, he showed that the amounts of heat generated in the two halves of the bar were unequal. He also showed that the effect was reversible.

Electron theory and the Thomson effect.—Since the electron density is greater at low temperatures than at high, it follows that we may expect an electromotive force whenever a temperature gradient exists in a conductor. Thus the simple electron theory accounts for the existence of the Thomson effect. Unfortunately, however, this theory would indicate that the e.m.f. is always directed from the region of high temperature to that where it is low, i.e. electrons move from the cold to the hot region, whereas experiment shows that this direction depends upon the material investigated.

THE MATHEMATICAL THEORY OF THERMOELECTRICITY

The Seebeck and Peltier effects.—Let A and B, Fig. 9·08 (*a*), be two dissimilar metals, one junction being hot while the other is cold. Then, in general, a thermoelectric current flows in this circuit as long as there is a temperature difference between the two junctions. ***It will be assumed that this current passes from* A *to* B, *across the hot junction.*** In Peltier's experiment a battery is placed in the circuit formed by A and B, cf. Fig. 9·08 (*b*), the two junctions being at the same temperature initially. It will be supposed that this current has the same direction as the thermoelectric current described above. Then the experimental fact discovered by Peltier is that heat is absorbed at the junction which was previously hot, and liberated at that which was cold. If the current is reversed the heating at the one junction is replaced by a cooling, and vice versa.

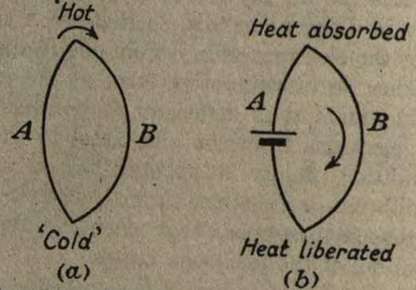


FIG. 9·08.—The Seebeck and Peltier effects.

The above experiment suggests that the thermoelectric current due to the Seebeck effect is maintained by the energy absorbed from the source less that supplied to the sink. Since there is a current flowing in the circuit there must be an electromotive force acting in it.

Let Π_{ab} denote the heat absorbed (in joules) per unit charge (in coulombs) when a small quantity of electricity passes from A to B. Π_{ab} is termed the ***Peltier coefficient;*** it is usually expressed in volts. This coefficient is not a constant for a given pair of metals but, as will be proved later, varies as the absolute temperature of the junction.

If in any particular instance Π_{ab} is negative, the minus sign indicates that heat is evolved when the current flows from A to B.

On the thermoelectric effect.—Let the arrows in Fig. 9·09 (*a*) denote the direction of the flow of current when heat is absorbed at the junction concerned. Suppose that the junctions are at temperatures T_2 and T_1, where $T_2 > T_1$. Then the Peltier coefficients under these conditions are as indicated. Let us consider the gain in energy of a charge δQ taken round the circuit in a clockwise direction.

Heat absorbed at the hot junction is $[\Pi_{ab}]_{T_2}\,\delta Q$ (work units).

Heat liberated at the cold junction is $[\Pi_{ab}]_{T_1}\,\delta Q$ (work units).

Hence the energy gained by the charge when it is taken once round the complete circuit is

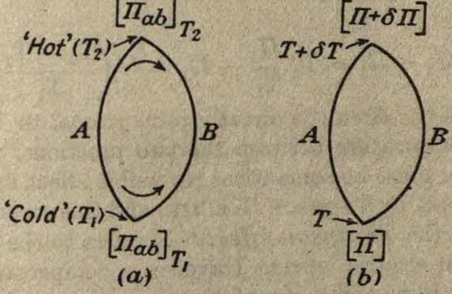
$$\{[\Pi_{ab}]_{T_2} - [\Pi_{ab}]_{T_1}\}\delta Q,$$

if it is assumed that the ohmic resistance of the circuit is negligible, and the Peltier effect is the only thermoelectric effect in the circuit.

If E is the electromotive force in the circuit, the energy gained when the charge δQ is taken round is $E.\delta Q$, i.e. if δW is the gain in potential energy, $E = \frac{dW}{dQ}$.

$$\therefore\ E = [\Pi_{ab}]_{T_2} - [\Pi_{ab}]_{T_1}.$$

Application of the principles of thermodynamics.—The circuit—cf. Fig. 9·09 (*b*)—now under consideration is such that it may be regarded as a heat engine in which energy is drawn from the source at a temperature $T + \delta T$ and given to the sink at a lower temperature T. In the process energy is expended. Such an engine is known as a Carnot engine, if the processes are reversible. Now the thermoelectric effects in the circuit considered are reversible, i.e. we have a reversible engine of the type contemplated in thermodynamics. According to the principles of that theory, the sum of the quantities



(*a*) (*b*)

FIG. 9·09.—Application of thermodynamics to a thermoelectric circuit.

$$\frac{\text{Heat absorbed}}{\text{Absolute temperature at which it is absorbed}}$$

for all the sources and sinks in a reversible cycle is zero, i.e. $\oint \frac{dq}{T} = 0.$

Let us apply the above principle to a circuit whose junctions are at temperatures T and (T + δT), the Peltier coefficients being $[\Pi_{ab}]_T$ and $[\Pi_{ab}]_{T+\delta T}$, respectively. Since no confusion can arise in the present discussion these coefficients will be denoted by Π and $\Pi + \delta\Pi$. Hence, by the principles of thermodynamics,

$$\frac{(\Pi + \delta\Pi)\delta Q}{T + \delta T} - \frac{\Pi\delta Q}{T} = 0,$$

i.e.
$$\frac{\Pi + \delta\Pi}{T + \delta T} - \frac{\Pi}{T} = 0, \text{ or } \delta\left(\frac{\Pi}{T}\right) = 0.$$

[Here we have to remember that if $y = \frac{u}{v}$, where u and v are functions of some variable, say x,

$$y + \delta y = \frac{u + \delta u}{v + \delta v}, \quad \text{i.e. } \delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}\Big].$$

Since $\delta\left(\frac{\Pi}{T}\right) = 0$, it follows that $\frac{\Pi}{T}$ is constant.

If we apply this result to a thermocouple whose junctions are at temperatures T_1 and T_2, ($T_1 < T_2$), we have

$$\frac{\Pi_2}{T_2} = \frac{\Pi_1}{T_1},$$

and therefore

$$E \equiv \Pi_2 - \Pi_1 = T_2 . \frac{\Pi_1}{T_1} - \Pi_1 = \Pi_1\left(\frac{T_2 - T_1}{T_1}\right),$$

i.e. the e.m.f. in the circuit is directly proportional to $T_2 - T_1$, the difference in temperature between the two junctions.) This is not so; hence there must be some other reversible effect in the circuit. This was predicted by THOMSON (KELVIN) and this second reversible effect is termed the ***Thomson effect.*** It refers to the evolution of energy when an electric current traverses a temperature gradient in a metal; it is the only one of the three thermoelectric effects which deals with a single material, so that it is sometimes referred to as the homogeneous thermoelectric effect.

(**The Thomson effect.**—This is defined in the following way. Let a charge δQ coulomb be taken from a point at temperature T in a homogeneous conductor to another point at temperature T + δT. Then the energy gained is $\sigma . \delta T . \delta Q$ joule, where σ is the Thomson coefficient for the material of the conductor when at a temperature T.) It is considered positive when there is an acquisition of energy by the charge under the above circumstances.

The apparatus shown in Fig. 9·10 (*a*) has been developed to demonstrate the existence of the Thomson effect in constantan; this alloy is selected since its Thomson coefficient has a large value.

A is a rod of constantan about 1·2 cm. in diameter and 18 cm. long. It has a fine screw thread cut along its whole length except near the ends where it is slightly reduced in diameter and fitted with brass terminals T_1 and T_2. B is a brass rod about 2·5 cm. in

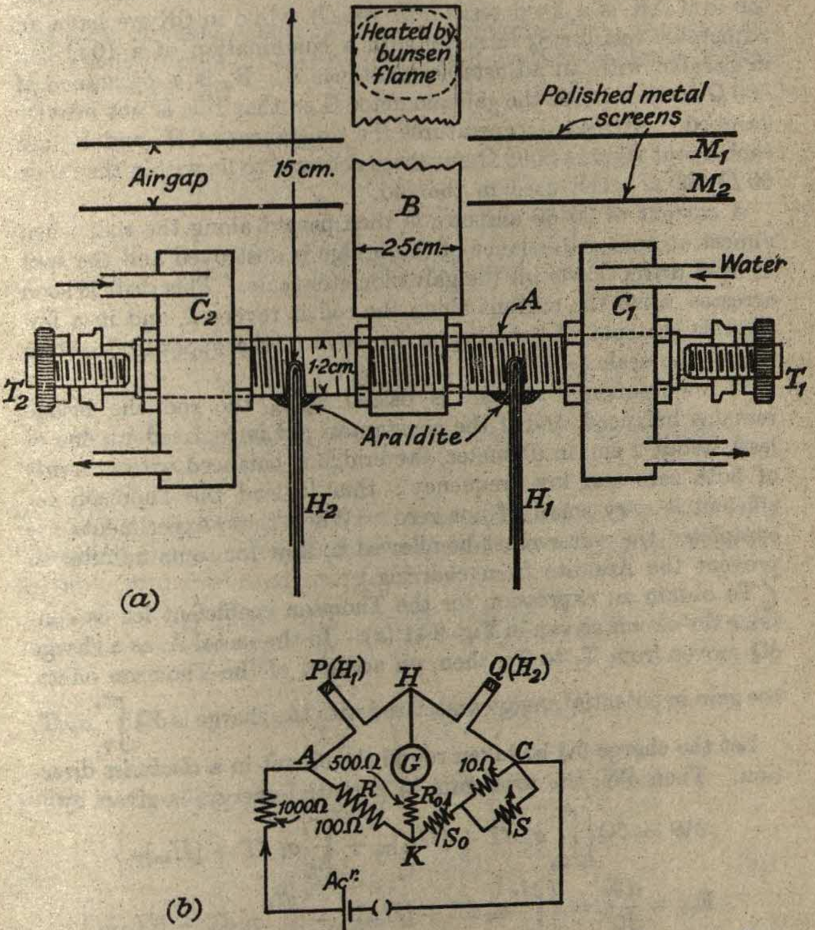


FIG. 9·10.—(*a*) Plan of apparatus for demonstrating the Thomson effect, (not to scale); (*b*) The electrical circuit.

diameter and screwed to carry the constantan rod; brass nuts hold this rigidly in position. C_1 and C_2 are brass chambers fitted to the rod and held by means of brass nuts in symmetrical positions with respect to its centre; a rapid stream of cold water passes continuously through these chambers. H_1 and H_2 are thermistors mounted with their elements on the axis of the rod and at equal distances from its mid-section; a small quantity of Araldite holds each in position. The bar B is heated with a bunsen flame, the main part of the assembly being screened from it with the aid of the polished metal plates M_1 and M_2.

H_1 and H_2 form the ratio arms P and Q of a Wheatstone bridge, the arm AK is a fixed resistor R(100 Ω), while in CK we have an adjustable resistor S_0 in series with a combination of a 10 Ω coil in parallel with an adjustable resistance S. R_0 is a resistance of 500 Ω in series with the galvanometer G so that this is not heavily damped. At room temperatures the resistances of H_1 and H_2 are each about 2000 Ω–5000 Ω but these decrease to lie within the range 50 Ω–100 Ω when used in the rod.

A current of 20–30 amperes is then passed along the rod, when almost at once the balance of the bridge is destroyed and the spot of light drifts slowly off the galvanometer scale. This drift is soon arrested when the current along the rod is reversed, and in a few seconds the spot of light begins to move in the opposite direction across the scale.

If low-frequency current is passed along the rod the bridge remains balanced, and if the constantan rod is replaced by one of lead, about 2 cm. in diameter, the bridge is balanced with currents of both zero and low frequency; thus in lead the Thomson coefficient is very small, if not zero. [When these experiments are completed the water must be allowed to flow for some minutes to prevent the Araldite from charring.]

To obtain an expression for the Thomson coefficient let us consider the circuit shown in Fig. 9·11 (*a*). In the metal A, as a charge δQ moves from T_1 to T_2, then, on account of the Thomson effect, the gain in potential energy associated with the charge is $\delta Q\int_{T_1}^{T_2}\sigma_a dT$.

Let the charge δQ be taken round the circuit in a *clockwise* direction. Then δW, the total gain in potential energy, is given by

$$\delta W = \delta Q\left\{\int_{T_1}^{T_2}\sigma_a\,dT + [\Pi_{ab}]_{T_2} + \int_{T_2}^{T_1}\sigma_b\,dT + [\Pi_{ba}]_{T_1}\right\}.$$

$$\therefore\ E_{ab} = \frac{dW}{dQ} = \left\{\int_{T_1}^{T_2}\sigma_a\,dT + [\Pi_{ab}]_{T_2} - \int_{T_1}^{T_2}\sigma_b\,dT - [\Pi_{ab}]_{T_1}\right\},$$

or

$$\delta E = \delta[\Pi_{ab}] + (\sigma_a - \sigma_b)\delta T,$$

where δE is written for δE_{ab}, and $\delta[\Pi_{ab}]$ for $[\Pi_{ab}]_{T+\delta T} - [\Pi_{ab}]_T$. Hence

$$\frac{dE}{dT} = \frac{d\Pi_{ab}}{dT} + (\sigma_a - \sigma_b) \quad . \quad . \quad . \quad \text{(i)}$$

If $P_{ab} = \dfrac{dE}{dT}$, the ***thermoelectric power*** or ***relative Seebeck*** *coefficient* for the two metals, we may write the above equation in the form

$$P_{ab} = \frac{d\Pi_{ab}}{dT} + (\sigma_a - \sigma_b), \qquad \text{(ii)}$$

and each of these equations has been obtained from the principle of the conservation of energy ; the small irreversible Joule heating effect is neglected.

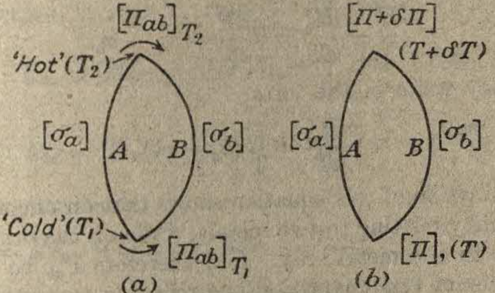


FIG. 9·11.—Application of thermodynamics to a thermoelectric circuit.

Now let us consider the Carnot cycle indicated in Fig. 9·11 (*b*) ; implicitly, the processes must be reversible. Applying the thermodynamic principle cited on p. 263 we have

$$\frac{\Pi + \delta\Pi}{T + \delta T} - \frac{\Pi}{T} + \frac{\sigma_a\,\delta T}{T} - \frac{\sigma_b\,\delta T}{T} = 0.$$

$$\therefore\ \delta\left(\frac{\Pi}{T}\right) + \left(\frac{\sigma_a - \sigma_b}{T}\right)\delta T = 0,$$

which in the limit gives $\dfrac{d}{dT}\left(\dfrac{\Pi}{T}\right) + \dfrac{\sigma_a - \sigma_b}{T} = 0,$

i.e.

$$\frac{1}{T}\frac{d\Pi}{dT} - \frac{1}{T^2}\Pi + \frac{\sigma_a - \sigma_b}{T} = 0 \qquad \text{(iii)}$$

From (i)

$$\frac{dE}{dT} = \frac{d\Pi}{dT} + (\sigma_a - \sigma_b) \qquad \text{(iv)}$$

Multiplying (iii) by T,

$$\frac{d\Pi}{dT} - \frac{\Pi}{T} + (\sigma_a - \sigma_b) = 0 \qquad \text{(v)}$$

From (iv) and (v)

$$\underline{\frac{dE}{dT} = \frac{\Pi}{T}}, \quad \text{i.e. } \frac{[\Pi_{ab}]}{T} = \frac{d(E_{ab})}{dT} = P_{ab} \qquad \text{(vi)}$$

Also $$\sigma_a - \sigma_b = \frac{\Pi}{\mathrm{T}} - \frac{d\Pi}{d\mathrm{T}} = \frac{d\mathrm{E}}{d\mathrm{T}} - \frac{d}{d\mathrm{T}}\left(\mathrm{T}\frac{d\mathrm{E}}{d\mathrm{T}}\right)$$
$$= \frac{d\mathrm{E}}{d\mathrm{T}} - \left(\frac{d\mathrm{E}}{d\mathrm{T}} + \mathrm{T}\frac{d^2\mathrm{E}}{d\mathrm{T}^2}\right).$$
$$\therefore\ \sigma_a - \sigma_b = -\mathrm{T}\frac{d^2\mathrm{E}}{d\mathrm{T}^2}. \qquad \text{(vii)}$$

Again, since $\mathrm{P} = \frac{\Pi}{\mathrm{T}}$,
$$\frac{\Pi}{\mathrm{T}} - \frac{d\Pi}{d\mathrm{T}} = \mathrm{T}\frac{d\mathrm{P}}{d\mathrm{T}} = \sigma_a - \sigma_b.$$

This we may write in the form
$$\frac{d\mathrm{P}_{ab}}{d\mathrm{T}} = \frac{1}{\mathrm{T}}(\sigma_a - \sigma_b).$$

An examination of this equation shows that on one side we have P_{ab}, a quantity relating to two metals, while on the other side each term refers to one metal only. Can therefore P_{ab} be regarded as the difference of two quantities, one referring to metal A and the other to metal B ? Since the third law of thermodynamics, cf. Vol. II, enables us to show that at the absolute zero of temperature both P_{ab} and $\frac{d\mathrm{P}_{ab}}{d\mathrm{T}}$ are zero, we may therefore write
$$\mathrm{P}_{ab} = \mathrm{P}_a - \mathrm{P}_b = \int_0^T \frac{\sigma_a d\mathrm{T}}{\mathrm{T}} - \int_0^T \frac{\sigma_b d\mathrm{T}}{\mathrm{T}},$$
where P_a is defined as the ***absolute Seebeck coefficient*** for the metal A ; similarly, for P_b. The absolute Seebeck coefficient for a metal at very low temperatures may be determined by joining it to a superconductor which is assumed to have negligible thermoelectric coefficients. In this way the absolute Seebeck coefficient for lead has been found (1958) ; the values for σ were determined by CHRISTIAN and others using the method of BORELIUS, cf. p. 291.

Law of intermediate temperatures.—It has been proved that
$$\mathrm{E}_{ab} = [\Pi_{ab}]_{\mathrm{T}_2} - [\Pi_{ab}]_{\mathrm{T}_1} + \int_{\mathrm{T}_1}^{\mathrm{T}_2}(\sigma_a - \sigma_b)d\mathrm{T}.$$

The first important property of E_{ab} becomes apparent if we choose some temperature T, such that $\mathrm{T}_2 > \mathrm{T} > \mathrm{T}_1$. Let E_{ab}'' and E_{ab}' denote the e.m.fs. for the temperature intervals $(\mathrm{T}_2 - \mathrm{T})$ and $(\mathrm{T} - \mathrm{T}_1)$, respectively. Then
$$\mathrm{E}_{ab}'' = [\Pi_{ab}]_{\mathrm{T}_2} - [\Pi_{ab}]_{\mathrm{T}} + \int_{\mathrm{T}}^{\mathrm{T}_2}(\sigma_a - \sigma_b)d\mathrm{T},$$
and $$\mathrm{E}_{ab}' = [\Pi_{ab}]_{\mathrm{T}} - [\Pi_{ab}]_{\mathrm{T}_1} + \int_{\mathrm{T}_1}^{\mathrm{T}}(\sigma_a - \sigma_b)d\mathrm{T}.$$

By addition we have

$$E_{ab}'' + E_{ab}' = E_{ab},$$

an equation showing that the e.m.f. of a thermocouple for any temperature interval $T_2 - T_1$ is the sum of the e.m.fs. corresponding to any smaller intervals into which the range $T_2 - T_1$ may be subdivided.

Law of intermediate metals.—Now let us consider the three thermocouples AB, BC, AC, the junctions in each instance being at temperatures T_2 and T_1 respectively, where $T_2 > T_1$. Then

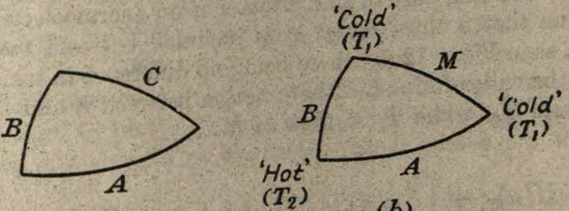
$$E_{ab} = [\Pi_{ab}]_{T_2} - [\Pi_{ab}]_{T_1} + \int_{T_1}^{T_2} (\sigma_a - \sigma_b)dT,$$

and
$$E_{bc} = [\Pi_{bc}]_{T_2} - [\Pi_{bc}]_{T_1} + \int_{T_1}^{T_2} (\sigma_b - \sigma_c)dT.$$

By addition, we have

$$E_{ab} + E_{bc} = [\Pi_{ab} + \Pi_{bc}]_{T_2} - [\Pi_{ab} + \Pi_{bc}]_{T_1} + \int_{T_1}^{T_2} (\sigma_a - \sigma_c)dT \quad . \quad . \quad . \quad \text{(i)}$$

To evaluate the first two terms on the r.h.s. of the above equation let us imagine that the three metals, A, B and C, form a closed circuit as in Fig. 9·12 (a), the temperature being the same at all points in the circuit. Then the resultant e.m.f. in this circuit must necessarily be zero, for otherwise a current would exist and energy be dissipated without any source of energy being present. Such a state of affairs would be inconsistent with the principle of the conservation of energy.



(a) (b)

FIG. 9·12.—The law of intermediate metals.

Since the temperature gradient is everywhere zero the Thomson e.m.fs. are zero, and we have, if a charge δQ is taken round the circuit in a clockwise direction,

$$(\Pi_{ab} + \Pi_{bc} + \Pi_{ca})\delta Q = 0, \quad \text{i.e. } \Pi_{ab} + \Pi_{bc} = \Pi_{ac}.$$

Hence from (i) we get

$$E_{ab} + E_{bc} = [\Pi_{ac}]_{T_2} - [\Pi_{ac}]_{T_1} + \int_{T}^{T_2} (\sigma_a - \sigma_c) dT$$

$$= E_{ac} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

Therefore, if the thermal e.m.fs. for each of two metals in combination with a third are known, the e.m.f. for the thermocouple formed by combining the two metals is obtained by subtraction, for equation (ii) may be written

$$E_{bc} - E_{ba} = E_{ac}.$$

It is necessary to note carefully that the equation must be written as above, since in actual practice the third metal is lead and the e.m.f. in a lead-metal circuit is considered positive when the thermoelectric current flows from the lead to the other metal through the hot junction. Thus for metals A and B in contact with lead (L) to form separate thermocouples, we have

$$E_{ab} + E_{bl} + E_{la} = 0, \quad \text{or} \quad E_{ab} = E_{al} + E_{lb},$$

from which it follows at once that

$$E_{lb} - E_{la} = E_{ab}.$$

The convenience of the above theorem is at once apparent, for it permits us to measure and tabulate for all metals their thermal e.m.fs. with respect to some metal as standard. This standard is lead, since the Thomson effect in lead at temperatures somewhat removed from the absolute zero of temperature is zero or negligibly small. The difference between two such values gives the e.m.f. for a thermocouple formed from the corresponding pair of metals.

To conclude this part of our discussion on thermoelectricity, let us imagine that a third metal M is included between two metals A and B, as in Fig. 9·12 (*b*), there being no difference in temperature between the ends of M. Let the junction between A and B be at a temperature T_2, while T_1 is that of M.

Then

$$E_{abm} = [\Pi_{ab}]_{T_2} + \int_{T_2}^{T_1} \sigma_b \, dT + [\Pi_{bm}]_{T_1} + 0 + [\Pi_{ma}]_{T_1} + \int_{T_1}^{T_2} \sigma_a \, dT$$

$$= [\Pi_{ab}]_{T_2} + [\Pi_{bm} + \Pi_{ma}]_{T_1} + \int_{T_1}^{T_2} (\sigma_a - \sigma_b) dT$$

$$= [\Pi_{ab}]_{T_2} - [\Pi_{ab}]_{T_1} + \int_{T_1}^{T_2} (\sigma_a - \sigma_b) dT$$

$$= E_{ab}.$$

Since this argument may be generalized to apply to any number of intermediate metals such as M, whose extremities are at the same temperature, it follows that the e.m.f. of a thermocouple is unaffected by the introduction of these metals under such conditions.

These last remarks are exceedingly important since they indicate that all instruments and connecting wires included in a thermoelectric circuit do not contribute to the e.m.f. provided that the temperatures are the same at each end of the wire or instrument.

The e.m.f. of a thermocouple.—Experiment shows that when the cold junction is at the temperature of melting ice, then for many pairs of metals and over a limited range of temperatures,

$$E = \alpha\theta + \tfrac{1}{2}\beta\theta^2, = \alpha(T - 273) + \tfrac{1}{2}\beta(T - 273)^2,$$

where α and β are constants for any one pair of metals, and θ is the temperature of the hot junction on the gas centigrade scale of temperature and T is that on the Kelvin scale. The restriction that the temperature of the cold junction is 0° C. does not affect the argument which follows since

$$\Big[E_{ab}\Big]_{\theta_1}^{\theta_2} = \Big[E_{ab}\Big]_{0}^{\theta_2} - \Big[E_{ab}\Big]_{0}^{\theta_1}$$
$$= \alpha(\theta_2 - \theta_1) + \tfrac{1}{2}\beta(\theta_2^2 - \theta_1^2),$$

where in this last equation θ_1 is a constant temperature, while θ_2 is a variable one.

These equations show that the relation between E_{ab} and temperature is a parabolic one, but whether or not the curve passes through the origin of coordinates depends on the scale of temperature and the temperature of the cold junction.

Thermoelectric power or the relative Seebeck coefficient.—This is denoted by the symbol P_{ab} and is defined by the equation

$$P_{ab} = \frac{dE_{ab}}{d\theta} = \alpha + \beta\theta = \alpha + \beta(T - 273).$$

Thus P_{ab} measures the slope of the thermoelectric curve, i.e. the curve obtained by plotting values of E_{ab} as ordinate and corresponding values of θ (or T) as abscissae. Such curves are shown in Fig. 9·13.

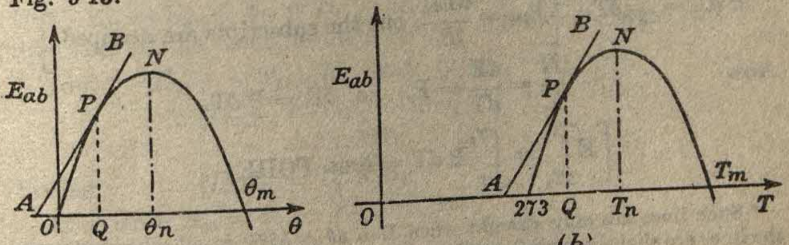


FIG. 9·13.—Thermoelectric curves and thermoelectric power.

The thermoelectric diagram.—This is a diagram constructed to show the relation between the relative Seebeck coefficient or thermoelectric power in a circuit comprising two metals and the temperature of the hot junction provided that the cold junction is kept at a fixed temperature. It is not necessary to specify this temperature, however, since the thermoelectric power is independent of the temperature of the cold junction. Usually one of the metals is lead since for this metal the Thomson effect is either zero, or negligibly small. Such diagrams were first constructed by TAIT, and are known as ***Tait diagrams.*** For purposes of theoretical discussion temperatures should be referred to the Kelvin scale of temperature.

Let it be stipulated that the current flows from lead to the metal at the hot junction. The straight line* A, in the Tait diagram shown in Fig. 9·14, represents the thermoelectric power P_{lm} in a lead-metal circuit as a function of T, the absolute temperature of the hot junction. Suppose that ordinates GF and HK corresponding to two temperatures T_1 and T_2 (on the Kelvin scale) and represented by OG and OH, respectively, are drawn; then the area FGHK measures the electromotive force in the lead-metal circuit when its junctions are at temperatures T_1 and T_2 respectively. To prove this, consider the element of area *amnb*. Its area is

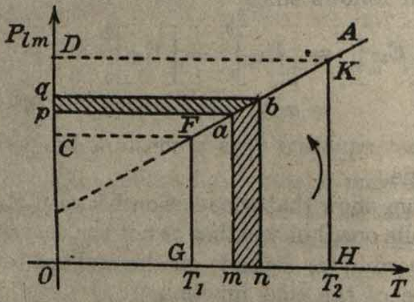


FIG. 9·14.—A thermoelectric diagram (or Tait diagram).

$$P\,\delta T = \frac{dE}{dT}\delta T. \quad \left[P_{lm} = \frac{dE_{lm}}{dT}, \text{ but the subscripts are dropped.}\right]$$

Now $$\frac{\Pi}{T} = \frac{dE}{dT} = P, \quad \text{i.e. } \delta E = P\,\delta T.$$

$$\therefore \Big[E\Big]_{T_1}^{T_2} = \int_{T}^{T_2} P\,dT = \text{area FGHK},$$

---

* Such lines are only straight when $E = \alpha\theta + \frac{1}{2}\beta\theta^2$; in all other instances the lines are slightly curved, but this makes no difference to the interpretation of the areas on a Tait diagram, which are now about to be discussed.

and the direction of the thermoelectric current is from the lead to the metal across the hot junction. This is represented on the diagram by the arrow drawn in an anticlockwise direction.

Again, if P_1 and P_2 are the thermoelectric powers corresponding to T_1 and T_2, we have

$$[\Pi_{lm}]_{T_2} = T_2\frac{dE_{lm}}{dT} = T_2P_2 = \text{area of rect. HKDO.}$$

Similarly, $$[\Pi_{lm}]_{T_1} = \text{area of rect. GFCO.}$$

Again, $$\sigma_l - \sigma_m = -T\frac{d^2E}{dT^2}.$$

$$\therefore\ \sigma_m = T\frac{d}{dT}\left(\frac{dE}{dT}\right), \quad [\because\ \sigma_l = 0]$$

$$= T\frac{dP}{dT}.$$

$$\therefore\ \sigma_m\,\delta T = T\,\delta P = \text{area of strip } abqp.$$

$$\therefore\ \int_{T_1}^{T_2}\sigma_m\,dT = \text{area of CFKD.}$$

$$\therefore\ \Big[E_{lm}\Big]_{T_1}^{T_2} = [\Pi_{lm}]_{T_2} - [\Pi_{lm}]_{T_1} - \int_{T_1}^{T_2}\sigma_m\,dT$$

$$= \text{area FGHK,}$$

as already proved. [N.B. This e.m.f. is such that the current flows from l to m across the hot junction, as shown by the arrow drawn in an anticlockwise direction.]

Suppose now that a second thermoelectric line is constructed on a Tait diagram. Let these lines be A and B in Fig. 9·15 (a), and let the metals be denoted by a and b respectively. Construct the ordinates shown. Then

$$\Big[E_{la}\Big]_{T_1}^{T_2} = \text{area } C_1C_2A_2A_1,$$

and $$\Big[E_{lb}\Big]_{T_1}^{T_2} = \text{area } C_1C_2B_2B_1.$$

Now $E_{ab} = E_{al} + E_{lb} = E_{lb} - E_{la} = \text{area } A_1A_2B_2B_1$, and the direction of this e.m.f. is indicated by the anticlockwise arrow.

Now $$[\Pi_{la}]_{T_1} = T_1.C_1A_1 \quad\text{and}\quad [\Pi_{lb}]_{T_1} = T_1.C_1B_1.$$

$$\therefore\ [\Pi_{ab}]_{T_1} = [\Pi_{al} - \Pi_{bl}]_{T_1} = [\Pi_{lb} - \Pi_{la}]_{T_1}$$

$$= T_1.A_1B_1.$$

Similarly, $$[\Pi_{ab}]_{T_2} = T_2.A_2B_2.$$

Also, since $E_{ab} = E_{lb} - E_{la}$, we have

$$P_{ab} = \frac{d}{dT}[E_{lb} - E_{la}] = P_{lb} - P_{la},$$

i.e. if any metal be taken as standard and the variation of thermoelectric power with temperature for a number of metals be plotted with respect to it, the thermoelectric power of two metals with respect to each other at a given temperature will be given simply by the difference of the respective ordinates for the thermoelectric lines of the two metals, each with respect to the standard metal.

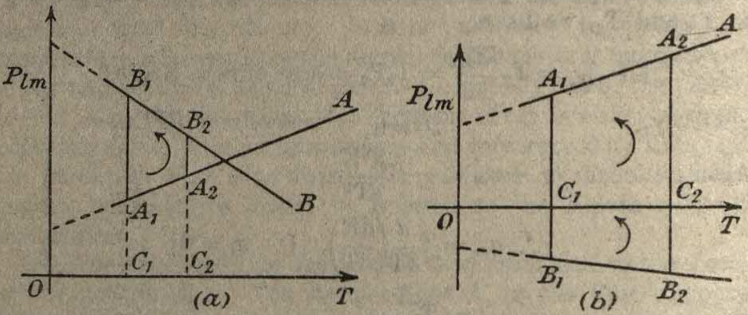


FIG. 9·15.—Tait diagrams.

[N.B. If it is assumed that $\sigma_l = 0$, $\sigma_m = T\frac{dP}{dT} = T\frac{d^2E}{dT^2}$.

Since T is essentially positive, σ_m has the same sign as $\frac{dP}{dT}$.]

Suppose now that the thermoelectric lines for A and B are as indicated in Fig. 9·15 (*b*). Then E_{la} is positive and is measured by the area $C_1C_2A_2A_1$, E_{lb} is negative and is measured by the area $C_1C_2B_2B_1$. What is E_{ba}? Now, in general,

$$E_{ab} + E_{bc} = E_{ac}.$$

$$\therefore E_{ab} = E_{al} + E_{lb} = E_{lb} - E_{la} = -[\text{area } A_1A_2B_2B_1].$$

$$\therefore E_{ba} = + \text{ area } A_1A_2B_2B_1.$$

Hence, in any instance in which the relative Seebeck coefficient or thermoelectric power is a linear function of the temperature of the hot junction, the e.m.f. of a thermocouple is equal to the product of the thermoelectric power at the mean temperature of the junctions and the difference in temperature between them.

The neutral temperature.—The temperature corresponding to the point at which two thermoelectric lines intersect is known as the ***neutral temperature***,* and it is such that at this temperature the thermoelectric power of the two metals is zero. Since $P = \frac{dE}{dT} = 0$, at the neutral temperature, it follows that at this temperature $|E|$ is a maximum.

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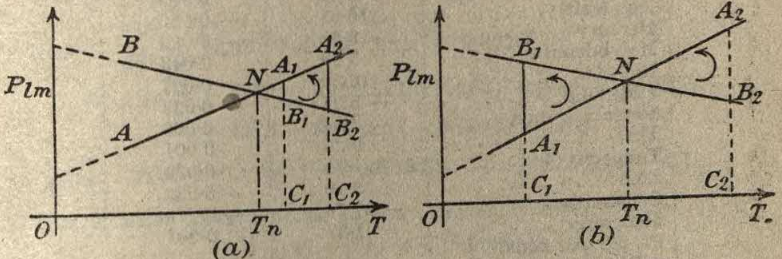
* Cf. also p. 282.

Let us suppose that the temperatures of the hot and cold junctions are each greater than the neutral temperature, T_n, in some definite case. If the Tait diagram for this instance is as shown in



(a) (b)

FIG. 9·16.—Tait diagrams and the neutral temperature.

Fig. 9·16 (*a*) how is the area $A_1A_2B_2B_1$ to be interpreted ? We have,

$$E_{la} = \text{area } C_1C_2A_2A_1, \quad E_{lb} = \text{area } C_1C_2B_2B_1$$

$$\therefore\ E_{ab} = E_{al} - E_{bl} = E_{lb} - E_{la}$$

$$= -\text{ area } A_1A_2B_2B_1,$$

and the current flows from *b* to *a* through the hot junction. The arrow on the diagram which represents this fact is still drawn in the anticlockwise direction because the line B is below A in the region under consideration.

The neutral temperature is further characterized by the fact that since $P_{ab} = 0$ at this temperature,

$$[\Pi_{ab}]_{T_n} = 0$$

Another important property is revealed by the fact that if the temperatures of the hot and cold junctions are equally removed from the neutral temperature and on opposite sides of it, cf. Fig. 9·16 (*b*), the electromotive force in the circuit is zero, for

$$\Big[E_{ab}\Big]_{T_1}^{T_2} = \Big[E_{ab}\Big]_{T_1}^{T_n} + \Big[E_{ab}\Big]_{T_n}^{T_2} = 0.$$

The numbers in the following table, when inserted in the expression $E = \alpha\theta + \frac{1}{2}\beta\theta^2$, give E in microvolts, when lead is one metal in the thermocouple. The positive direction of the current is such that it flows from lead to the metal at the hot junction.

Table of Thermoelectric Constants

Metal	α (μV.deg.$^{-1}$ C.)	β (μV.deg.$^{-2}$ C.)
Antimony	35·58	0·146
Bismuth	− 74·42	0·032
Cadmium	3·06	0·029
Copper	2·76	0·012
Gold	2·90	0·009
Iron (soft)	16·65	− 0·030
Mercury	− 8·81	− 0·033
Molybdenum	5·89	0·043
Nickel	− 16·30	− 0·027
Platinum	− 3·04	− 0·033
Silver	3·34	0·008
Tin	0·23	−0·001
Tungsten	1·59	0·034
Zinc	3·10	− 0·032
Constantan	− 37·76	− 0·079
Manganin	1·37	0·001
Pt (90 per cent) —Rh (10 per cent)	7·00	0·0064

Example.—Calculate, using the above table, Π_{ab} and $(\sigma_a - \sigma_b)$ for a bismuth-silver thermocouple at 0° C.

We have

$$E_{\text{Pb.Bi}} = [-74{\cdot}42\theta + \tfrac{1}{2}(0{\cdot}032\theta^2)]\ \mu\text{V}.$$

$$E_{\text{Pb.Ag}} = [3{\cdot}34\theta + \tfrac{1}{2}(0{\cdot}008\theta^2)]\ \mu\text{V}.$$

$$\therefore\ E_{\text{Bi.Ag}} = E_{\text{Bi.Pb}} - E_{\text{Ag.Pb}} = E_{\text{Pb.Ag}} - E_{\text{Pb.Bi}}$$

$$= [77{\cdot}76\theta - \tfrac{1}{2}(0{\cdot}024\theta^2)]\ \mu\text{V}. \quad = \text{E (say)}.$$

Now $\dfrac{d\text{E}}{d\text{T}} = \dfrac{d\text{E}}{d\theta} = 77{\cdot}76 - 0{\cdot}024\theta.$

$$\therefore\ \Big[\Pi_{\text{Bi.Ag}}\Big]_{\text{T}=273} = 273\left(\frac{d\text{E}}{d\theta}\right)_{\theta=0} = (273 \times 77{\cdot}76)\ \mu\text{V}.$$

$$= 2{\cdot}12 \times 10^{-2}\ \text{V}.$$

$$= 2{\cdot}12 \times 10^{-2}\ \text{joule.coulomb.}^{-1}$$

Also
$$\sigma_{\text{Bi}} - \sigma_{\text{Ag}} = -(\sigma_{\text{Pb}} - \sigma_{\text{Bi}}) + (\sigma_{\text{Pb}} - \sigma_{\text{Ag}})$$

$$= \text{T}\left[\frac{d^2E_{\text{Pb.Bi}}}{d\theta^2} - \frac{d^2E_{\text{Pb.Ag}}}{d\theta^2}\right]$$

Hence, at 0° C.,

$$\sigma_{\text{Bi}} - \sigma_{\text{Ag}} = 273[0{\cdot}032 - 0{\cdot}008]\ \mu\text{V. deg.}^{-1}\ \text{C.}$$

$$= 6{\cdot}6\ \mu\text{V. deg.}^{-1}\ \text{C.}$$

[Alternatively, we could have written

$$\Big[\sigma_{\text{Bi}} - \sigma_{\text{Ag}}\Big]_{\theta=0} = -\left[\text{T}\frac{d^2E_{\text{Bi.Ag}}}{d\text{T}^2}\right]_{\text{T}=273} = -273[-0{\cdot}024]$$

$$= 6{\cdot}6\ \mu\text{V. deg.}^{-1}\ \text{C.}]$$

Example.—A certain thermocouple, with one junction at 0° C., has an e.m.f. given by

$$E = [87{\cdot}276\,\theta - 14{,}527\{1 - \exp(-0{\cdot}00253\,\theta)\}]\ \mu V.$$

Determine the Peltier coefficient at 0° C., and show that the difference between the Thomson coefficients is practically stationary in a small temperature range in the neighbourhood of 122° C.

We have

$$\sigma_a - \sigma_b = -T\frac{d^2E}{dT^2} = -(273 + \theta)\frac{d^2E}{d\theta^2}.$$

Now $$\frac{d^2E}{d\theta^2} = 14{,}527(0{\cdot}00253)^2 \exp(-0{\cdot}00253\,\theta).$$

$$\therefore\ \sigma_a - \sigma_b = -(273 + \theta)[14{,}527(0{\cdot}00253)^2 \exp(-0{\cdot}00253\,\theta)].$$

$$= -(273 + \theta)A \exp(-0{\cdot}00253\,\theta),\ \text{say}.$$

$$\therefore\ \frac{d}{d\theta}[\sigma_a - \sigma_b] = -A \exp(-0{\cdot}00253\,\theta)[1 - 0{\cdot}00253(273 + \theta)].$$

This is zero, corresponding to a stationary value of $(\sigma_a - \sigma_b)$, if

$$0 = 1 - (273 + \theta)(0{\cdot}00253),$$

i.e.
$$395{\cdot}3 = 273 + \theta, \quad \text{or} \quad \theta = 122°\ \text{C}.$$

Also

$$[\Pi]_{\theta=0} = \left[T\frac{dE}{dT}\right]_{273} = 273[87{\cdot}276 - 14{,}527 \times 0{\cdot}00253 \exp(0)]\ \mu V.$$

$$= 273[87{\cdot}28 - 36{\cdot}76]\ \mu V.$$

$$= 1{\cdot}38 \times 10^{-2}\ V.$$

This example is appended to show that expressions, other than that used in the rest of the chapter, may be found to express the relation between E_{ab} and the temperature θ. [Cf. also Ex. 9·27.]

Thermodynamics of the steady state.—In order to develop a mathematical theory and thereby explain certain observed facts, it is necessary to leave the complexity of the real world and replace it by an idealized situation capable of being represented by mathematical analysis. One of the most successful ideas thus introduced into science has been that of ***equilibrium***; this is a static and time-invariant state of a system in which all macroscopic quantities remain unaltered and no spontaneous process occurs. Such a conception is necessarily an ideal one, for no system of observable bodies is ever in a state of absolute equilibrium, being subject to small changes such as may originate from the penetration of the system by high-frequency radiations or by cosmic rays.

Similarly, equilibrium theory is not applicable to a system across whose boundary there is a flow of electricity or of energy. In the case of heat flow, however, steady flow methods have been developed for the measurement of thermal conductivity, cf. Vol. II. In such methods the temperature at each point finally attains a steady value, i.e. the system remains unchanged in time but the occurrence of a dissipative force prevents us from regarding the system as more than one in quasi-thermodynamical equilibrium.

Since the first quarter of the present century the conception of a second type of idealized state, which is also independent of time, has come to be accepted. In contrast to the first type termed ***equilibrium,*** which is the time-invariant condition of a ***closed state,*** in the second type the time-invariant condition of a system which is ***open to its environment*** is termed the ***steady state.*** When the flux from the surroundings approaches in value to zero we have a system tending to a condition of absolute equilibrium.

In developing the mathematical theory of thermoelectricity in this chapter, the methods used by THOMSON (Kelvin) in 1854 have been followed; in these the irreversible processes of heat conduction and of Joulean heating, both of which increase the entropy of the system, have been neglected. In applying the second law of thermodynamics only to the reversible processes in such a system Thomson was fully aware he was using a procedure which could not be justified on contemporary theory; experimental data subsequently obtained, however, were in such reasonable accord with the equations Thomson had obtained by his quasi-thermodynamical arguments, that these were accepted. As late as 1934 Bridgman wrote, '*As long as there is any doubt as to the legitimateness of neglecting the irreversible processes of Joulean heating and thermal conduction, the comparison of values obtained by direct and indirect methods is not superfluous.*'

Thomson used the same method respecting the variation of the e.m.f. of a voltaic cell, cf. p. 669. In 1878 similar methods were used by HELMHOLTZ in connexion with the theory of concentration cells, and in 1889 by NERNST in his study of diffusion. On each occasion the neglect of the irreversible processes was fully appreciated.

Onsager's theory.—In 1931 ONSAGER introduced into thermodynamics a new principle known as ***the principle of microscopic reversibility.*** It is auxiliary to the first and second laws, and according to it 'Under equilibrium conditions any molecular process, and the reverse of that process, will be taking place on the average at the same rate . . .'; this implies that at molecular level equilibrium is never maintained by a cyclic process.

Onsager proceeded to establish his theory by introducing the so-called ***thermodynamic equations of motion.*** These express an assumed linearity between the rates of the various types of process (current density, energy flow, diffusion, etc.) and the so-called ***thermodynamic forces,*** which are regarded as the causes of the corresponding currents. Such forces are proportional to the respective gradient of potential, temperature, concentration, etc.

Thus if $\vec{J}$ is taken as the rate of flow and $\vec{X}$ as the force, there exists an empirical relation of the form

$$\vec{J} = L\vec{X},$$

where L is a scalar quantity. When there is a flow of energy between two ' vessels ' and also a flow of electricity, we should have

$$\vec{J_1} = L_1\vec{X_1}, \quad \text{and} \quad \vec{J_2} = L_2\vec{X_2},$$

if the processes take place separately. Actually they occur simultaneously and so, retaining the assumption of linearity, Onsager writes

$$\vec{J_1} = L_{11}\vec{X_1} + L_{12}\vec{X_2},$$

and

$$\vec{J_2} = L_{21}\vec{X_1} + L_{22}\vec{X_2}.$$

The coefficients L_{11} and L_{22} are related to the thermal and electrical conductivities respectively, whereas the coefficients L_{12} and L_{21} represent a possible coupling between the two flows ; **Eckart** assigns to each the term ***drag coefficient*** and Onsager showed that $L_{12} = L_{21}$. These reciprocal relations of Onsager express a certain symmetry in the mutual interference of two irreversible processes occurring simultaneously in a system. Thermoelectric effects may be viewed as the result of mutual interference between energy flow and electric current in a system.

In 1948 **Callen** used Onsager's theory to place the subject of thermoelectricity on a more certain basis. The final equations obtained are identical with those obtained nearly a century previously by Thomson. The calculations are too involved to be given here but the original paper is in the *Physical Review* for 1948.

EXPERIMENTAL THERMOELECTRICITY

Experimental determination of thermal e.m.fs.—Since the thermal e.m.f. which it is proposed to measure is small, a potentiometer wire having a small potential drop per unit length along it is required. Suppose that this wire AB has a resistance r ohms and it is placed in series with a resistance R ohms and a cell D of e.m.f. E. The potential drop across the wire is equal to the current along the wire multiplied by the resistance of the wire, viz.

$$\frac{Er}{R + r}.$$

If E is of the order 2 volts, r of the order 10 ohms, and R of the order 2000 ohms, the potential drop across the potentiometer wire will be 10^{-2} volt, or 10^{-4} volt.cm.$^{-1}$ if the wire is one metre long.

To help maintain a steady current in the potentiometer wire AB, Fig. 9·17, the resistance R is divided into two parts R_1 and R_2; R_1 is fixed in any given determination and R_2 is variable. A standard cell S and galvanometer G together with a protective resistance X, 10,000 Ω, are arranged as shown and with the keys K_1 and K_2 closed, the value of R_2 is adjusted so that there is no deflexion of G. The potential drop across R_1 is then equal to the e.m.f. of S on open circuit since this cell is supplying no current. If necessary, when an approximately correct balance has been obtained, the key K_3 may be closed to short circuit X; this permits the galvanometer to be used at its maximum sensitivity. K_2 and K_3 are then opened.

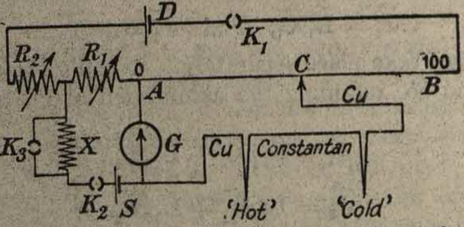


FIG. 9·17.—Experimental determination of thermal e.m.fs.

The thermocouple is arranged, as shown, and, with K_2 open, its e.m.f. determined by finding the point C on the wire AB corresponding to no deflexion of the galvanometer. If E is the e.m.f. of the standard cell, the current in the main circuit is $\frac{E}{R_1}$, i.e. the drop in potential across the potentiometer wire is $r\left(\frac{E}{R_1}\right)$. The potential difference between A and C is therefore

$$r.\frac{E}{R_1}.\frac{AC}{AB}.$$

This is the e.m.f. of the thermocouple.

[N.B. If $E = 1{\cdot}0186$V, it is convenient to make $R_1 = 1019\,\Omega$ so that the current in the potentiometer is 1·000 mA.; the arrangement is easily modified to give currents of 0·5 or 2·0 mA. Also it is not necessary to know R_2 and when once the potential drop across R_1 has been made equal to the e.m.f. of S on open circuit, the constancy of the current along the potentiometer wire may be tested and maintained by keeping R_1 constant and adjusting R_2 so that the deflexion of G is zero when K_2 is closed.]

To investigate how the e.m.f. of the thermocouple varies with the temperature of its hot junction one junction of the thermocouple is placed in a test-tube and surrounded by melting ice and the other heated to different known temperatures; in each instance the position of the sliding contact C, when the galvanometer deflexion is zero, is noted. The thermal e.m.f. in the circuit is in each instance equal to the potential drop along the corresponding portion of the wire AC.

Thermoelectric curves.—The curve obtained by plotting the e.m.f. in a thermocouple against the temperature of the hot junction is known as a ***thermoelectric curve***. Generally, one junction is kept at 0° C. In many instances the curve is almost a parabola with its axis vertical; sometimes only a portion of the curve can be obtained either because one of the metals melts, or rapid oxidation occurs.

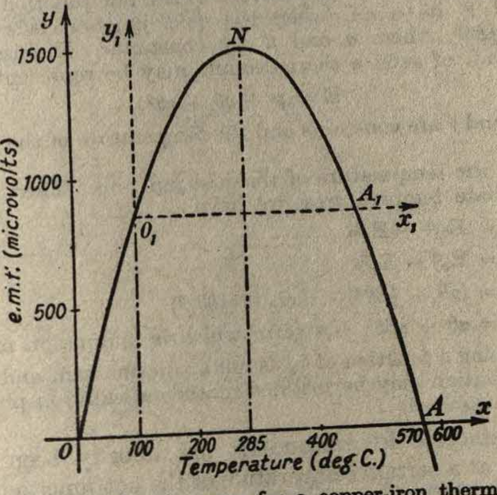


FIG. 9·18.—The thermoelectric curve for a copper-iron thermocouple with one junction at 0° C.

For a copper-iron thermocouple, for which when one junction is at 0° C. $E_{Cu.Fe} = [10{\cdot}34\ \theta - \frac{1}{2}(0{\cdot}0366)\ \theta^2]\ \mu V.$, the curve is as shown in Fig. 9·18. If the temperature of the cold junction is raised to 100° C. (say), the corresponding thermoelectric curve is obtained by transferring the origin to O_1, whose abscissa with respect to O is 100° C., and using O_1x_1 and O_1y_1 as the axes of coordinates.

Practical determination of the constants in the equation to a thermoelectric curve.—If the equation expressing the relation between the e.m.f. of a thermocouple and the temperature θ of its hot junction, when the other is kept at 0° C. and α and β are constants, is

$$E = \alpha\theta + \tfrac{1}{2}\beta\theta^2,$$

it may be written

$$\frac{E}{\theta} = \alpha + \tfrac{1}{2}\beta\theta.$$

It follows at once that if a series of corresponding values of E and θ is obtained, then by plotting $\left(\frac{E}{\theta}\right)$ against θ a straight line, whose slope is $\frac{1}{2}\beta$ and whose intercept on the $\left(\frac{E}{\theta}\right)$-axis is α, should be obtained.

If the equation to the curve is $E = a\theta^n$, where a and n are constants, then we have

$$\log E = \log a + n \log \theta,$$

and the method described in detail on p. 257 should be used to discover accurate values for the constants [cf. Ex. 9·29, p. 300].

Example.—If θ is the temperature of the hot junction of a thermocouple and E its e.m.f. when the cold junction is at 0° C., then $E = \alpha\theta + \frac{1}{2}\beta\theta^2$, where α and β are constants. Discuss whether or not the e.m.f. of such a thermocouple may be represented by

$$E = p + q\theta + r\theta^2,$$

where p, q and r are constants and the temperature of the cold junction is *not* 0° C.

Let θ_0 be the temperature of the cold junction. Then from the law of intermediate temperatures, we have

$$E_0^{\theta} = E_{\theta_0}^{\theta} + E_0^{\theta_0},$$

i.e.

$$\begin{aligned} E_{\theta_0}^{\theta} &= E_0^{\theta} - E_0^{\theta_0} \\ &= (\alpha\theta + \tfrac{1}{2}\beta\theta^2) - (\alpha\theta_0 + \tfrac{1}{2}\beta\theta_0^2) \\ &= \alpha\theta + \tfrac{1}{2}\beta\theta^2 + \text{a term which is a function of } \theta_0. \end{aligned}$$

Thus p, being a function of θ_0, is not a true constant and so while the proposed equation may be justified mathematically, in physics its use is somewhat limited.

Thermoelectric inversion.—It has already been stated [cf. p. 274] that at a certain temperature of the hot junction for a given pair of metals $\frac{dE}{d\theta}$, i.e. the thermoelectric power may become zero, i.e. E is either a maximum or a minimum. The temperature at which this occurs is known as the ***neutral temperature.*** If the curve can be obtained at still higher temperatures then it may eventually cut the axis of temperature, i.e. the e.m.f. will be zero in spite of the fact that there is a temperature difference between the junctions of the thermocouple. The temperature of the hot junction (other than 0° C.) when E becomes zero is known as the ***inversion temperature.*** If θ_0 is the temperature of the cold junction, θ_n the neutral temperature and θ_m the inversion temperature, it is clear that when the relation between E and θ is parabolic then

$$(\theta_m - \theta_0) = 2(\theta_n - \theta_0).$$

Note on some thermocouples used in practice.—When a thermocouple is used to measure a temperature one junction is placed in contact with the object whose temperature is to be determined while the other is kept at 0° C. ; a calibrated microammeter enables the temperature θ to be obtained directly. For precision work the e.m.f. developed must be measured by a potentiometer method and any additional junction pairs must be kept at 0° C. For example, in an iron-manganin thermocouple for which $\beta \fallingdotseq 0$, so that the relation between E and θ is almost linear—a fact which enables a graph to be used quite easily—these two metals are joined together and the other ends of the wires are joined separately to copper wires. The copper wires lead to a potentiometer, while the copper-iron and copper-magnanin junctions are kept at 0° C. ; it is the iron-manganin junction which is heated. The arrangement of the wires is shown in Fig. 9·19 (*a*) and (*b*).

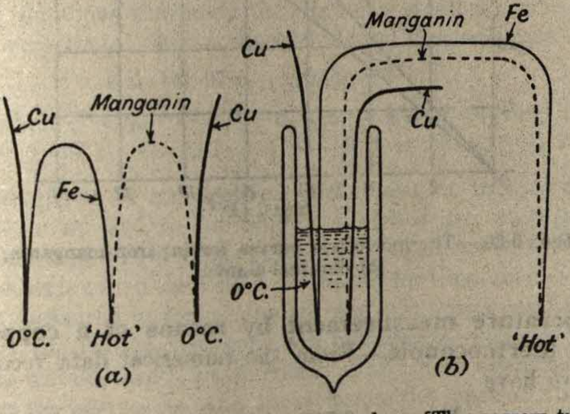


FIG. 9·19.—The mounting of thermocouples. [The narrow tubes carrying the wires and insulating them are not shown.]

Copper-constantan is another thermocouple which may be used very satisfactorily in the temperature range − 200° C. to 400° C. It is because of the large thermal e.m.f. between copper and constantan that this alloy is never used in resistance boxes designed for precision work ; the coils are made of manganin because the thermal e.m.f. between copper and manganin is small, although the temperature coefficient of increase of resistance is not quite so small as for constantan.

For the temperature range 0° C. to 1750° C. platinum joined to a platinum-rhodium alloy [13 per cent Rh] is used ; up to 1400° C. a chromel-alumel thermocouple is very useful and for this combination of alloys the thermoelectric curve is very nearly a straight line. [Chromel is 90 per cent Ni ; 10 per cent Cr ; Alumel is 95 per cent Ni, 5 per cent Al with traces of Si and Mn.]

For purposes of comparison thermoelectric curves for iron-manganin and for chromel-alumel are shown in Fig. 9·20. The linearity of the chromel-alumel curve is very striking.

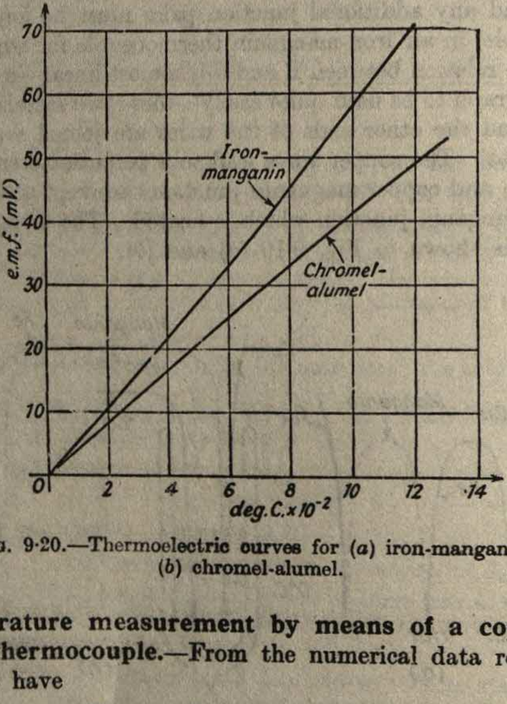


FIG. 9·20.—Thermoelectric curves for (*a*) iron-manganin, (*b*) chromel-alumel.

Temperature measurement by means of a copper-constantan thermocouple.—From the numerical data recorded on p. 276 we have

$$E_{\text{Pb Cu}} = 2{\cdot}76\theta + \tfrac{1}{2}(0{\cdot}012)\theta^2,$$

and
$$E_{\text{Pb Cons}} = -\,37{\cdot}76\theta - \tfrac{1}{2}(0{\cdot}079)\theta^2,$$

so that
$$E_{\text{Cu Cons}} = E_{\text{Cu Pb}} + E_{\text{Pb Cons}} = -\,E_{\text{Pb Cu}} + E_{\text{Pb Cons}}$$
$$= -\,40{\cdot}52\theta - \tfrac{1}{2}(0{\cdot}091)\theta^2 = E \text{ (say)}.$$
$$\therefore \frac{dE}{d\theta} = -\,40{\cdot}5 - 0{\cdot}091\theta.$$

Hence the relative Seebeck coefficient is zero when $\theta = -\,444°$ C. This is the neutral temperature for this particular thermocouple and is unattainable.

Suppose now, however, that a copper-constantan thermocouple, with one junction at 0° C., is used as a centigrade thermometer. A temperature ϕ on this scale is given by

$$\frac{\phi}{100} = \frac{E - E_0}{E_{100} - E_0} = \frac{E}{E_{100}},$$

since $E_0 = 0$.

Hence
$$\phi = \frac{E}{\gamma},$$
where $\gamma = \frac{E_{100}}{100}$. If therefore we write $E = \alpha\theta + \frac{1}{2}\beta\theta^2$, we have
$$\phi - \theta = \frac{E}{\gamma} - \theta = \frac{\alpha\theta + \frac{1}{2}\beta\theta^2}{\alpha + 50\beta} - \theta$$
$$= \frac{\frac{1}{2}\beta}{\alpha + 50\beta}.\theta(\theta - 100),$$
so that, by proceeding as on p. 193, a value for θ should theoretically always be obtainable when ϕ is known.

Unfortunately, however, for a copper-constantan thermocouple $\phi - \theta$ is never small except when $\theta = 0°$ or 100° C. Hence θ is best deduced from a large-scale graph showing how E varies with θ. It is for this reason that an iron-manganin thermocouple is preferred to the above thermocouple since β is small, i.e. $\phi - \theta$ never becomes very large. For a chromel-alumel thermocouple
$$E = [40{\cdot}97\theta - \tfrac{1}{2}(0{\cdot}0068)\theta^2]\mu V.,$$
so that when $\phi = 500°$ C., $\phi - \theta$ is only about 1·6 deg. C.; and when $\phi = 1000°$ C., the value of $\phi - \theta$ does not exceed 7 deg. C.

Measurement of the Peltier coefficient at the junction of two metals at 0° C.—The energy absorbed per unit (positive) charge when a small quantity of electricity passes across the junction of two metals is called the Peltier coefficient for those metals. If the energy is measured in joules, the current in amperes, and the time in seconds, the Peltier coefficient will be given in volts. Generally, one of the metals is lead, and if energy is absorbed when the current flows from the lead to the other metal across the junction, the Peltier coefficient according to our convention with regard to the sign of E is positive.

To measure the Peltier coefficient at 0° C.—for a lead-constantan junction, for example—the junction is placed in the inner tube of a Bunsen ice-calorimeter and covered with a small quantity of insulating oil to ensure good thermal contact with the calorimeter. Let a current of I amperes be passed through the junction for t seconds. If R is the effective resistance of the couple, then
$$\text{Heat evolved} = \frac{I^2Rt - \Pi It}{J} = kx_1,$$
where Π is the Peltier coefficient, J is the mechanical equivalent of heat, viz. 4·18 joule.cal.$^{-1}$, k is the constant of the calorimeter (determined by an electrical method) and x_1 is the distance through which the mercury recedes.

Now let the current be reversed for t seconds. Then

$$\frac{\mathrm{I}^2\mathrm{R}t + \Pi\mathrm{I}t}{\mathrm{J}} = kx_2,$$

where x_2 is the distance the mercury recedes in this instance. Hence,

$$\Pi = \frac{k(x_2 - x_1)\mathrm{J}}{2.\mathrm{I}t} \text{ volt.}$$

[NOTE.—Although the Joule heating could be made small and perhaps negligible by using thick leads, this would be a very undesirable procedure in practice since the flow of heat by conduction along the leads to the interior would be augmented and so make steady conditions well nigh impossible.]

Caswell's method for measuring the Peltier coefficient for two metals.—Fig. 9·21 (a) is purely to show the principle of the method adopted. A and B are two Dewar flasks containing equal masses of an insulating oil such as kerosene—or rather the masses of oil are adjusted so that the thermal capacities of the vessels and their contents are equal. Let us suppose that copper and bismuth are the two metals for which the Peltier coefficient is to be determined. Suitable lengths of thick wire are soldered together and the junctions placed in the calorimeter vessels. The liquid is well

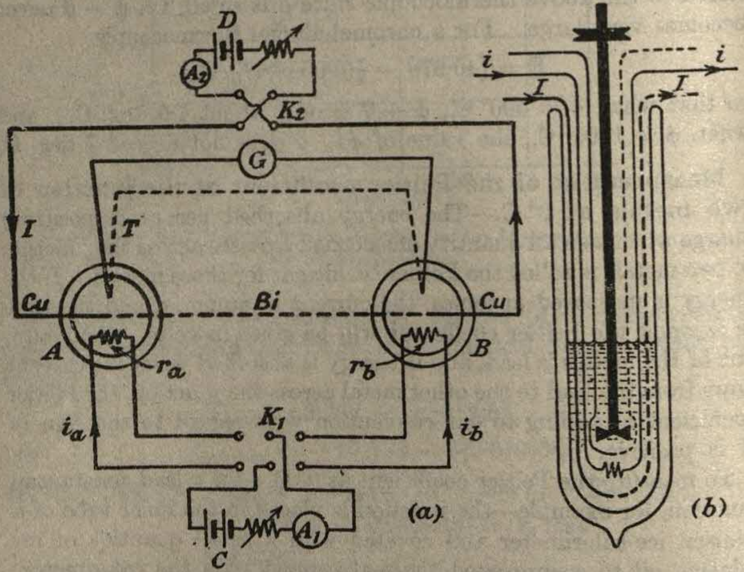


FIG. 9·21.—Experimental determination of the Peltier coefficient for a Cu–Bi thermocouple. [$\Pi_{\mathrm{Cu.Bi}}$ is negative.]

stirred and any difference in temperature between the two calorimeters is detected with the aid of a copper-constantan thermocouple T, in series with a sensitive galvanometer G. A battery D sends a current through the copper-bismuth wires; its magnitude is recorded by A_2. By means of a key K_2 this current can be reversed.

Now at one of the junctions heat will be absorbed—at the other it will be liberated so that the temperatures of the two liquids will become different. To equalize these temperatures small resistances r_a and r_b are inserted as shown and energy can be dissipated in one or other of these resistances. Let the thermal capacities of the two vessels be equal and let I be the current through the copper-bismuth system.

Now heat is absorbed on account of the Peltier effect when the current passes from bismuth to copper. Then if Π is the numerical value of the Peltier coefficient and R_a, R_b, the effective resistances of the thermocouples in A and B respectively, the rates of gain of heat are

$$I^2R_a + \Pi I + \alpha \quad \text{and} \quad I^2R_b - \Pi I + i_b{}^2r_b + \alpha,$$

where α is a small term representing the net gain due to stirring, etc. It is assumed constant. The above rates were made equal—the deflexion of G was zero—so that $I^2(R_b - R_a) = 2\Pi I - i_b{}^2r_b$.

When the current is reversed

$$I^2(R_b - R_a) = -2\Pi I + i_a{}^2r_a.$$

From these two equations we have

$$\Pi = \frac{i_a{}^2r_a + i_b{}^2r_b}{4I}.$$

The method of mounting the different wires, etc., in the Dewar flask is indicated in Fig. 9·21 (*b*). Some of the results obtained were

$$\Pi_{\text{Cu.Bi}} = -\ 0{\cdot}01613\ \text{V.,} \quad \text{at } 18^\circ\ \text{C.}$$
$$\Pi_{\text{Cu.Ag}} = -\ 0{\cdot}00003\ \text{V.,} \quad \text{at } 18^\circ\ \text{C.}$$

Thermoelectric cooling.—In 1838, LENZ, making use of the newly discovered Peltier effect, showed that a droplet of water placed in a small depression at the junction between bismuth and antimony rods could be frozen by the passage of an electric current across the junction, the direction of the current being from bismuth to antimony, cf. p. 258. The success of this experiment depended, in part at least, on the low ambient temperatures which often exist in Leningrad. In recent years, pairs of semiconductors have been produced for which the Peltier coefficient is large.

Many solids are semiconductors, i.e. their behaviour is intermediate between that of good conductors and that of good insulators, and this remarkable feature is due to the presence of impurities which disturb the regularity of the crystal lattice on which the atoms are normally arranged. Some solids are semiconductors even in the absence of such imperfections; these are the so-called ***intrinsic semiconductors*** in contrast to the ***extrinsic or impurity semiconductors*** mentioned above, but they, too, may have their properties greatly modified by the introduction of impurities.

The simplest type of impurity semiconductor has a very small fraction (10^{-6} to 10^{-10}) of the atoms in its crystal lattice replaced by atoms of an element whose valency differs by one or two units from that of an atom in the undisturbed crystal. These substituted atoms produce no significant distortion of the lattice and appear to occupy the positions in the lattice of those atoms which they have ejected. When the substituted atoms have a valency greater than that of the displaced atoms, i.e. they have more electrons, they are known as ***donors,*** whereas if the valency is less we have ***acceptor atoms,*** a name arising from the fact that such atoms have an insufficient number of electrons to maintain normal relations with neighbouring atoms. Thus, essentially, there are two types of semiconductor known respectively as the n-type and p-type. When the conduction in a semiconductor is due primarily to electrons it belongs to the n-type. On the other hand, in the p-type of semiconductor, the current is due to the transfer of ***holes,*** a name implying a region from which an electron has been removed, i.e. an electron has been ejected from a covalent bond. Thus a hole is equivalent to a positively charged body but it must be emphasized that holes have no real existence; they only represent the absence of electrons.

The particular semiconductor with which we have to deal is bismuth telluride, Bi_2Te_3; atomic numbers 83 and 52 respectively. To prepare it, an ingot, formed by direct fusion of the two elements (in correct proportions), is crushed into small pieces, some of which are then placed in a long silica boat and melted in a vacuum. The melt thus produced is allowed to solidify when, by means of high-frequency induction heating, a short molten zone is formed at one end of the miniature ingot. This zone is caused to move slowly along the ingot, after which the central portion is found to possess uniform thermoelectric properties; it is of the p-type. To produce bismuth teluride of the n-type a small and carefully measured amount of iodine, atomic number 53, is added to the crushed pieces and a new miniature ingot produced. It is thought that the iodine atoms are substituted for tellurium atoms within the bismuth telluride lattice, and it should be noted that a normal iodine atom has one more electron than a tellurium atom, so that the necessary extra electrons for the production of an n-type semiconductor are available. The semiconductor is said to have been doped with iodine.

Since the p- and n- types of a material such as bismuth telluride have relative Seebeck coefficients or thermoelectric powers which are positive and negative respectively, a Tait-diagram for them is as shown in Fig. 9·22 (a); the thermoelectric current in a thermocouple made from these semiconductors is directed from the n-type to the p-type across the hot junction. If therefore, by using a cell as shown in Fig. 9·22 (b), a current is caused to pass in this direction across such a junction, this will be cooled.

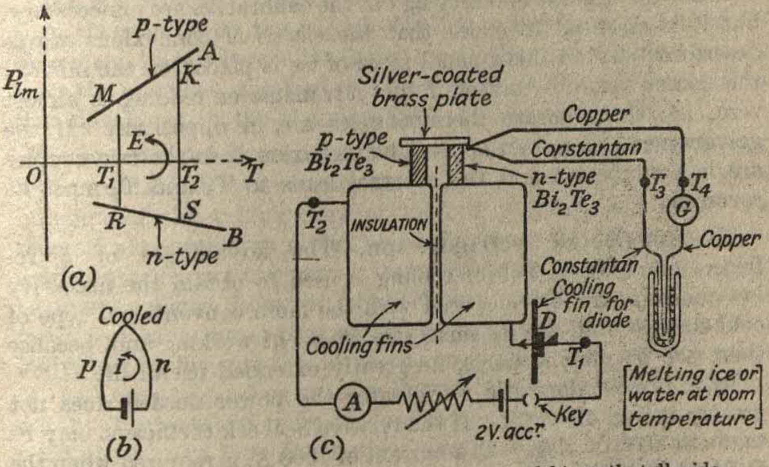


FIG. 9·22.—Tait diagrams for p- and n-types of bismuth telluride.

To demonstrate such cooling two short lengths of the p- and n-types of bismuth telluride are arranged as shown in Fig. 9·22 (c) to form a thermocouple; their upper ends are joined together by a brass plate, silvered on its upper surface to form a mirror. Each lower end of the thermocouple is soldered to a system of vanes with large blackened cooling surfaces; each system is insulated from the other. In the diagram each set of vanes is represented by a plate. T_1 and T_2 are two terminals arranged so that currents up to 10 A. may be sent across the upper junction, a germanium diode, D, with an adequate cooling fin being incorporated in the circuit so that only the passage of current in the correct direction through the thermocouple is permitted. When this takes place the brass plate is cooled very rapidly and hoar frost soon appears on it—a small droplet of water placed on the plate quickly freezes.

The same instrument may be used to determine the dew-point. The current is adjusted so that the silvered plate cools slowly when the formation of dew upon it shows that the dew-point has been reached, and a value for it is determined as the mean of the two temperatures at which dew forms and disappears.

To measure such temperatures one junction of a copper-constantan thermocouple is placed in thermal but not electrical contact with the mirror; T_3 and T_4 are terminals which permit a high-resistance galvanometer G to be placed in series with the thermocouple, but if this is to be calibrated an external copper-constantan thermocouple may be added to the circuit as shown. Its junction is kept in melting ice while the temperature of the instrument may be maintained at any desired and known temperature within the range 0° to 30° C. Details for carrying out the calibration are unnecessary, but it is essential to check that the electrical connexions are as described. To do this a small piece of ice is placed on the mirror; the connexions are correct if the galvanometer reading is almost zero, i.e. the separate thermocouples are in opposition. If the galvanometer shows an appreciable deflexion then the two couples are in conjunction and the external leads to T_3 and T_4 must be reversed.

Thermoelectric refrigeration.—The advantages of a refrigerator in which Peltier cooling is used to obtain the necessary low temperature, as compared with the more conventional type of cold storage unit, are its small size, its silent working and, because there are no moving parts, its greatly extended useful life.

In a typical domestic refrigerator the power needed does not greatly exceed 40 watts. If the relative Seebeck coefficient may be taken as 400 μV. deg.$^{-1}$ C. a current of 1000 A. is required when the Peltier effect is used to produce the cooling; thus, theoretically, a single thermocouple would be sufficient provided each element in it had a large cross-sectional area and a short length. To reduce the current to a reasonable value, say 5 A., two hundred couples, arranged in series, are required. The essential structure of a practical form of cooling unit is shown in Fig. 9·23. The thermo-

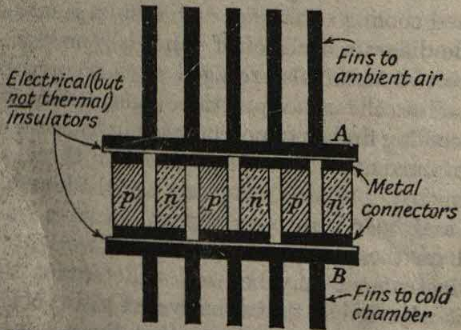


FIG. 9·23.—The structure of a thermoelectric refrigerator.

elements are connected by flat copper or aluminium strips so as to form a rectangular array The faces of the copper connectors are ground flat and pressed against the flat surfaces of two large metal slabs A and B to each of which cooling fins are attached. These slabs must be insulated electrically from the copper connecting strips but at the same time the thermal contact must be good. IOFFE has shown that these conditions are satisfied if thin mica sheets, coated with a silicone varnish containing aluminium powder, are interposed. The metal slabs are drawn together by bolts arranged round their periphery.

The fins attached to the hot face A of the cooling unit are larger than those entering the cooled chamber. This is a necessary requirement since whereas the latter merely have to abstract energy from the chamber, the former have to pass this energy, as well as that developed in the thermocouples, to the surroundings.

Experimental determination of the Thomson coefficient for a metal.—By measuring the e.m.f. of a thermocouple as a function of the temperature of its hot junction and using the theoretical equation on p. 268, the difference between the Thomson coefficients for the two metals comprising the thermocouple may be found. Since, however, the Thomson coefficient is a characteristic of each metal it is desirable that absolute values for the individual coefficients should be known. When the absolute value of the Thomson coefficient, σ, is known for one metal, then values for all other metals may be deduced from experiments such as those cited above. In 1922 BORELIUS devised a suitable method which, in conjunction with KEESOM and JOHANSSON, was improved in 1928. Essentially the method, which may be used at low temperatures, is as follows.

The metal under investigation is in the form of a wire about 0·1 cm. in diameter ; a piece about 7 cm. long is stretched in a vacuum between two massive leads maintained at a selected constant temperature. When a steady current is passed along the wire, the Joule heating effect raises its temperature ; it is a maximum at the centre of the wire so that in one half the Thomson effect produces a cooling effect, while in the other half there is a heating effect. As shown in Fig. 9·24 (a), a pair of platinum-constantan thermocouples is used to measure the temperature of the wire at two points H_1 and H_2, equally distant from O, the centre of the wire. The wire on test is coated with a thin layer of enamel lacquer, while shellac serves to improve the insulation between the wire and a thermocouple, and to hold it in position. Actually the temperatures of the cold junctions are kept at the same temperature as the massive leads, so that it is an excess temperature ϕ which is really measured.

Two essential measurements are necessary; for the first, in which the thermocouples are in series, the necessary connexions are shown by the thick lines between the mercury cups of the reversing key and it should be noted that, when these connexions are made, a resistor R_0 is added to the circuit so that the sensitivity of the galvanometer is reduced. When the second measurement is made, the thermocouples are in opposition; hatched lines show the connexions, and under these conditions R_0 is no longer operative.

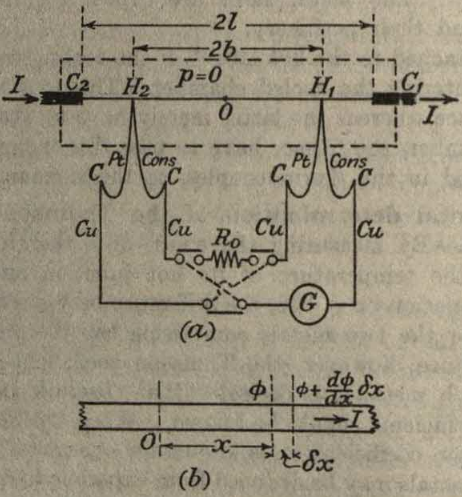


FIG. 9·24.—Determination of the absolute value of the Thomson coefficient of a metal at a fixed temperature.

Temperature differences due to asymmetry in the positions of H_1 and H_2 may be eliminated by reversing the direction of the heating current and repeating the observations. In the theory which follows it will be assumed that energy is lost only by conduction along the wire and κ, the thermal conductivity of its material, together with R, the resistance per unit length of the wire, are constant.

If S is the cross-sectional area of the wire, then since the total amount per second of energy developed within or entering and leaving the element shown in Fig. 9·24 (b) is zero, we have

$$-\kappa S\frac{d\phi}{dx} + \kappa S\frac{d}{dx}\left(\phi + \frac{d\phi}{dx}\delta x\right) + I^2R\delta x - I\sigma\delta\phi = 0,$$

i.e.

$$\kappa S\frac{d^2\phi}{dx^2} - I\sigma\frac{d\phi}{dx} + 0 = -I^2R.$$

For the complementary function, we have

$$\phi = A \exp \alpha x + B,$$

where A and B are constants and $\alpha = \frac{I\sigma}{\kappa S}$, while for the particular integral

$$\phi = \frac{I^2R}{\alpha}x + \frac{I^2R}{\alpha^2}.$$

The complete solution is therefore

$$\phi = A \exp \alpha x + B + \frac{I^2R}{\alpha}x + \frac{I^2R}{\alpha^2},$$

and the constants A and B may be found by using the facts that if $2l$ is the length of the wire, then $\phi = 0$ for both $x = -l$ and $x = l$. These end conditions give

$$0 = A \exp(-\alpha l) + B - \frac{I^2R}{\alpha}l + \frac{I^2R}{\alpha^2},$$

and $$0 = A \exp(\alpha l) + B + \frac{I^2R}{\alpha}l + \frac{I^2R}{\alpha^2}.$$

Subtracting and adding these equations in turn, we find

$$A = -\frac{I^2Rl}{\alpha}.\frac{1}{\sinh \alpha l},$$

and $$0 = A \cosh \alpha l + B + \frac{I^2R}{\alpha^2}.$$

Hence, after some reduction,

$$\phi = \frac{I^2Rl}{\kappa A\alpha \sinh \alpha l}\left[-\exp \alpha x + \cosh \alpha l + \frac{x}{l}\sinh \alpha l\right].$$

If H_1 and H_2 are the points $x = b$, $x = -b$, we have

$$\phi_{x=b} = \ldots \left[-\exp \alpha b + \cosh \alpha l + \frac{b}{l}\sinh \alpha l\right],$$

and $$\phi_{x=-b} = \ldots \left[-\exp(-\alpha b) + \cosh \alpha l - \frac{b}{l}\sinh \alpha l\right].$$

If $\phi_1 = \phi_{x=b}$, and $\phi_2 = \phi_{x=-b}$, we find

$$\frac{\phi_2}{\alpha b\phi_1} = \frac{\frac{1}{\alpha l}\sinh \alpha l - \frac{1}{\alpha b}\sinh \alpha b}{\cosh \alpha l - \cosh \alpha b}$$

$$= \tfrac{1}{3}, \text{ when } (\alpha l)^4 \rightarrow 0.$$

Hence $$\sigma = \frac{3\phi_2\kappa S}{\phi_1 I b}.$$

In these investigations it was found that the Thomson coefficient,

σ, was related to the apparent, i.e. measured value, σ_A, by the relation

$$\sigma = \sigma_A + \frac{C}{I^2},$$

where C is a constant and I the current through the wire. This was accounted for by assuming that slight inhomogeneities in the wire introduced a Peltier effect. It was also found that at very low temperatures the thermal conductivity of a pure metal varied so much with temperature that the expression for σ in terms of κ, etc., was no longer applicable. The following are some of the results obtained; the alloys contained 0·37 atomic per cent gold.

THOMSON COEFFICIENTS [μ-joule. coul.$^{-1}$ deg.$^{-1}$ $\equiv$ μV deg.$^{-1}$]

Metal	Temperature (° K.)	
	200	300
Copper	0·80	1·52
Copper + gold. . .	2·03	2·33
Silver + gold . . .	0·98	1·38

EXAMPLES IX

9·01. A heating coil, of resistance 21 ohms, immersed in oil contained in a calorimeter of total thermal capacity 200 cal.deg.$^{-1}$ C., is arranged in series with a copper voltameter. When a steady current is passed through the circuit for a certain time it is found that 0·0987 gm. of copper is deposited on the cathode of the voltameter, and that when the temperature of the calorimeter is equal to that of its surroundings the rate of rise of temperature is 0·045 deg. C. sec.$^{-1}$. Assuming the electrochemical equivalent of copper to be 0·000329 gm.coulomb.$^{-1}$, find the time of flow of the current. [224 sec.]

9·02. A torch bulb takes a current of 0·333 amp. from a battery of e.m.f. 4·5 volts, the potential difference across the lamp being 3·6 volts. Calculate values for the resistance of the filament and for the internal resistance of the battery.

If the battery costs 10*d.* and has a total working life of 100 minutes, what is the cost, in pounds sterling per kilowatt-hour, of the energy it supplies to the filament? [£16·7.]

9·03. The air, specific heat at constant pressure = 1·00 joule.gm.$^{-1}$ deg.$^{-1}$ C., which is used to remove the heat losses from a generator, rises in temperature from θ_1° C. to θ_2° C. as it is used. If the rate of heat loss is P kW., and the density of air at s.t.p. is 0·001293 gm.cm.$^{-3}$, find the volume of air in cubic metres which must pass through the generator per second to carry away the heat losses; the volume should be referred to θ_1° C. and the existing barometric height, H cm. of mercury.

Hence calculate the rate of flow of air in metre.^{3}sec.$^{-1}$ needed to carry off the losses of a 5000 kW. generator having an efficiency of 0·90 at full load, assuming that the air temperature rises from 10° C. to 30° C., H = 76·2, and the volume of air is to be measured at 10° C. and 76·2 cm. of mercury pressure.

$$\left[0{\cdot}775\frac{P}{\theta_2 - \theta_1}\cdot\frac{(273 + \theta_1)}{273}\cdot\frac{76}{H},\ 20{\cdot}0 \text{ metre.}^3\text{sec.}^{-1}.\right]$$

9·04. Define *electromotive force, internal resistance of a cell.*

Describe and explain an experiment to measure the internal resistance of an accumulator. A cell, e.m.f. E and internal resistance B, is connected to a resistance R. Show that W, the rate of dissipation of energy in R, is a maximum when R = B, also that

$$B = \frac{E^2}{4W_{\max}}.$$

Experimentally it was found that, for a certain battery, W varied with R according to the following table:

R (ohm.) . .	0·5	1·0	1·5	2·0	2·5	3·0
W (watt.) . .	0·62	0·83	0·90	0·90	0·88	0·85

Determine graphically the maximum rate of dissipation of energy, $W_{\max}$ and the corresponding value of R. If E = 2·5 volt. calculate a value for B, and comment on your result.

[0·91 watt., 1·74 ohm.]

9·05. (*a*) Electrons pass from a hot cathode across a vacuum space to a collecting plate which is maintained at a potential of 10 volts. relative to the cathode. Find the velocity of the electrons when they reach the plate assuming that their specific charge is $-1{\cdot}76 \times 10^7$ e.m.u. gm.$^{-1}$; they start from rest at the cathode. (*b*) If the cathode stream in a vacuum tube bears a current of 5 mA. and its velocity is one-tenth that of light, calculate, in calories, the heat developed per second by the bombardment of the anode, assuming that no other energy transformation occurs.

[(*a*) $1{\cdot}88 \times 10^8$ cm.sec.$^{-1}$, (*b*) 3·04 cal.sec.$^{-1}$.]

9·06. Give a detailed account of a steady flow method for determining the specific heat of water.

A wire used in a steady flow calorimeter carried a current of 4·45 amp. when the potential difference across it was 5·69 volt. The water flowed at a rate of 60·9 gm.min.$^{-1}$ and the observed rise in temperature was 5·95 deg. C. When no energy was dissipated in the wire the water at the exit apparently had a temperature excess of 0·10 deg. C. over the incoming water. When using a slower rate of flow of 16·9 gm.min.$^{-1}$ the observed temperature rise was 6·20 deg. C.; the current was 2·45 amp. and the potential difference 3·13 volt. Obtain values for the specific heat of water (*a*) neglecting heat losses, (*b*) making a correction for them.

[4·26 joule.gm.$^{-1}$deg.$^{-1}$ C.; 4·18 joule.gm.$^{-1}$deg.$^{-1}$ C.]

9·07. A tangent galvanometer and a coil of insulated resistance wire, immersed in a calorimeter containing water, are joined in series and a steady current, which gives a deflexion of $\tan^{-1} 0{\cdot}4$ on the galvanometer, is passed for 6 min. 58 sec. The temperature of the calorimeter and its contents rises from 15·80° C. (room temperature) to 22·51° C. while the current is passing and begins to fall at a rate 0·15 deg. C. min.$^{-1}$ when the current is switched off. If the galvanometer has 2 turns of wire of 15·7 cm. radius, H_0 = 0·20 oersted and the calorimeter of thermal capacity 9·5 cal. deg.$^{-1}$ C. contains 90·5 gm. of water, calculate a value for the resistance of the coil. Give a diagram of the electrical circuit and suggest any modifications which would lead to a more accurate result. [7·24 ohm.]

9·08. Six wires, each of resistance 2 ohms, connect the vertices of a hexagon ABCDEF. Each vertex is connected to the centre of the hexagon by a wire of resistance 1 ohm. A current of 2 amp. enters the system at A and leaves it at D. Find the drop in potential between A and D and the power dissipated in the network.
[2·22 V., 4·44 W.]

9·09. The rate of emission of energy from an electric fire is $a\theta^4$ watts; the resistance of its heater element is $(30 + b\theta^2)$ ohms where a and b are constants independent of the temperature θ of the heater. The fire is operated from a d.c. supply whose voltage, V, can be varied. The current is 4 A. when V = 200 volts. Show that in order to change the power to P, the voltage must be altered to $\sqrt{30P + 0{\cdot}71P^{1\cdot5}}$. Show that the current is then $\dfrac{\sqrt{30P + 0{\cdot}71P^{1\cdot5}}}{30 + 0{\cdot}71P^{0\cdot5}}$.

9·10. A No. 16 s.w.g. copper wire in an electric circuit is protected by a fuse wire No. 22 s.w.g. Assuming the room temperature to be 15° C. and the following constants for copper and for fuse wire, estimate how long it will take for the fuse wire to melt for a short-circuit current of 20 A. Also estimate the temperature of the copper wire when the fuse 'blows'. [Neglect all heat losses.]

	Copper	Fuse wire
Resistivity × 10^6, [ohm.cm.] . .	2·0	4·0
Density, [gm.cm.$^{-3}$]	9·0	7·0
Specific heat, [cal.gm.$^{-1}$ deg.$^{-1}$ C.] .	0·1	0·05
Melting point, [°C.]	—	190
Heat of fusion, [joule.gm.$^{-1}$]. . .	—	25

No. 16 s.w.g. = 1·63 mm. diam.
" 22 " = 0·71 mm. "

[2·50 sec., 27° C.]

9·11. A thermoelectric circuit consists of two equal bars of copper and iron with the ends joined and the junctions maintained respectively at 19° C. and 21° C. Compare the heat dissipated on account of the Joule effect with that conducted along the bars. The thermoelectric power at 20° C. may be taken as 14 μV. deg.$^{-1}$ C.; the resistivities of copper and iron as 1·6 and 9·8 microhm.cm., and the thermal conductivities as 0·9 cal.cm.$^{-1}$ sec.$^{-1}$ deg.$^{-1}$ C. and 0·2 cal.cm.$^{-1}$ sec.$^{-1}$ deg.$^{-1}$ C. respectively. [7·4 × 10^{-6}.]

9·12. Two semicircular rings, one of copper and one of constantan, are joined together to form a ring whose mean diameter may be taken as 14 cm. Each metal has a cross-sectional area of 2 cm.^{2}. If one junction is kept at 0° C. and the other at 200° C., estimate the current that will flow in the ring and the magnetic field strength at its centre.

[Assume $\chi_{Cu} = 2 \times 10^{-6}$ ohm.cm., $\chi_{cons} = 50 \times 10^{-6}$ ohm.cm., and use the data given on p. 276.] [17·4 A., 1·56 oersted.]

9·13. Describe and explain how to determine by experiment the relation between the e.m.f., E, of a copper-iron thermocouple and the temperature, θ, of its hot junction, the cold junction being maintained at 0° C.

The relation between E and θ for a certain thermocouple is given in the following table:

θ° C. . .	10	30	60	100
E (μV) .	98	282	528	800

If the relation is $E = \alpha\theta + \frac{1}{2}\beta\theta^2$, where α and β are constants, determine their values. [$\alpha = 10$ μV. deg.$^{-1}$ C., $\beta = -0{\cdot}04$ μV. deg.$^{-2}$ C.]

9·14. A thermocouple, in series with a fixed resistor and a galvanometer, was used as a thermometer to determine the melting point of an alloy. The cold junction was kept in melting ice and when the hot junction was placed in water at 100° C. and then in molten lead at 327° C. the galvanometer deflexions were 76 and 219 scale divisions respectively. The galvanometer reading corresponding to the melting point of the alloy was 175 scale divisions. Assuming a relation of the form $E = \alpha\theta + \frac{1}{2}\beta\theta^2$, where α and β are constants and θ is the temperature of the hot junction, obtain a value for the melting point of the alloy. [250° C.]

9·15. The thermoelectric power of iron with respect to lead is 16·65 μV. deg.$^{-1}$ C. at 0° C. and changes by $-$ 0·030 μV. for one degree C. increase in temperature. For copper and lead the corresponding numbers are 2·76 and 0·012 respectively. Construct the thermoelectric lines for these metals and determine

(a) the neutral temperature for iron-copper,
(b) the e.m.f. of an iron-copper thermocouple whose junctions are kept at 100° C. and 200° C. respectively,
(c) the values of the Peltier coefficient for the two temperatures and the difference between the Thomson coefficients for the two metals at 100° C.

Verify your conclusions by an analytical method.
[331° C., 0·759 mV., 5·73 mV., 2·60 mV., 15·6 μV. deg.$^{-1}$ C.]

9·16. Determine the e.m.f. of a silver-iron thermocouple with junctions at 20° C. and 80° C. The thermoelectric powers of silver and iron may be taken as $3{\cdot}34 + 0{\cdot}008\theta$ and $16{\cdot}65 - 0{\cdot}030\theta$ respectively, where θ is the temperature in degrees centigrade. [Unit is 1 μV. deg.$^{-1}$ C.] [685 μV.]

9·17. The current sensitivity of a moving-coil galvanometer is proportional to the square root of its resistance provided the distribution of mass in the coil is kept constant. Determine the resistance of the galvanometer you would select to measure the small thermoelectric e.m.f. of a couple if its resistance and that of its associated leads amounts to 15 ohms. [15 ohm.]

9·18. The thermoelectric power of a lead-copper circuit is

$$(2{\cdot}76 + 0{\cdot}012\theta)\ \mu\text{V. deg.}^{-1}\ \text{C.};$$

for a lead-iron circuit it is

$$(16{\cdot}65 - 0{\cdot}030\theta)\ \mu\text{V. deg.}^{-1}\ \text{C.}$$

Calculate the e.m.f. in a copper-iron circuit whose junctions are at 50° C. and 250° C. respectively. What is the neutral temperature? [1·518 mV., 331° C.]

9·19. Assuming that the thermoelectromotive force in a certain thermocouple is given by

$$\text{E} = (17{\cdot}35\theta - 0{\cdot}025\theta^2)\ \mu\text{V.},$$

where θ is the temperature (centigrade) of one junction, the other being maintained at 0° C., find out as much as you can about the state of affairs in the circuit when $\theta = 100°$ C.

Establish all essential formulæ.

[$\text{E}_0{}^{100} = 1485\ \mu$V., $\Pi_{100} = 4{\cdot}61$ mV., $\Pi_0 = 4{\cdot}73$ mV., $\sigma_1 - \sigma_2 = 0{\cdot}05\text{T}$.]

9·20. Describe and explain how you would measure the e.m.f. in a circuit comprising two wires, of different materials, whose junctions are kept at different temperatures. In a certain instance with one junction at 0° C., the following observations were made :—

Temp. of 'hot' junction (° C.)	50	100	180	250	300	400
e.m.f. (μV.)	175	300	396	375	300	0

Find an expression which shall represent the thermoelectric power of the above pair of metals. [$(4{\cdot}0 - 0{\cdot}020\theta)\ \mu$V. deg.$^{-1}$ C.]

9·21. If the thermoelectric powers of copper and of constantan are $(2{\cdot}76 + 0{\cdot}012\theta)\ \mu$V. deg.$^{-1}$ C. and $(-\ 37{\cdot}76 - 0{\cdot}079\theta)\ \mu$V. deg.$^{-1}$ C. respectively, find the e.m.f. in a copper-constantan circuit whose junctions are at 50° C. and 200° C. If the circuit consists of a copper bar in the form of a U 20 cm. × 2 cm.^{2} whose ends are silver-soldered to a constantan bar 5 cm. × 2 cm.^{2}, estimate the current that will flow if the resistivities of copper and of constantan may be taken as 2×10^{-6} ohm.cm. and 50×10^{-6} ohm.cm. respectively.

[− 7·79 mV., 54 A.]

9·22 (*a*). Using the data in the table given on p. 276, show that for a platinum/platinum-rhodium thermocouple the neutral temperature is −255° C.

(*b*) Prove that $\Pi = \text{T}\dfrac{d\text{E}}{d\text{T}}$ and hence show that if $\text{E} = \alpha\theta + \frac{1}{2}\beta\theta^2$, where the symbols have their usual meanings, then

$$\Pi = a + b\theta + c\theta^2,$$

where a, b and c are constants; determine their values.

[$273\alpha,\ \alpha + 273\beta,\ \beta$.]

9·23. For a chromel-alumel thermocouple, whose cold junction is maintained at 0° C., the e.m.f. is 18·37 μV. and 44·24 μV., when the hot junction is at 444·6° C. and 1083° C. respectively. Calculate values when the hot junction is at 600° C. for

(*a*) the thermal e.m.f.,
(*b*) the thermoelectric power,
(*c*) the Peltier coefficient,
(*d*) the Thomson coefficient.

[24·71 μV., 0·04076 μV. deg.$^{-1}$ C., 35·58 μV., $\sigma_1 - \sigma_2 = 6{\cdot}43 \times 10^{-4}$ μV. deg.$^{-1}$ C.]

9·24. By means of the formula

$$E = \alpha(\theta_2 - \theta_1) + \tfrac{1}{2}\beta(\theta_2^2 - \theta_1^2), \; [\theta_2 > \theta_1],$$

show that the e.m.f. in a thermoelectric circuit may be represented as the difference of two terms, each of which involves, apart from a common constant, only one of the temperatures θ_1 and θ_2 together with the neutral temperature, θ_n.

$$\left[\mathrm{E} = \beta\left\{\theta_2\left(\frac{\theta_2}{2} - \theta_n\right) - \theta_1\left(\frac{\theta_1}{2} - \theta_n\right)\right\}.\right]$$

9·25. What is a thermoelectric diagram? Indicate the information which can be obtained from it.

In an experiment with a copper-iron thermocouple with one junction at 0° C., the following results were obtained with the hot junction at θ° C.—

θ	20	200	320
e.m.f. (mV.) . .	0·204	1·356	1·427

Assuming a parabolic law, find (*a*) the thermoelectric power at 100° C., (*b*) the neutral temperature for the couple. (G)

[6·8 μV. deg.$^{-1}$ C., 279° C.]

9·26. Prove $\sigma_a - \sigma_b = -\mathrm{T}\dfrac{d^2\mathrm{E}}{d\mathrm{T}^2}$, where the symbols have their usual meanings.

If, for a certain thermocouple $\mathrm{E} = a\theta + b\theta^2$, where θ° C. is the temperature of the hot junction, the cold junction being at 0° C., $a = 10$ μV. deg.$^{-1}$ C. and $b = -\frac{1}{40}$ μV. deg.$^{-2}$ C., calculate

(*a*) the temperature at which E = 0,
(*b*) „ „ „ „ Π is a maximum, (Π is the Peltier coefficient)
(*c*) „ „ „ „ E is solely due to the Thomson effect.

[(a) 0° C., 400° C., (*b*) − 36·5° C., (*c*) − 73° C.]

9·27. A certain thermocouple, with one junction at 0° C. and the other at θ° C., has an e.m.f. given by

$$\mathrm{E} = [100\theta - 15000\{1 - \exp(-0{\cdot}0025\theta)\}]\ \mu\mathrm{V}.$$

Determine the Peltier coefficient at 0° C., and show that the difference between the two Thomson coefficients is practically stationary in a small temperature range in the neighbourhood of 127° C.

[17·1 mV.]

9.28. The e.m.f., E, of a certain thermocouple may be represented by the equation $E = a\theta + b\theta^2 + c\theta^3$, where a, b and c are constants and θ° C. is the temperature of the hot junction. The cold junction is kept at 0° C. If suffixes denote corresponding values of E and θ show that

$$\left(\frac{E_2}{\theta_2} - \frac{E_1}{\theta_1}\right)\left(\frac{1}{\theta_2 - \theta_1}\right) = b + c(\theta_2 + \theta_1).$$

Hence suggest a method for finding the constants b and c. The following table refers to a constantan–copper thermocouple.

θ	100° C.	200° C.	300° C.
E(μV) . . .	4460	9680	15,420

Obtain values for a, b and c.

[$a = 40\ \mu$V. deg.$^{-1}$ C., $b = 5{\cdot}0 \times 10^{-2}\ \mu$V. deg.$^{-2}$ C., $c = -4{\cdot}0 \times 10^{-5}\ \mu$V. deg.$^{-3}$ C.]

9·29. For a chromel-alumel thermocouple with one junction at 0° C. and the other at θ° C., the following observations were obtained.

θ	0	20	100	200	300	500	700	900	1000
e.m.f., E (mV.)	0	0·08	3·38	7·40	11·49	19·92	28·42	36·62	40·60

Assuming $E = a\theta^n$, where a and n are constants, plot a large scale logarithmic graph and determine values for these constants.

[a = 0·0200 mV., n = 1·175.]

9·30. Use the data given on p. 276 to show that for a nickel-silver thermocouple $E = 19{\cdot}64\theta + 0{\cdot}0175\theta^2$. Assuming that the Peltier coefficient is given by $\Pi = (1282 + 6{\cdot}98\theta + 0{\cdot}0837\theta^2)$ μ-cal. coul.$^{-1}$, obtain a value for the melting point of ice on the Kelvin scale of temperature. [272·8° K.]

9·31. In a thermocouple constructed from lead and a semiconductor, the thermoelectric power is given by $\frac{dE}{d\theta} = a + \frac{b}{273 + \theta}$, where a and b are constants, and θ is the temperature of one junction, the other being kept at 0° C. Obtain an expression for the rate of absorption of energy in the semiconductor when a current of I amp. flows through it and the temperature difference between its ends is one degree on the centigrade scale. $\left[\frac{bI}{273 + \theta}\text{ watt.}\right]$

CHAPTER X

ELECTRICAL MEASURING INSTRUMENTS

Forces and torques on rigid circuits.—The following discussion is limited to a pair of circuits in order to simplify the necessary analysis as much as possible. In addition, each circuit will be assumed rigid so that only relative motion between the circuits may occur.

Let one circuit be fixed with respect to the observer and suppose that the other circuit suffers a small displacement δx, cf. Fig. 10·01.

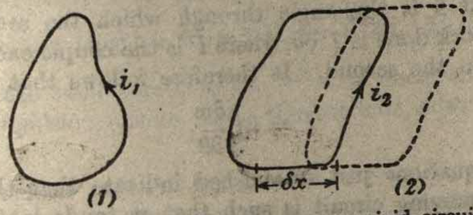


FIG. 10·01.—Forces and torques on rigid circuits.

Let F be the force on the second circuit in the direction of x increasing so that F δx is the work done on it. Since each circuit is rigid, their respective self inductances, l_1 and l_2, remain constant but the mutual inductance, m, varies with x. If the currents in the two circuits are i_1 and i_2 respectively, the flux of magnetic induction through (i) is

$$\Phi_1 = l_1 i_1 + m i_2,$$

and the e.m.f. induced in (i) when i_1, i_2 and m each vary with time, is

$$e_1' = -\left[l_1\frac{\partial i_1}{\partial t} + \frac{\partial}{\partial t}(m i_2)\right] = -\left[l_1\frac{\partial i_1}{\partial t} + m\frac{\partial i_2}{\partial t} + i_2\frac{\partial m}{\partial t}\right].$$

If e_1 is the applied e.m.f. in the circuit (i) and r_1 its resistance

$$e_1 + e_1' = r_1 i_1,$$

or

$$l_1\frac{\partial i_1}{\partial t} + m\frac{\partial i_2}{\partial t} + i_2\frac{\partial m}{\partial t} + r_1 i_1 = e_1.$$

Similarly, the value of e_2 may be written down. The rate of working in the two circuits is therefore

$$\begin{aligned} P &= e_1 i_1 + e_2 i_2 \\ &= l_1 i_1 \frac{\partial i_1}{\partial t} + l_2 i_2 \frac{\partial i_2}{\partial t} + m i_1 \frac{\partial i_2}{\partial t} + m i_2 \frac{\partial i_1}{\partial t} + 2 i_1 i_2 \frac{\partial m}{\partial t} + r_1 i_1^2 + r_2 i_2^2, \end{aligned}$$

the last two terms representing the Joule effect.

Now the energy stored in the magnetic field associated with the two circuits is given by

$$W = \tfrac{1}{2} l_1 i_1^2 + m i_1 i_2 + \tfrac{1}{2} l_2 i_2^2, \qquad \text{[cf. p. 238]}$$

$$\therefore \frac{\partial W}{\partial t} = l_1 i_1 \frac{\partial i_1}{\partial t} + l_2 i_2 \frac{\partial i_2}{\partial t} + i_1 i_2 \frac{\partial m}{\partial t} + m i_1 \frac{\partial i_2}{\partial t} + m i_2 \frac{\partial i_1}{\partial t}.$$

But the rate of working by the e.m.fs. applied to the two circuits is equal to

$$\text{(the rate at which the force F does work)} + \frac{\partial W}{\partial t} + r_1 i_1^2 + r_2 i_2^2.$$

$$\therefore \mathrm{F} \frac{\partial x}{\partial t} = i_1 i_2 \frac{\partial m}{\partial t}, \quad \text{or} \quad \mathrm{F} = i_1 i_2 \frac{\partial m}{\partial x}.$$

Similarly, if θ is the angle through which the second circuit rotates, the work done is $\Gamma\,\delta\theta$, where Γ is the couple exerted by the first circuit on the second. It therefore follows that

$$\Gamma = i_1 i_2 \frac{\partial m}{\partial \theta}.$$

The two equations just established indicate that the force or torque on a moving circuit is such that m tends to increase.

Effect of current on current.—From Ampère's law of the equivalence of a current circuit to a uniform magnetic shell we should expect that forces (and/or couples) would exist between two circuits carrying electric currents. Such effects may easily be produced and their magnitudes calculated from the forces (and/or couples) between the equivalent magnetic shells. Thus, consider two circular circuits mutually at right angles as indicated in Fig. 10·02 (a), where AB represents a large circular coil and CD a small one, each coil having one turn only. Now the magnetic field at the centre of AB is $\dfrac{2\pi i_1}{r_1}$, where r_1 is its radius and i_1 the current flowing in it, and the magnetic moment of the equivalent magnetic shell for the small coil CD is $\mu \alpha i_2$, where μ is the permeability of the medium in which the coil CD lies, α its area and i_2 is the current in it. Hence CD will experience a couple $\dfrac{2\pi\mu i_1 i_2 \alpha}{r_1}$ tending to twist its plane into that of AB. [If the two coils are inclined at an angle θ as in Fig. 10·02 (b), the couple is $\frac{2\pi\mu a i_1 i_2 \cos\theta}{r_1}\Big]$.

Again, a linear current, i_1, produces a magnetic field $\frac{2i_1}{r}$ at a distance r from it; since the magnetic induction is μ times this, a second linear current i_2, parallel to i_1, cf. Fig. 10·02 (c), will experience per unit length a force i_2B, or $\frac{2\mu i_1 i_2}{r}$. If the two currents flow in the same direction the force on each wire is one of attraction; repulsion ensues when the currents flow in opposite directions.

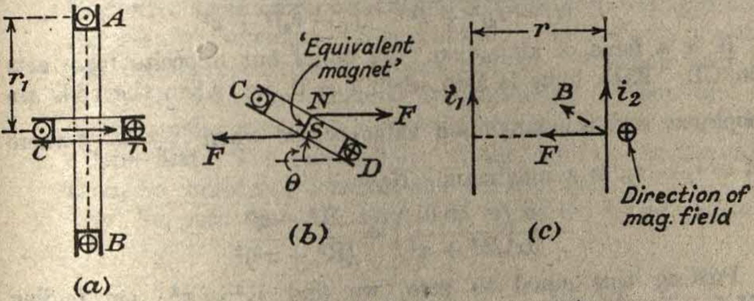


FIG. 10·02.—Effect of current on current.

Coaxial circular coils.*—(a) First let us consider two circular coaxial coils of very nearly the same radius and situated at a small distance apart as indicated in Fig. 10·03 (a). If μ is the permeability of the medium surrounding the coils, the force on each unit length of either coil may be taken as $\frac{2\mu i_1 i_2}{\text{AC}}$ and for an element at A of the coil AB this force acts along AC. The component of this, normal to

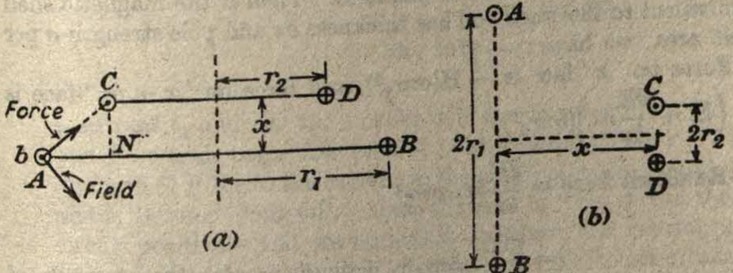


FIG. 10·03.—Coaxial circular coils.

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* Each coil has one turn only.

the axis of the coil, taken all round the coil will, by symmetry, vanish, but for unit length the component parallel to the axis is

$$\frac{2\mu i_1 i_2}{\mathrm{AC}} \cdot \frac{\mathrm{CN}}{\mathrm{AC}} = \frac{2\mu i_1 i_2 x}{(\mathrm{AC})^2}.$$

For the whole coil, since the total length $= 2\pi r_1$, the force is

$$\frac{2\mu i_1 i_2 x . 2\pi r_1}{(r_1 - r_2)^2 + x^2} = \frac{4\pi\mu i_1 i_2 r_1 x}{(r_1 - r_2)^2 + x^2}.$$

It is a force of attraction. An equal but opposite force acts on CD. Each force is zero when $x = 0$, i.e. when the coils are coplanar and the maximum value occurs when $\dfrac{x}{\mathrm{R}^2 + x^2}$, where $\mathrm{R} = r_1 - r_2$, is a maximum. Now

$$\frac{\partial}{\partial x}\left(\frac{x}{\mathrm{R}^2 + x^2}\right) = \frac{\mathrm{R}^2 - x^2}{(\mathrm{R}^2 + x^2)^2}.$$

Putting this equal to zero, we find $\mathrm{R}^2 = x^2$. On finding $\dfrac{\partial^2}{\partial x^2}\left(\dfrac{x}{\mathrm{R}^2 + x^2}\right)$ and substituting R^2 for x^2, the result is negative and therefore $\mathrm{R}^2 = x^2$ or $(r_1 - r_2)^2 = x^2$ corresponds to a maximum. The force between the coils is therefore a maximum when

$$x = \pm (r_1 - r_2)$$

and its value is then

$$\frac{2\pi\mu i_1 i_2 r_1}{(r_1 - r_2)}.$$

(*b*) When the coil r_2, cf. Fig. 10·03 (*b*), is so small that the variation of magnetic field over its cross-section due to the current in the coil r_1 is negligible, let H be this field. Then if the magnetic shell equivalent to the small coil has thickness δx and pole strength σ per unit area, we have :—

Force on 'x'-face is $-\mathrm{H}(\sigma\pi r_2^2)$, and force on '$x + \delta x$'-face is $+\left(\mathrm{H} + \dfrac{\partial \mathrm{H}}{\partial x}\delta x\right)(\sigma\pi r_2^2)$.

$$\therefore \text{Resultant force} = \frac{\partial \mathrm{H}}{\partial x}\delta x . \sigma\pi r_2^2$$

$$= \frac{\partial \mathrm{H}}{\partial x} \cdot \mu i_2 \pi r_2^2, \quad [\because \mu i_2 = \sigma\delta x, \text{ the strength of the shell.}]$$

μ being the permeability of the surrounding medium.

Now

$$\mathrm{H} = \frac{2\pi r_1^2 i_1}{(r_1^2 + x^2)^{1.5}}.$$

$$\therefore \frac{\partial \mathrm{H}}{\partial x} = -\frac{6\pi r_1^2 i_1 x}{[r_1^2 + x^2]^{2.5}}.$$

∴ Force tending to increase the distance between the coils is

$$\frac{-6\pi^2\mu r_1^2 r_2^2 i_1 i_2 x}{\{r_1^2+x^2]^{1\cdot5}}.$$

This is zero when $x = 0$ and a maximum when $x = \frac{1}{2}r_1$.

The forces and couples on a small circular coil lying with its centre on the axis of a larger circular coil, both coils carrying steady currents.—This problem, when the coils are circular, has already been solved by one method; it is now proposed to verify the solutions obtained by considering the potential energy of the smaller coil (not necessarily circular) in the magnetic field of the first. The permeability of the surrounding medium will again be considered constant and denoted by μ.

(*a*) Suppose that AB, Fig. 10·04 (*a*), is the larger coil carrying a current i_1 and CD the smaller coil carrying a current i_2. Let O and P be their respective centres; let us write $OP = x$. The field strength at P due to the current in the large coil is $2\pi i_1 r_1^2(r_1^2 + x^2)^{-1\cdot5}$ and is directed along OP. Since the lines of force thread the second circuit positively and are normal to its plane, the mutual potential energy is obtained by multiplying the magnetic induction associated with the field by $-i_2\pi r_2^2$ [cf. p. 125]. [If CD is a small coil of area S, the multiplying factor is $-i_2$S.]

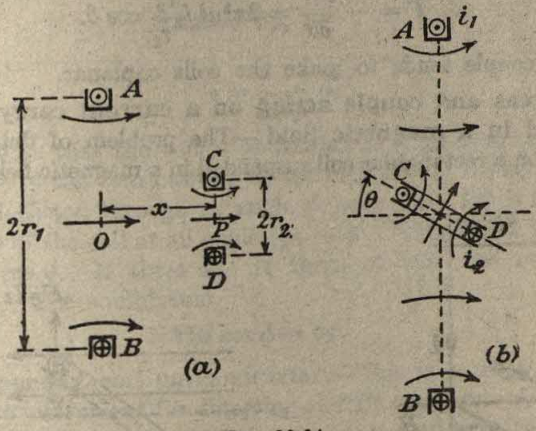


FIG. 10·04.

$$\therefore\ W = -2\pi^2\mu i_1 i_2 r_1^2 r_2^2 (r_1^2 + x^2)^{-1\cdot5}.$$

∴ Force tending to increase x is

$$-\frac{\partial W}{\partial x} = -6\pi^2\mu r_1^2 r_2^2 i_1 i_2 x(r_1^2 + x^2)^{-2\cdot5}.$$

The circuits therefore attract one another when the currents are in the same sense and repel one another when the senses are contrary.

(*b*) The value of the couple on a small coil situated at the centre of a large coil has already been obtained. Its value may also be calculated by considering the potential energy of the small coil in the field of the first. Thus if AB, Fig. 10·04 (*b*), is a single turn of radius r_1, the field at its centre is $\frac{2\pi i_1}{r_1}$. Since the magnetic induction is $\frac{2\pi\mu i_1}{r_1}$, where μ is the permeability of the surrounding medium, the potential energy of the system is

$$-\frac{2\pi\mu i_1 i_2}{r_1}\sin\theta.\pi r_2^2 = \text{W}.$$

$$\therefore \Gamma = -\frac{\partial \text{W}}{\partial\theta} = 2\pi^2\mu i_1 i_2\frac{r_2^2}{r_1}\cos\theta.$$

Thus the couple tends to make the coils coplanar.

The forces and couple acting on a current carrying coil suspended in a magnetic field.—The problem of determining the forces on a rectangular coil suspended in a magnetic field is very

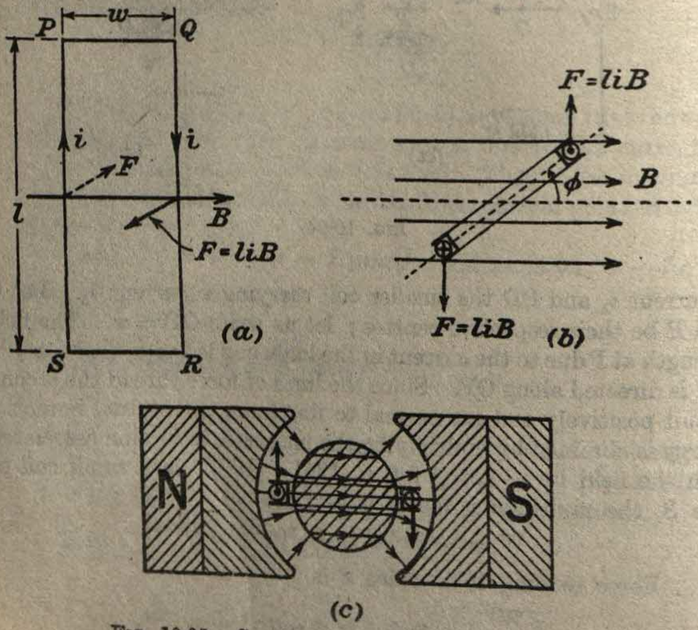


FIG. 10·05.—Couple on a coil in a magnetic field.

important since upon its solution depends the principle of many current measuring instruments. Let PQRS, Fig. 10·05 (*a*), be a fixed rectangular coil of length l and breadth w placed in a magnetic field whose direction is always in the plane of the coil. This field is a radial one and is obtained by using the system shown in Fig. 10·05 (*c*). If the current flows in the direction indicated each vertical side of the wire will experience a force $li\text{B}$, where B is the magnetic induction at points on the vertical sides of the coil. These forces act in the directions shown. They constitute a couple whose moment is $li\text{B}.w = i\text{AB}$, where A is the area of the coil.

If the coil is made of N turns of wire each with an area A, the couple becomes $\text{NAB}i$.

For equilibrium this couple must be balanced by the couple due to the relative twist ϕ between the ends of the suspension. If b is the restoring couple per unit twist in the suspension, then for equilibrium

$$\text{NAB}i = b\phi,$$

i.e.

$$i = \frac{b}{\text{NAB}}\phi.$$

Now suppose that the coil is suspended in a uniform magnetic field parallel to the zero position of the coil, cf. Fig. 10·05 (*b*). If the coil is deflected through an angle ϕ, each force $li\text{B}$ is inclined to the plane of the coil at an angle $(\frac{1}{2}\pi - \phi)$. The couple is therefore $li\text{B}.w \cos \phi$. If there are N turns of wire the couple is $\text{NAB}i \cos \phi$. For equilibrium

$$\text{NAB}i \cos \phi = b\phi.$$

The suspended-coil galvanometer.—This is a most reliable and sensitive instrument for detecting electric currents. A narrow coil, Fig. 10·06 (*a*), consisting of many turns of wire (only the frame on which these are wound is shown) is suspended from a movable head by a fine phosphor bronze strip between the poles of a strong magnet. When a current flows through the coil, forces act in contrary directions on the opposite sides of the coil, i.e. a couple acts on the coil which rotates until the twist in the suspension produces an equal and opposite couple. To make the couple acting on the coil as large as possible for a given current the coil must consist of many turns of wire and be placed in a strong magnetic field; the method of winding is shown in Fig. 10·06 (*b*). To increase the deflexion the suspension is made very fine so that the torsional couple per unit angular displacement is small. In order to concentrate the field and arrange that it is always parallel to the plane of the coil for all positions of the latter not far removed from its position of rest a cylindrical piece of soft iron is fixed midway between the poles of the magnet, as shown in Fig. 10·06 (*b*). The current enters the coil through the suspension and leaves through a very fine spiral below the coil. This spiral is made of very fine wire and consists of several relatively large turns so that it shall exert only a small restoring couple on the coil. A mirror rigidly attached to the framework of the coil or to the lower and thicker portion of the suspending wire—there must be no relative motion between the mirror and the coil—enables any small deflexion to be measured. Let us assume that ϕ is the steady angular deflexion and that b is the couple in the strip due to unit (radian) difference of twist between its ends. Then $b\phi$ is the couple in the present instance. But the moment of the forces on the coil is $i\mathrm{NAB}$, if it consists of N equal turns each of area A. When equilibrium is attained this must be equal and opposite to the couple due to the twist in the suspension, i.e.

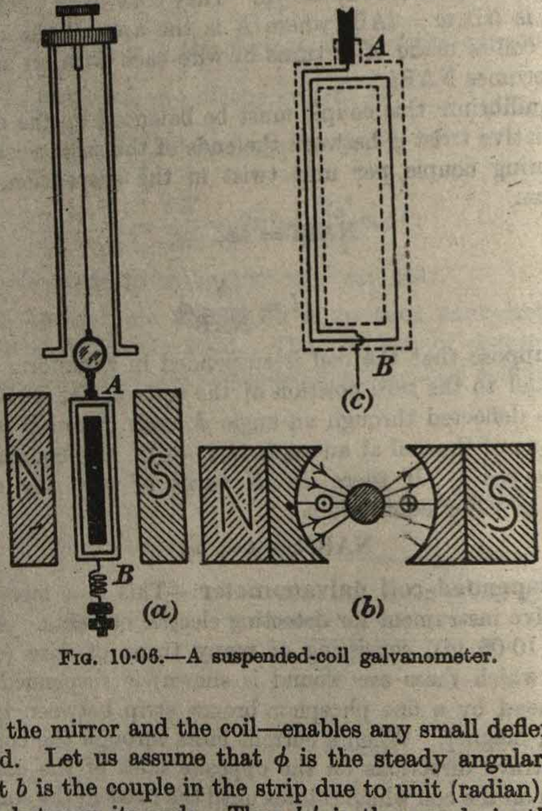


FIG. 10·06.—A suspended-coil galvanometer.

$$i\mathrm{NAB} = b\phi, \quad \text{or} \quad i = \frac{b\phi}{\mathrm{NAB}}.$$

In modern moving coil galvanometers of great sensitivity special care is taken to use for the coil copper wire which is as near as possible entirely free from traces of iron. In addition, in order to make the coil light no frame is used to carry the actual coil. Instead, the wire, as it is wound on a former, is passed through a Bakelite varnish such as 'Damarda'. When the winding is completed, the whole is heated to about 100° C. to drive off all solvent and then the framework is removed. The coil is then very rigid and requires no supporting frame when used as part of a galvanometer.

Such a feature is very desirable when the galvanometer is used ballistically [cf. p. 330], but when the coil forms part of a moving coil microammeter, ammeter, voltmeter, etc., then it is usual to wind the coil on a brass framework so that the motion becomes almost critically damped.

Example.—The current sensitivity of a given type of galvanometer may be taken as being directly proportional to the square root of its resistance, R. What would be the resistance of the galvanometer you would select to measure the e.m.f. of a thermocouple, if the resistance of the thermocouple and its connexions to the galvanometer is 12 ohms ?

The current sensitivity is defined as $\lim \frac{\delta\phi}{\delta I}$, i.e. $\frac{d\phi}{dI}$, where $\delta\phi$ is the change in deflexion due to a change δI in the current. Since the sensitivity in this instance is proportional to $R^{\frac{1}{2}}$ we may write $\frac{d\phi}{dI} = kR^{\frac{1}{2}}$, where k is a constant. On integrating, we obtain

$$\phi = kR^{\frac{1}{2}}I,$$

the integration constant being zero.

If E is the e.m.f. of the thermocouple, then

$$I = \frac{E}{12 + R},$$

and this current will produce a deflexion ϕ given by

$$\phi = kIR^{\frac{1}{2}} = k.\frac{E}{12 + R}.R^{\frac{1}{2}}.$$

Thus $\phi = \alpha\left(\frac{R^{\frac{1}{2}}}{12 + R}\right)$, where α is a constant.

$$\therefore \frac{d\phi}{dR} = \alpha\left[\frac{\frac{1}{2}R^{-\frac{1}{2}}}{12 + R} - \frac{R^{\frac{1}{2}}}{(12 + R)^2}\right].$$

This is zero, corresponding to a maximum value of ϕ, if

$$\tfrac{1}{2}R^{-\frac{1}{2}} - \frac{R^{\frac{1}{2}}}{12 + R} = 0.$$

i.e. $$\frac{1}{2} - \frac{R}{12 + R} = 0,$$

i.e. $$R = 12 \text{ ohms}.$$

Thus a galvanometer with a resistance of 12 ohms is required.

Example.—An accumulator, e.m.f. E and negligible internal resistance, is connected to two resistors AC and CB in series. The resistance of AC is $\frac{R}{n}$, while that of CD is $\left(1 - \frac{1}{n}\right)R$, where n is a constant. Across AC there is placed a galvanometer (resistance G) in series with a thermocouple; the e.m.f. of the thermocouple is E_0 and it has a resistance R_0. Obtain an expression for the galvanometer current.

A galvanometer with interchangeable suspensions is available. If its sensitivity S is proportional to $G^{\frac{1}{2}}$, find the optimum value for G.

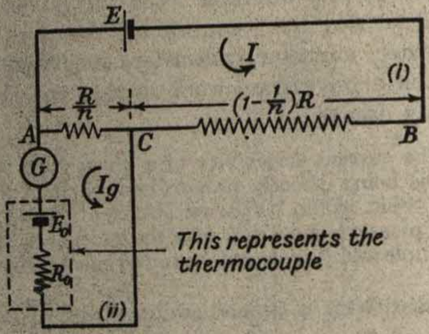


FIG. 10·07.

Suppose the circuit is arranged as shown in Fig. 10·07. Let I and I_g be the cyclic currents indicated. Then for the loop (i) we have

$$E = (I - I_g)\frac{R}{n} + I\left(1 - \frac{1}{n}\right)R$$

$$= IR - \frac{R}{n}I_g.$$

For the loop (ii),

$$-E_0 = I_g(G + R_0) + (I_g - I)\frac{R}{n}.$$

$$\therefore \quad I_g = \frac{(E - nE_0)n}{(n-1)R + n^2(R_0 + G)},$$

which may be written in the form

$$I_g = \frac{A}{B + CG},$$

where A, B and C are constants.

Now $S = \frac{\phi}{I_g}$, where ϕ is the galvanometer deflexion.

$$\therefore \quad \phi = \frac{kA\sqrt{G}}{B + CG},$$

if $S = kG^{\frac{1}{2}}$, where k is a constant.

$$\therefore \quad \frac{d\phi}{dG} = kA\left[\frac{\frac{1}{2\sqrt{G}}(B + CG) - C\sqrt{G}}{(B + CG)^2}\right],$$

this is zero if $B + CG = 2CG$,

i.e.
$$G = \frac{B}{C} = \frac{(n-1)R + n^2R_0}{n^2}.$$

Since $\phi = 0$ when $G = 0$ and also when $G = \infty$, and ϕ is essentially positive, the above condition corresponds to a maximum deflexion.

The tangent galvanometer.—It has been shown [cf. p. 132] that the magnetic field, H, at the centre of a circular coil carrying a current i, is equal to $\frac{2\pi Ni}{a}$, where N is the number of turns each of radius a. A tangent galvanometer is an instrument in which use is made of the above formula in order to measure a current in electromagnetic units. The coil is placed in the magnetic meridian; at the centre of the coil there is suspended a compass needle free to turn in a horizontal plane about a vertical axis through its centre of gravity. The length of the needle must be small compared with the radius of the coil.

When a steady current passes through the coil the magnetic field it produces at the centre is compared with H_0, the horizontal component of the earth's magnetic field. If ϕ is the deflexion we have the usual relation for two magnetic fields mutually perpendicular, viz.

$$H = H_0 \tan \phi$$

Since
$$H = \frac{2\pi Ni}{a},$$

we have
$$i = \frac{aH_0}{2\pi N} \tan \phi.$$

This formula is often written $i = \frac{H_0}{G} \tan \phi$, where G is the galvanometer constant; G is the magnetic field at the centre of the coil per e.m.u. of current in it. It may also be written $i = k \tan \phi$, where k is the reduction factor in e.m.u. for the galvanometer. If I is the current in amperes, then

$$\frac{I}{10} = \frac{H_0}{G} \tan \phi, \quad \text{or} \quad I = \frac{10H_0}{G} \tan \phi.$$

This may be written

$$I = K \tan \phi,$$

where $K = \frac{10H_0}{G}$ is the reduction factor for the galvanometer. It is expressed in amperes and for a given galvanometer is not a constant, except at one particular location, since it involves H_0.

The sensitivity and accuracy attainable with a tangent galvanometer.—If ϕ is the angular deflexion of the needle when the current is i, the sensitivity is defined as $\frac{d\phi}{di}$. For a tangent galvanometer $i = k \tan \phi$, so that $\frac{d\phi}{di} = k^{-1} \cos^2 \phi$, i.e. the sensitivity is a maximum when $\phi = 0$. This conclusion is of little practical importance.

In dealing with a tangent galvanometer it is more important to consider a quantity which will be called 'the accuracy of the galvanometer'. If $\delta\phi$ is the change in deflexion when the current is increased from i to $i + \delta i$, then $\frac{\delta i}{i}$ is the fractional change in the current, and

$$\lim \frac{\delta\phi}{\left(\frac{\delta i}{i}\right)}, \quad \text{or} \quad i\frac{d\phi}{di},$$

is called the 'accuracy of the galvanometer'; let it be denoted by α. Then

$$\alpha = i\frac{d\phi}{di} = k \tan \phi . k^{-1} \cos^2 \phi = \tfrac{1}{2} \sin 2\phi.$$

Since the maximum value of $\sin 2\phi$ is 1, it follows that the accuracy will be a maximum when $2\phi = \frac{\pi}{2}$, i.e. $\phi = \frac{\pi}{4}$.

Example.—Prove that if a tangent galvanometer is used to measure current from a cell, e.m.f. E volts and internal resistance B ohms, the instrument, for which $I = K \tan \phi$, will be used with maximum accuracy if it is shunted by a resistance S such that

$$S = \frac{KR_g(B + R)}{E - K(R_g + B + R)},$$

where R_g is the resistance in ohms of the galvanometer and R that of a coil in series with the cell.

If I is the current from the cell,

$$I = \frac{E}{R + B + \frac{R_g S}{R_g + S}}.$$

Hence the current through the galvanometer, I_g, is given by

$$I_g = \frac{S}{R_g + S}.I = \frac{ES}{(R + B)(R_g + S) + R_g S}.$$

If this produces a deflexion $\frac{\pi}{4}$, $I_g = K$ and the instrument will be used with maximum accuracy. Hence

$$ES = K[(R + B)(R_g + S) + R_g S],$$

or

$$S = \frac{KR_g(B + R)}{E - K(R_g + B + R)}.$$

The Helmholtz system of galvanometer coils.—The strength of the magnetic field at a point on the axis of a circular coil carrying a current is given by the relation

$$\mathrm{H} = \frac{2\pi i a^2}{(a^2 + x^2)^{1\cdot5}},$$

where a is the radius of the coil, i the current and x the distance of the point from the centre of the coil. If the coil has N closely wound turns of wire, the above expression becomes

$$\mathrm{H} = \frac{2\pi \mathrm{N} i a^2}{(a^2 + x^2)^{1\cdot5}} = \frac{\mathrm{A}}{(a^2 + x^2)^{1\cdot5}} \text{ (say).}$$

The rate at which this field changes with x is given by $\frac{d\mathrm{H}}{dx}$. It is important to find whether there is any region over which the above rate of change is constant, for, if this is so, by superimposing two fields due to currents in equal circular coils it should be possible to obtain a uniform magnetic field over a considerable region. If the space rate of change of H with x, i.e. $\frac{d\mathrm{H}}{dx}$, is constant, then $\frac{d^2\mathrm{H}}{dx^2}$ will be zero. Differentiating the expression for H with respect to x we obtain

$$\frac{d\mathrm{H}}{dx} = \mathrm{A}\frac{d}{dx}(a^2 + x^2)^{-1\cdot5} = -3\mathrm{A}x(a^2 + x^2)^{-2\cdot5}.$$

Differentiating again,

$$\frac{d^2\mathrm{H}}{dx^2} = -3\mathrm{A}[(a^2 + x^2)^{-2\cdot5} - 5x^2(a^2 + x^2)^{-3\cdot5}].$$

This is zero, if $\frac{d\mathrm{H}}{dx}$ is constant. Equating the above expression to zero, we have

$$5x^2(a^2 + x^2)^{-1} = 1,$$
$$\therefore x = \pm \tfrac{1}{2}a.$$

The above analysis shows that the rate of change of the field is constant at the above positions. This fact is utilized in the construction of a Helmholtz tangent galvanometer. Such an instrument is not very important today, but the system of coils finds an important application in accurate determinations of H_0 and H_V, and also in the absolute determination of the ohm. The system consists of two coaxial coils, each of radius a and comprising the same number of turns in each. The distance between the centres of the coils is a. When the current through the coils is the same and adjusted so that the north pole of one coil faces the south pole of the other, there will be a region midway between the coils where the field strength is uniform, for as we move along the axis from the mid-point any diminution in the field strength due to one coil is compensated by the increase in the field due to the other coil. If $x = \frac{a}{2}$, we have

$$H = \frac{2\pi Nia^2}{\left[a^2 + \frac{a^2}{4}\right]^{1\cdot 5}} = \frac{16}{5\sqrt{5}} \cdot \frac{\pi Ni}{a},$$

as the numerical value of the field due to the current in each coil at the point considered. Since the coils are arranged so that the actual directions of the fields are the same, the total field at the centre of the system is double the above, i.e.

$$H = \frac{32}{5\sqrt{5}} \cdot \frac{\pi Ni}{a}.$$

If the planes of the coils are in the magnetic meridian, and H_0 is the horizontal component of the earth's magnetic field, and ϕ the angle of deflexion of a small magnet placed at the centre of the coils, $H = H_0 \tan \phi$,

i.e. $$i = \frac{5\sqrt{5}}{32\pi N} aH_0 \tan \phi.$$

The manner in which the field due to each coil varies and the region over which the combined field is uniform is shown in Fig. 10·08.

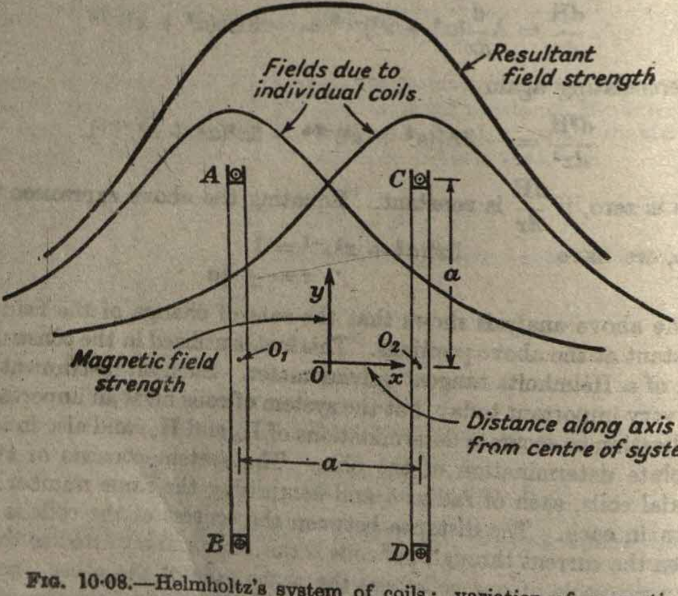
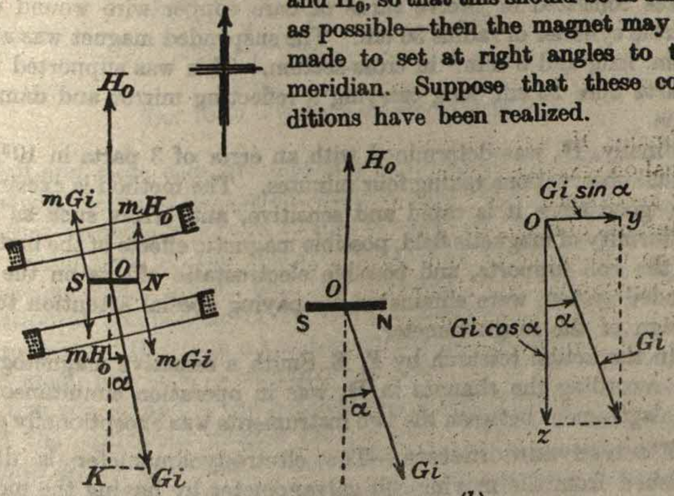

Resultant field strength
Fields due to individual coils
A
C
a
y
Magnetic field strength
O_1
O_2
O
x
Distance along axis from centre of system
a
B
D

FIG. 10·08.—Helmholtz's system of coils; variation of magnetic field strength along its axis.

The Schuster magnetometer.—The most accurate method for determining the horizontal component of the earth's magnetic field was proposed by Schuster in 1914. The actual research was carried out by F. E. Smith and completed in 1923. The principle of this instrument, known as the Schuster magnetometer, is as follows. Two equal coils are arranged at a distance apart equal to their common radius as in the Helmholtz galvanometer. A small magnet is suspended by a quartz fibre so that its centre is on the axis of the coils and midway between them. When a current is sent through the coils in the same direction a magnetic field is produced which is uniform over a considerable region in the neighbourhood of the magnet. Suppose that the planes of the coils are normal to the magnetic meridian. Then the magnetic field due to the current in them is parallel to the direction of H_0. If the sense of the field is the same as that of H_0, then the magnet remains undeflected for all values of the field due to the current. If the current through the coils is reversed, the sense of the magnetic field due to it will be opposite to that of H_0. Let G be the field per unit current (e.m.u.) at the centre of the coil system due to a current in the coils. If the current is i, the magnetic field is Gi. As long as Gi is less than H_0, the magnet will continue to point to the magnetic north, but when Gi exceeds H_0 the magnet swings round through 180° so that its north pole now points to the south. It continues to do this for all values of $Gi > H_0$. If, however, the coil system is rotated through a small angle—its magnitude depends on the difference between Gi and H_0, so that this should be as small as possible—then the magnet may be made to set at right angles to the meridian. Suppose that these conditions have been realized.



(a) (b)

Fig. 10·09.—The principle of a Smith-Schuster magnetometer.

Let NS, Fig. 10·09 (*a*), be the suspended magnet so that when deflected it is at right angles to H_0. Let the normal to the plane of the coils make an angle α with OK. Let m be the pole strength of the magnet and i the current in the coils. Then the magnet is in equilibrium under the action of the two couples indicated. Since the moments of these are equal when equilibrium has been reached, we have

$$mH_0 . 2l = mGi . 2l \cos \alpha,$$

if $2l$ is the length of the magnet. Hence

$$H_0 = Gi \cos \alpha.$$

The formula $H_0 = Gi \cos \alpha$ may also be obtained in the following way. Let the magnetic fields H_0 and Gi be directed as indicated in Fig. 10·09 (*b*). The field Gi may be resolved, cf. Fig. 10·09 (*c*), into two rectangular components, $Gi \cos \alpha$ and $Gi \sin \alpha$, respectively. If, by adjusting the value of the current, $Gi \cos \alpha$ is made equal to H_0, then the field $Gi \sin \alpha$ will cause a small magnet NS to set perpendicularly to H_0. Thus $H_0 = Gi \cos \alpha$ is the condition required.

[In passing, it should be noted that if the field Gi makes an angle $-\alpha$ with H_0, then NS will be rotated through an angle π from the position shown, since the component of Gi along Oy is reversed.]

The value of the current was adjusted so that α was small when NS was in the desired position. The current was determined by 'weighing' it with the aid of an ampere balance. The coils themselves consisted of twelve turns of bare copper wire wound on a marble cylinder of radius 30 cm. The suspended magnet was about 1 cm. long and 5 mm.^{2} in cross-section. This was supported by a quartz fibre 25 cm. long carrying a reflecting mirror and damping vane.

Finally, H_0 was determined with an error of 3 parts in 10^5, the actual observations taking four minutes. The method is exceptionally good since it is rapid and sensitive, and errors such as non-uniformity of magnetic field, possible magnetic effects of the material of the coil supports, and possible electrostatic effects on the suspended system were eliminated by paying special attention to the design of the magnetometer.

In the actual research by F. E. Smith a sensitive magnetograph for recording the changes in H_0 was in operation simultaneously. The agreement between the two instruments was exceptionally good.

Electrodynamometers.—The electrodynamometer is distinguished from the moving-coil galvanometer by having the moving member suspended not in the field of a permanent magnet but in that due to a system of fixed coils which are traversed by the current. Electrodynamometers may be either absolute or secondary standard instruments. The coils may be arranged in various ways but in an absolute electrodynamometer the geometry of the system must be such that the constant of the instrument can be calculated from it.

As described at first in this chapter it may appear that an electrodynamometer is primarily an instrument for the measurement of current, but at a later stage in the chapter it will be shown how an electrodynamometer may be adapted to the measurement of power.

Weber's electrodynamometer.—This instrument may be regarded as a standard tangent galvanometer, i.e. a Helmholtz galvanometer, with its two coils, A_1 and A_2, Fig. 10·10 (*a*), in which a small coil C, suspended at the centre of the system by a bifilar suspension, replaces the magnet. When no current is passing through the coils, which are arranged in series, the planes of the small and large coils are mutually perpendicular. The passage of a current through the system tends to turn the small suspended coil so that its plane is parallel to that of each fixed coil. The deflexion is determined by the relative magnitude of the couple due to the electromagnetic effects and the restoring couple due to H_0, the horizontal component of the earth's magnetic field, and to the suspension. If i (e.m.u.) is the current, the magnetic moment of the small coil is $M = \pi a^2 N\mu i = Si$ (say), where N is the number of turns in it, and μ the permeability of the surrounding medium. Let the equivalent magnet have a pole strength m and magnetic length $2l$, i.e. $M = 2ml$. Then if G is the field per unit current at the centre of the large coils (i.e. the Helmholtz system), the magnetic field is Gi and the couple due to it is

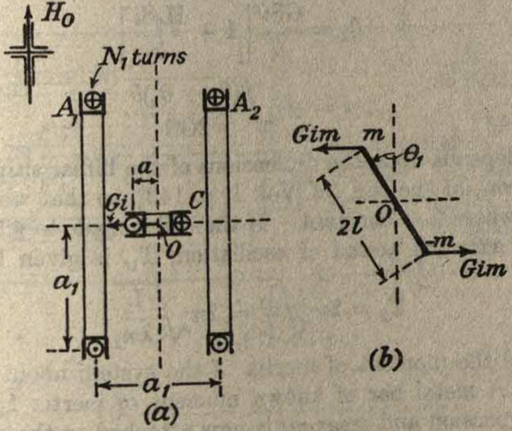


FIG. 10·10.—Weber's electrodynamometer.

$$Gim.2l\cos\theta_1 = MGi\cos\theta_1,$$
$$= GSi^2\cos\theta_1,$$

where θ_1 is the deflexion; this equation is easily obtained by considering the equivalent system shown in Fig. 10·10 (*b*). The restoring couple due to H_0 is $(N\pi a^2\mu i)H_0 \sin\theta_1$, or $SiH_0 \sin\theta_1$, and if θ_1 is small this may be written $SiH_0\theta_1$; due to the suspension it is $b\theta_1$, where b is a constant. Now when $\theta_1 \rightarrow 0$, $\cos\theta_1 \rightarrow 1$. Hence

$$\theta_1 = \frac{Gi^2S}{SiH_0 + b} = \frac{GSi^2}{b}\left[1 - \frac{H_0Si}{b}\right], \quad \left[\because \frac{SiH_0}{b} \rightarrow 0\right].$$

When the current is reversed in each coil let θ_2 be the steady deflexion; the deflecting couple due to the current and to the bifilar suspension will each act as before, but the couple due to the earth's magnetic field will be reversed. Under such conditions

$$\theta_2 = \frac{GSi^2}{b}\left[1 + \frac{H_0Si}{b}\right].$$

$$\therefore i = +\sqrt{\frac{(\theta_1 + \theta_2)b}{2GS}}.$$

Now b depends upon the dimensions of the bifilar suspension and the mass, m_1, of the coil, [cf. Vol. I, p. 143], so that we may write $b = \lambda m_1$, where $\lambda =$ constant. If the coil is made to vibrate about a vertical axis the period of oscillation, T_1, is given by

$$T_1 = 2\pi\sqrt{\frac{I_1}{b}} = 2\pi\sqrt{\frac{I_1}{\lambda m_1}},$$

where I_1 is the moment of inertia of the system about its axis of rotation. A metal bar of known moment of inertia I_2 about the axis of suspension and mass m_2 is now attached to the coil and T_2, the new period, observed. Then

$$T_2 = 2\pi\sqrt{\frac{I_1 + I_2}{\lambda(m_1 + m_2)}},$$

so that λ, and hence b, can be found.

The value of b can also be found by passing a known direct current through the system; when b is known the electrodynamometer may be used to measure the r.m.s. value of an alternating current since the current is reversed simultaneously in all the coils, so that the mean couple due to the earth's magnetic field is zero. If θ is the steady deflexion under such conditions

$$(i)_{\text{r.m.s.}} = \sqrt{\frac{b\theta}{GS}},$$

or, in practical units,

$$I = 10(i)_{\text{r.m.s.}} = 10\sqrt{\frac{b\theta}{GS}} = \sqrt{\frac{b\theta}{G_0S_0}},$$

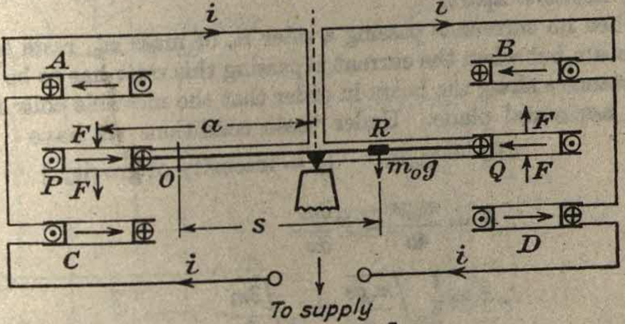
where G_0 is the field per ampere at the centre of the Helmholtz coil system and

$$M = Si = S\left(\frac{I}{10}\right) = S_0 I,$$

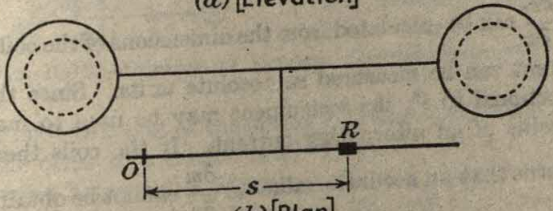
i.e. $$10G_0 = G, \text{ and } 10S_0 = S.$$

Kelvin's ampere balance.—Kelvin designed several instruments for the accurate measurement of current, the principle of which depends upon the mutual action between the fixed and movable portions of a circuit through which a current is passing.

A diagrammatic representation of a Kelvin balance for measuring currents from 0·01 to 1·0 e.m.u. is shown in Fig. 10·11 (*a*) and (*b*).



(*a*) [Elevation]



(*b*) [Plan]

FIG. 10·11.—The principle of Kelvin's ampere balance.

It is a gravity controlled, astatic, zero reading dynamometer instrument. The instrument consists essentially of six circular coils, four fixed and two movable, these latter being carried on a beam which can rotate about a horizontal axis as in an ordinary physical balance. The fulcrum is represented conventionally by a knife-edge but in an actual ampere balance there is no knife-edge, the beam being suspended by two flexible copper ribbons each consisting of a large number of fine copper wires. These wires carry the current. The coils are connected in series and the same current, *i*, flows through each; the directions of the current are shown conventionally as well as by the horizontal arrows which indicate the direction of the current at any instant in each half of each coil which is nearer to an observer. Consequently the coils P and Q, which are the movable ones, are acted on by the four forces F as indicated; for example, the upper force F on P arises from the repulsion due to the currents in A and P, while the lower force F on P is due to the attraction between the currents in P and C.

If W is the mutual potential energy of the coils A and P,

$$F=-\frac{\partial W}{\partial x}=+i^2\frac{\partial m}{\partial x}, \left[\begin{array}{l}\because \text{ in general, } W=-i_1\Psi \\ \quad =-i_1(mi_2)=-i^2m, \\ \text{since here } i_1=i_2=i.\end{array}\right]$$

where m is the mutual inductance of the coils A and P and x is their distance apart.

When no current is passing a rider R, of mass m_0, rests at O on the beam but when the current is passing this rider has to be shifted a distance s along the beam in order that the movable coils shall be in a horizontal plane. Under these conditions we have

$$4Fa=m_0gs. \qquad [g=\text{intensity of gravity.}]$$

$$\therefore F=\frac{m_0gs}{4a}=i^2\frac{\partial m}{\partial x}.$$

$$\therefore i=\frac{1}{2}\sqrt{\frac{m_0gs}{a}}\div\sqrt{\frac{\partial m}{\partial x}}.$$

Now $\frac{\partial m}{\partial x}$ can be calculated from the dimensions of the coils and thus the current can be measured in absolute units. Since the force F is proportional to i^2, the instrument may be used to measure the r.m.s. value of an alternating current. If the coils then have so many turns that an accurate value for $\frac{\partial m}{\partial x}$ cannot be obtained from a knowledge of the geometry of the system, the balance may be calibrated by using known direct currents; preferably the values of these currents should be deduced from observations on the potential differences across standard resistances.

Secondary electrodynamometers: Siemens' electrodynamometer.—As a matter of fact one single principle underlies all electromagnetic instruments, i.e. the current carrying circuit tends to enclose as large a magnetic field as possible. In the moving needle instrument the needle turns so that more of its lines of force pass through the coil, while in the moving coil instrument the coil sets itself so as to enclose as much as possible of the field of the magnet. In the soft-iron instrument the soft iron moves so as to increase the magnetic flux produced by the coil and in the dynamometer the moving coil turns so that its magnetic effect increases that of the fixed coils.

The Siemens' electrodynamometer, shown diagrammatically in Fig. 10·12 (*a*), consists essentially of two coils one of which, ABCD, is fixed while the other UVWX, suspended by a plaited silk thread, is movable. Details of the suspension are shown in Fig. 10·12 (*b*) and initially the planes of the coils are mutually perpendicular—cf. Fig. 10·12 (*c*). One end of the spring S is attached to the movable coil and the other to a torsion head, T, which carries a pointer P_1 moving over a horizontal uniformly graduated circular scale. The ends W and X of the movable coil dip into mercury cups M_1, M_2, arranged vertically above each other and on the vertical axis of the coil. A pointer, P_2, is rigidly attached to the movable coil but by means of two stops R_1 and R_2 its range of deflexion is limited to a few degrees each side of the position of balance. The instrument is levelled and the same relative position of the coils is maintained during any calibration and subsequent use. When a steady current passes through the coils the moving coil *always* tends to move in a counterclockwise direction and the torsion head and P_1 are moved in the opposite direction until the pointer P_2 is at the zero mark. Since the movable coil is always restored exactly to its initial position the torque on it is rigidly proportional to the square of the current and hence, if the restoring couple due to the spring is proportional to the angle through which the torsion head T is rotated,

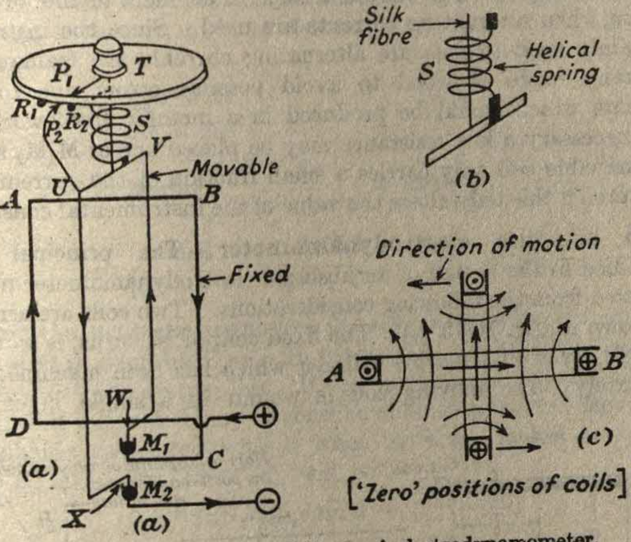


Fig. 10·12.—Principle of Siemens' electrodynamometer.

then $\theta \propto I^2$, or $I = k\sqrt{\theta}$, where k is almost a constant for a given instrument; it is not quite a constant since no spring is perfect. The instrument is therefore calibrated using direct currents of known magnitude. When this calibration is being carried out it is desirable to orientate the instrument so that the local magnetic field does not affect it. This is accomplished by rotating the instrument about a vertical axis until the movable coil is not deflected when a large current is sent through that coil *only*. The effect of the local field is zero when alternating currents are used. Since the instrument is mainly used to measure alternating currents the framework is generally made of wood to avoid possible errors due to eddy currents which would be produced in a metallic framework.

If necessary a low resistance may be placed across M_1M_2 so that the movable coil only carries a small fraction of the current to be measured; this only alters the value of the instrumental constant k.

An absolute electrodynamometer.—The principal ideas embodied in the design of an absolute electrodynamometer may be gathered from the following considerations. Two coils are arranged as shown in Fig. 10·13 (*a*). The fixed coil, of N_1 turns, is wound in a single layer on a marble cylinder which has been machined very accurately. The moving coil is wound in a single layer on a carefully machined porcelain cylinder. The axes of the coils are mutually perpendicular and their centres coincide at O.

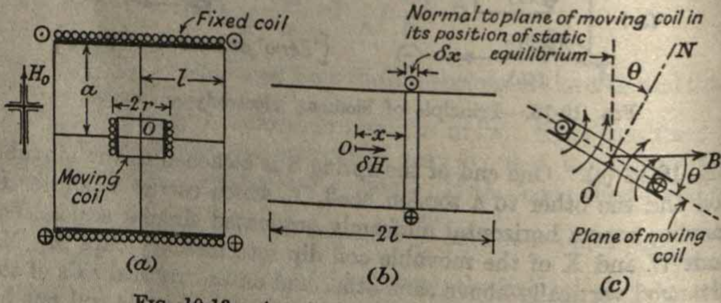


FIG. 10·13.—An absolute electrodynamometer.

The smaller coil is suspended by a torsion wire and the current is taken to it by means of two mercury cups, the leads being arranged so that any disturbing effect due to them is a minimum. The suspension wire is the most troublesome feature in the instrument for its elastic properties must be so permanent that they can be known to a very high order of accuracy. A well-aged phosphor bronze wire has been found most satisfactory. If i (e.m.u.) is the current in the coils, arranged in series, the couple acting on the movable coil is given by

$$\Gamma = i^2\frac{\partial m}{\partial \theta},$$

where θ is the angle between the axes and m is the mutual inductance for the system.

To obtain an approximate value for m let us assume that the moving coil is a plane one of radius r. If n is the number of turns per cm. in the fixed coil the contribution to the magnetic field, H, at O due to the current in an element of width δx, cf. Fig. 10·13 (*b*), is given by

$$\delta \mathrm{H} = \frac{2\pi a^2 in\,\delta x}{(a^2 + x^2)^{\frac{3}{2}}}.$$

$$\therefore\ \mathrm{H} = 2\pi a^2 ni\int_{-l}^{+l}\frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = 2\pi ni\left[\frac{x}{(a^2 + x^2)^{\frac{1}{2}}}\right]_{-l}^{+l}$$

$$= \frac{4\pi nil}{(a^2 + l^2)^{\frac{1}{2}}} = \frac{2\pi \mathrm{N}_1 i}{(a^2 + l^2)^{\frac{1}{2}}}. \qquad [\mathrm{N}_1 = 2nl.]$$

Now the magnetic induction, B, associated with the magnetic field at O is given by $\mathrm{B} = \mu_a\mathrm{H}$, where μ_a is the permeability of the air in which the system lies. If the movable coil is small the field in which it is situated may be considered uniform so that the number of linkages associated with it when its plane makes an angle θ with B, as in Fig. 10·13 (*c*), will be

$$[(\pi r^2)2\pi \mathrm{N}_1\mathrm{N}_2\mu_a i \sin\theta] \div (a^2 + l^2)^{\frac{1}{2}},$$

where N_2 is the number of turns on the small coil.

$$\therefore\ m = \frac{2\pi^2\mu_a r^2\mathrm{N}_1\mathrm{N}_2\sin\theta.}{(a^2 + l^2)^{\frac{1}{2}}}$$

$$\therefore\ \Gamma = i^2\,\frac{\partial m}{\partial \theta} = \frac{2\pi^2\mu_a r^2 i^2\mathrm{N}_1\mathrm{N}_2\cos\theta}{(a^2 + l^2)^{\frac{1}{2}}}.$$

To measure the current the moving coil is restored to its position of static equilibrium by rotating the head which carries the suspension through an angle ϕ. The couple so applied is $b\phi$, where b is the couple per unit twist in the suspension. Under such conditions $\mathrm{I} = b\phi$, $\theta = 0$, and we have

$$\frac{2\pi^2\mu_a r^2\mathrm{N}_1\mathrm{N}_2 i^2}{(a^2 + l^2)^{\frac{1}{2}}} = b\phi.$$

Gray has pointed out that if the radius of each coil to its length is as 1 : 3, all the terms in the series for m between the first and seventh vanish so that m is as above for all practical purposes when this additional condition is satisfied.

Some disadvantages of dynamometers used to measure direct currents.—Although dynamometers may be used as precision instruments in connexion with the measurement of the r.m.s. value of an alternating current, cf. p. 507, they have several disadvantages when compared with the permanent-magnet (moving-coil) type of instrument as a means for measuring direct currents. The more important are as follows :—

(*a*) Since no iron is present, the strength of the magnetic field obtainable is small. This means that a relatively large number of ampere-turns is required on the movable coil in order to obtain the necessary deflecting couple. Consequently there is a somewhat massive moving coil and the loss of power is considerable. The 'torque per unit mass of the suspended system' is small and this means that errors due to friction and internal heating assume an important role.

(*b*) Dynamometers are expensive.

(*c*) Since the deflecting torque per unit current is not constant the scale is not linear ; it varies approximately as the square of the current.

Power measuring instruments : wattmeters.—A wattmeter is an instrument which indicates by a single reading the rate of dissipation of energy in any part of an electrical circuit. One of the earliest types of torsion wattmeter was simply a modified Siemens' electrodynamometer; the fixed coil and the moving coil are initially at right angles and to eliminate the effect of the earth's horizontal magnetic field on the moving coil, when currents of zero frequency are being used, this is placed with its plane at right angles to the magnetic meridian. The fixed coil is of low resistance and, as shown in Fig. 10·14, carries the same current as that which flows in the main circuit. The moving coil is known as the 'volt-coil'; it is

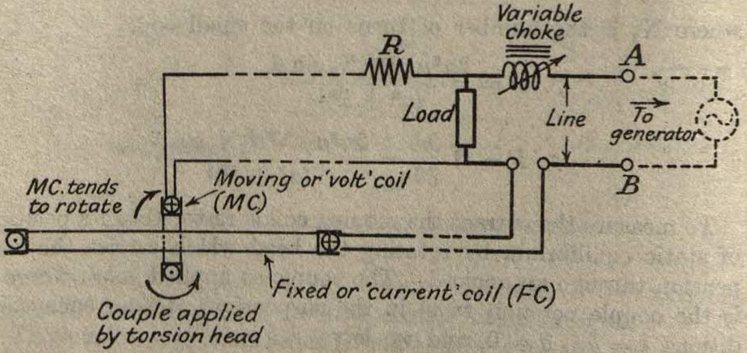

Variable choke
R
A
Load
Line
To generator
B
MC. tends to rotate
Moving or 'volt' coil (MC)
Fixed or 'current' coil (FC)
Couple applied by torsion head

FIG. 10·14.—Principle of a wattmeter.

connected in series with a high resistance R and the combination thus formed is joined as a shunt to that part of the circuit in which the power is to be measured. In an ideal wattmeter the volt-coil should have negligible inductance and capacitance so that its reactance is small compared with the sum of its resistance and that of the resistor R in series with it, in order to ensure that the current flowing through the volt-coil is in phase with the impressed voltage. In some cases a low value for the reactance is secured by winding only a portion of the coil inductively, the remainder being made non-inductive by double winding of the wire. If v is the instantaneous voltage across the volt-coil and i the instantaneous current in the 'load'-coil the mean power is $\frac{1}{T}\int_0^T vi\,dt$ watts. The in-

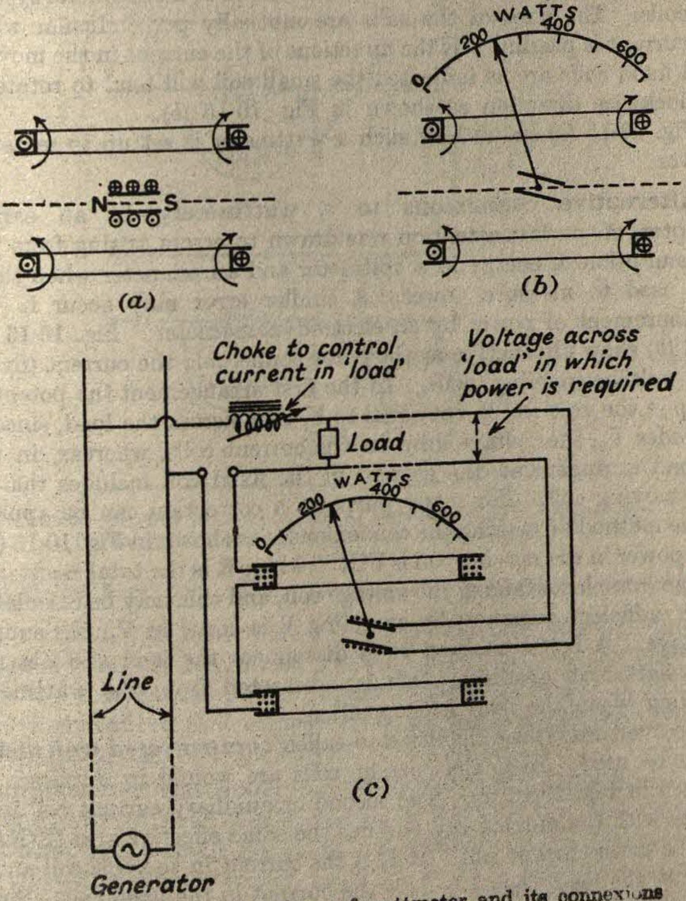

WATTS
0
200
400
600
N
S
(a)
(b)
Choke to control current in 'load'
Voltage across 'load' in which power is required
Load
WATTS
0
200
400
600
Line
(c)
Generator

FIG. 10·15.—Another form of wattmeter and its connexions [shown for use with a.c.].

stantaneous torque will be kvi, where k is a constant, so that the mean torque will be $\frac{1}{T}\int_0^T kvi\,dt$, and hence is proportional to the mean rate at which energy is dissipated in the load. When the moving coil is restored to its zero position, i.e. its plane is normal to that of the fixed coil, the couple is proportional to the angle θ through which the torsion head is rotated, i.e.

$$P = K\theta. \qquad [K = \text{constant.}]$$

The constant K may be determined by using a d.c. circuit, the voltage drop across the load and the current in it being measured by d.c. instruments.

In another form of wattmeter, cf. Fig. 10·15 (*a*), the small moving coil is suspended with its centre at the centre of a Helmholtz system of coils. The axes of the coils are mutually perpendicular when no current is passing. If the directions of the current in the moving and fixed coils are as indicated the small coil will tend to rotate in a clockwise direction as shown in Fig. 10·15 (*b*).

Fig. 10·15 (*c*) shows how such a wattmeter is set up to measure power.

Alternative connexions to a wattmeter.—In an earlier chapter, cf. p. 251, attention was drawn to errors arising from the consumption of energy in a voltmeter and an ammeter when these are used to measure power. A similar error may occur in the measurement of power by an electrodynamometer. Fig. 10·16 (*a*) and (*b*) show alternative methods for connecting the current (fixed) and voltage (moving) coils. In the first arrangement the potential drop in the volt-coil is not equal to V, that across the load, since it includes V_0, the voltage drop in the current coils, whereas, in the second arrangement, the current in the fixed coil includes that in the moving coil. For many purposes a correction can be applied if the method of making the connexions is as shown in Fig. 10·16 (*b*); the power in the moving coil is V^2R^{-1}, where R is the total resistance in the branch containing the voltage coil, and this may be calculated with sufficient accuracy by assuming V is equal to V_s, the supply voltage. A better method is to disconnect the load and observe the wattmeter reading; this is subtracted from the wattmeter reading when the load is in position.

To overcome this difficulty a so-called ***compensated wattmeter*** may be used. In it the current coils are wound in duplicate as shown in Fig. 10·16 (*c*). The second or auxiliary current coil is in series with the moving coil and has the same effective area $[\Sigma(NA)]$ as the main current coil. If I_0 is the current in the volt-coil and I that in the load, then $I + I_0$ is the current in the fixed coil. Since the current I_0 flows in opposite directions through the above coils it makes no contribution to the mutual potential energy of the system and hence to the torque on the moving coil.

[Alternatively, one may consider that the magnetic induction at a selected point in the region of the moving coil has two components, say $\vec{B}$ and $\vec{B}_0$, due respectively to the components I and I_0 of the current in the fixed coil. The current I_0 in the duplicate winding produces a field with induction $-\vec{B}_0$; thus the field in which the moving coil swings is determined solely by I. Hence the auxiliary coil acts as a compensator; it has a magnetomotive force, cf. p. 397, which exactly cancels that component of the main magnetomotive force due to I_0, the volt-coil current.

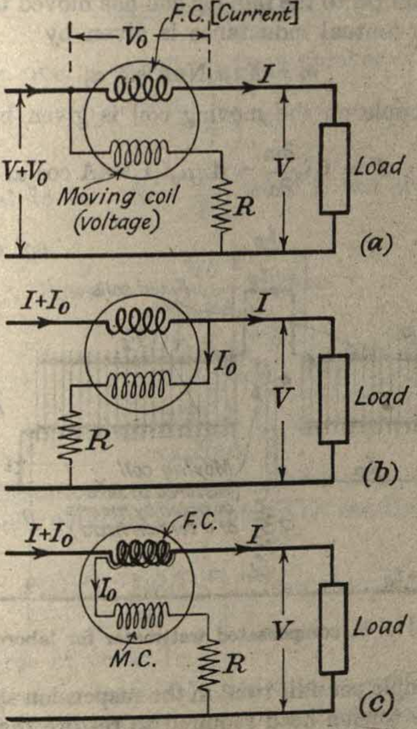


FIG. 10·16.—(a) and (b) Errors due to connexions in a wattmeter, (c) the principle of a compensated wattmeter.

A compensated wattmeter for laboratory use.—This instrument consists of two fixed coils F_1 and F_2, cf. Fig. 10·17 (a), arranged as a solenoid with a small gap at its mid-section. A small moving coil C, suspended from a torsion head and with an effective cross-sectional area NA, is free to rotate about a vertical axis in the field due to the current in the fixed coils. In general, let i_1 and i_2 (absolute units) be the currents in the fixed and moving coils respectively. With the usual notation $\mathrm{B} = 4\pi\mu_a n i_1$, if the effect of the gap in the long solenoid is neglected, and

$$\Psi = \mathrm{NAB} \sin \alpha = 4\pi\mu_a \mathrm{NA} n i_1 \sin \alpha,$$

when the normal On to the moving coil has moved through an angle α. Hence the mutual inductance is given by

$$m = 4\pi\mu_a \mathrm{N} n \mathrm{A} \sin \alpha,$$

so that the couple on the moving coil is given by

$$\Gamma = i_1 i_2 \frac{\partial m}{\partial \alpha} = 4\pi\mu_a i_1 i_2 \mathrm{N} n \mathrm{A} \cos \alpha.$$

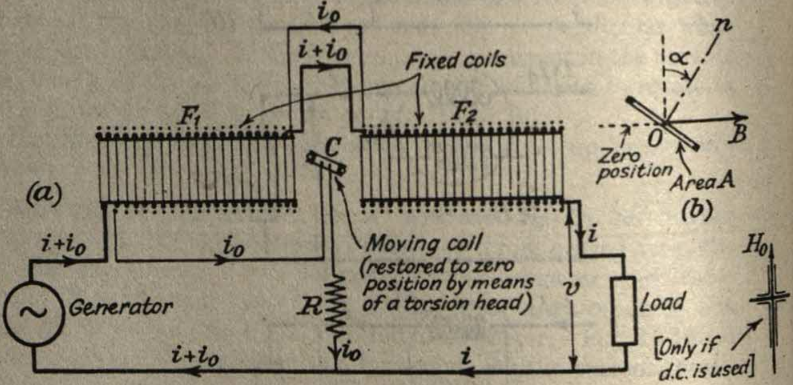


FIG. 10·17.—A compensated wattmeter for laboratory use.

If b is the couple per unit twist in the suspension and θ the angular rotation of the torsion head required to restore the moving coil to its position of static equilibrium (when $\alpha = 0$),

$$4\pi\mu_a i_1 i_2 \mathrm{N} n \mathrm{A} = b\theta.$$

Since θ is proportional to the product of the currents in the two coils, the instrument may be used to measure various electrical quantities.

In order to compensate for the effect of the current flowing in the moving coil when power is being measured, the solenoid is wound with a second coil having the same number of turns and cross-section as the main coil; it is arranged in series with C. Thus the field in which C moves is due entirely to the current through the load.

If v is the instantaneous potential difference across the load and r (e.m.u.) the resistance of the link containing the moving coil, we have $i_2 = \frac{v}{r}$, provided the coil reactance is negligible; this has been called i_0. Hence, writing $i = i_1$, the instantaneous couple on the moving coil when in its position of static equilibrium, i.e. $\alpha = 0$, is

$$4\pi\mu_a \mathrm{N}n\mathrm{A}i\left(\frac{v}{r}\right).$$

Its mean value over a period, or large number of periods, is

$$\frac{4\pi\mu_a \mathrm{N}n\mathrm{A}}{r}\cdot\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} vi\,dt,$$

and this is equal to $b\theta$. Now $\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} vi\,dt$ is the mean power in erg.sec.$^{-1}$ in the load.

$$\therefore\ \text{Power} = \frac{br\theta}{4\pi\mu_a \mathrm{N}n\mathrm{A}}\,\text{erg.sec.}^{-1}$$

$$= \frac{b(\mathrm{R}\times 10^9)\theta}{4\pi\mu_a \mathrm{N}n\mathrm{A}}\cdot\frac{1}{10^7}\,\text{watt.}$$

$$= \frac{25b\mathrm{R}\theta}{\pi\mu_a \mathrm{N}n\mathrm{A}}\ \text{watt.}$$

where R is the resistance in ohms equivalent to r e.m.u. of resistance.

[Using the results obtained in Chapter XIX the dimensions of this expression are

$$\left[\frac{\mathrm{ML^2T^{-2}}.\mu\mathrm{LT^{-1}}}{\mu\mathrm{L^{-1}L^2}}\right] = [\mathrm{ML^2T^{-3}}],$$

i.e. they are those of power.]

Ballistic galvanometers.—In order to measure the quantity of electricity which passes each point in a closed circuit in a very short interval of time, ballistic galvanometers have been designed. The galvanometer is inserted in the circuit and it evaluates $\int_0^{\Delta t} i\,dt = q$, the quantity of electricity which has passed in time Δt. Such galvanometers are of two types, known respectively as moving-coil and moving-magnet instruments. Suppose that the coil (or magnet) is suspended and free to move; if it is given an impulse while at rest in its equilibrium position, it will be caused to oscillate and the amplitude of the oscillation may be considered as a measure of the impulse.

[Note: Before proceeding to give the theory of ballistic galvanometers it is useful to refresh ourselves with respect to two important theorems in mechanics.

(*a*) If two equal and opposite forces, F, acting at a distance r

apart, so that they constitute a couple of moment $\Gamma = \mathrm{F}r$, cause a body to rotate about an axis through an angle $\delta\theta$ in time δt, we have

$$\begin{aligned}\mathrm{F}r\,\delta\theta = \Gamma\,\delta\theta &= \text{work done by couple}\\ &= \text{increase in the kinetic energy of the rotating body}\\ &= \delta(\tfrac{1}{2}\mathrm{I}\omega^2) = \mathrm{I}\omega\,\delta\omega,\end{aligned}$$

I being the moment of inertia of the body about its axis of rotation.

Dividing throughout by δt, we have, in the limit,

$$\Gamma\omega = \mathrm{I}\omega\dot{\omega}, \quad \text{or} \quad \Gamma = \mathrm{I}\ddot{\theta}, \qquad [\dot{\omega} = \ddot{\theta}].$$

i.e. the couple acting on the body is equal to the rate of change of its angular momentum (Iω).

(*b*) Since $\Gamma = \mathrm{I}\dot{\omega}$, we may write

$$\Gamma\,\delta t = \mathrm{I}\,\delta\omega = \mathrm{F}r\,\delta t.$$

If the couple acts for a time Δt, we have

$$\int_0^{\Delta} \mathrm{F}r\,dt = \mathrm{I}[(\omega + \Delta\omega) - \omega] = \mathrm{I}\Delta\omega,$$

where $\Delta\omega$ is the increment in the angular velocity. If r is constant the above equation becomes

$$r\int_0^{\Delta_t} \mathrm{F}\,dt = \mathrm{I}\Delta\omega,$$

i.e. the moment of the impulse is equal to the change in angular momentum.]

The theory of ballistic galvanometers.—(*a*) ***Moving-coil ballistic galvanometer.***—It will be assumed that the sides of the coil are vertical and the magnetic field* in which it lies is radial. Suppose, for the present, that the coil consists of a single turn. Let B be the value of the magnetic induction at points on the vertical sides of the coil. Then if i is the current (e.m.u.) flowing and l the length of the vertical side of the coil, the force on each side of the coil will be $li\mathrm{B}$. If these forces act for a time δt, the impulse given to the coil will be $2li\mathrm{B}\,.\,\delta t$. If the 'transient current' lasts for a time Δt, the total impulse is

$$2\int_0^{\Delta t} li\mathrm{B}\,dt = 2l\mathrm{B}q, \qquad [\because \int_0^{\Delta t} i\,dt = q].$$

$\therefore$ Moment of the impulse about the axis of rotation

$$\begin{aligned}&= 2l\mathrm{B}q \times \text{semi-width of coil}\\ &= \mathrm{AB}q,\end{aligned}$$

where A = area of the coil, i.e.

$$\mathrm{I}_0\omega_0 = \mathrm{AB}q,$$

---

* The magnetic field is only 'radial' in the air-gap.

where I_0 is the moment of inertia* of the coil about its axis of rotation and ω_0 the angular velocity with which the coil begins to rotate. If the coil has N equal turns, the above equation becomes

$$I_0\omega_0 = NABq \quad . \quad . \quad . \quad . \quad \text{(i)}$$

If we could observe ω_0 the above equation would enable us to calculate q when I, NA and B were known. Since ω_0 cannot be observed directly, an indirect method is used as follows.

The initial kinetic energy of the moving system is $\frac{1}{2}I_0\omega_0^2$ and this is expended in twisting the suspension until the angular deflexion of the coil attains its maximum value and the coil is momentarily at rest. If b is the torsion constant for the suspension, the work done for an additional rotation $\delta\theta$ is $b\theta . \delta\theta$.

$\therefore$ Energy spent in twisting the suspension is $\int_0^{\theta_0} b\theta \, d\theta = \frac{1}{2}b\theta_0^2$,

where θ_0 is the angular deflexion when the coil is at rest (momentarily). Hence $I_0\omega_0^2 = b\theta_0^2$.

Now T, the period of oscillation of the coil, is known to be given by

$$T = 2\pi\sqrt{\frac{I_0}{b}}.$$

From the above equations we have

$$N^2A^2B^2q^2 = I_0^2\omega_0^2 = I_0 b\theta_0^2$$
$$= \frac{T^2}{4\pi^2}b . b\theta_0^2.$$

Hence $$q = \frac{b}{NAB}\cdot\frac{T}{2\pi}\cdot\theta_0, \quad \left(\text{or } q = \frac{b}{NAB}\cdot\frac{T}{\pi}\cdot\tfrac{1}{2}\theta_0\right) \quad . \quad . \quad \text{(ii)}$$

Since, in general, the values of N, A and B are unknown, it is desirable to eliminate them from equation (ii); to do this it is necessary to observe the steady deflexion ϕ when a known steady current i is flowing through the galvanometer. Then, cf. p. 307,

$$i = \frac{b}{NAB}\phi,$$

so that

$$q = \frac{T}{2\pi}\cdot\frac{i}{\phi}\cdot\theta_0 \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

This equation is true in any consistent set of units; in particular, if Q is the charge in coulombs and I the current in amperes, .

$$Q = \frac{T}{2\pi}\cdot\frac{I}{\phi}\cdot\theta_0 \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

---

* I_0 is the symbol used to denote a moment of inertia; confusion with I, the symbol for a current in amperes, is thereby avoided.

(*b*) ***Suspended-magnet ballistic galvanometer.***—Let G be the galvanometer constant, so that if i is the current in its coils at any instant, the field at the centre is Gi. Hence if m is the pole-strength of a small magnet suspended at the centre of the coil, the instantaneous force experienced by each pole will be Gim. If this acts for a time δt, the impulse will be G$im\,\delta t$. If the 'transient current' lasts for a time Δt, the total impulse on each pole will be

$$\int_0^{\Delta t} \mathrm{G}im\,dt = \mathrm{G}mq. \qquad \left[\because q = \int_0^{\Delta t} i\,dt.\right]$$

The moment of the impulse is G$mq(2l)$, [$2l$ = magnetic length]

$$= \mathrm{GM}q = \mathrm{I}_0\omega_0, \qquad [\mathrm{M} = 2ml.]$$

where I_0 is the moment of inertia of the magnet about its axis of rotation and ω_0 is the initial angular velocity of the magnet. Again ω_0 must be determined indirectly. Suppose that the impulse is over before the magnet has moved from its zero position; then the energy acquired by the magnet is $\frac{1}{2}\mathrm{I}_0\omega_0^2$. The needle will rotate until the controlling field brings it to rest, i.e. until the work done against the field by the needle is equal to $\frac{1}{2}\mathrm{I}_0\omega_0^2$.

When the magnet has been moved through an angle θ, cf. Fig. 4·03, p. 101, the work done against the field is

$$2(\mathrm{AP})m\mathrm{H} = \mathrm{MH}(1 - \cos\theta).$$

If θ_0 is the maximum deflexion of the needle when it will be momentarily at rest, we have

$$\tfrac{1}{2}\mathrm{I}_0\omega_0^2 = \mathrm{MH}(1 - \cos\theta_0).$$

$$\therefore\ q^2 = \frac{\mathrm{I}_0}{\mathrm{MH}}\cdot\frac{4\mathrm{H}^2}{\mathrm{G}^2}\sin^2\tfrac{1}{2}\theta_0$$

But the period of the magnet is given by

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}_0}{\mathrm{MH}}}.$$

$$\therefore\ q = \frac{\mathrm{H}}{\mathrm{G}}\cdot\frac{\mathrm{T}}{\pi}.\sin\tfrac{1}{2}\theta_0.$$

If i is the steady current giving a deflexion ϕ, then, cf. p. 311,

$$\mathrm{G}i = \mathrm{H}\tan\phi,$$

and hence

$$q = \frac{\mathrm{T}}{\pi}\cdot\frac{i}{\tan\phi}\cdot\tfrac{1}{2}\sin\theta_0.$$

If θ and ϕ are each small, we have

$$q = \frac{\mathrm{T}}{\pi}\cdot\frac{i}{\phi}\cdot\tfrac{1}{2}\theta_0.$$

As before, this equation is true in any consistent set of units, so that, in practical units,

$$Q = \frac{T}{\pi} \cdot \frac{I}{\phi} \cdot \tfrac{1}{2}\theta_0$$
$$= \frac{T}{2\pi} \cdot \frac{I}{\phi} \cdot \theta_0.$$

Damping : logarithmic decrement per cycle.—In practice the motion of a suspended coil or magnet is never purely simple harmonic because the amplitude decreases with time. The main contributory causes for this so-called damping of the oscillation are the resistance of the air to the motion of the system and electromagnetic damping. Let $\theta_1, \theta_2, \theta_3, \ldots$ be successive maximum angular deflexions as the needle swings, i.e. the first maximum deflexion and then the deflexions at the end of each successive cycle of the motion. It is found, cf. p. 336, that

$$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = \frac{\theta_3}{\theta_4} = \ldots = \rho, \text{ (say)}$$

where ρ is a constant. Since $\ln \theta_1 - \ln \theta_2 = \text{etc.} = \ln \rho = \delta$, say, the constant difference δ between the logarithms of the amplitudes of successive oscillations on the same side of the zero is termed the *logarithmic decrement per cycle.*

Now a damped oscillatory motion may be represented by the projection of a rotating vector whose initial position is OA, Fig. 10·18, upon OB, where OB is a straight line normal to OA. In practice successive amplitudes always decrease in a constant ratio, so that it is necessary to imagine that the rotating vector OA shrinks with time. Since the impulse was given to the moving system before it left its equilibrium position, the shrinkage of the vector OA that has actually occurred before θ_1 is observed will be that which occurs in a quarter of a cycle.

Now $\dfrac{\theta_1}{\theta_2} = \exp \delta$, so that for a quarter of a cycle $\dfrac{\theta_0}{\theta_1} = \exp\left(\dfrac{\delta}{4}\right)$.

$$\therefore \ \theta_0 = \theta_1 \exp \frac{\delta}{4} = \theta_1[1 + \tfrac{1}{4}\delta], \quad [\because \ \delta \text{ is small in practice}].$$

Hence the formula obtained for q, the quantity of electricity discharged through a ballistic galvanometer, becomes

$$q = \frac{T}{\pi} \cdot \frac{i}{\phi} \cdot \tfrac{1}{2}\theta_1\left(1 + \frac{\delta}{4}\right),$$

where θ_1 is the maximum observed deflexion and δ is the logarithmic decrement per cycle. If λ is the logarithmic decrement per half-cycle and this is still used in connexion with ballistic galvanometers, then $\lambda = \frac{1}{2}\delta$, so that

$$q = \frac{T}{\pi}\cdot\frac{i}{\phi}\cdot\tfrac{1}{2}\theta_1\left(1+\frac{\lambda}{2}\right).$$

Ballistic galvanometers—alternative theory.—(*a*) ***Moving-coil pattern.*** Since the lines of magnetic induction in the air-gap are nearly radial, the couple on the coil is almost independent of the angle θ which its plane makes with a plane through the centres of the pole pieces and it is proportional to the current, say κi (actually it is iNAB). The restraining couple due to twisting of the suspension is $b\theta$, where b is a constant, and when the current is steady, the coil takes up a position for which these couples balance. Hence, if ϕ is the steady deflexion,

$$\kappa i = b\phi, \quad \text{or} \quad i = \frac{b}{\kappa}\phi.$$

[κ = NAB, in our previous notation.]

When the current is a transient one the instantaneous value of the total couple tending to increase the deflexion is $\kappa i - b\theta$, so that, if I_0 is the moment of inertia of the coil about the axis of suspension, the equation to its motion is

$$I_0\ddot{\theta} = \kappa i - b\theta,$$

when damping is neglected.

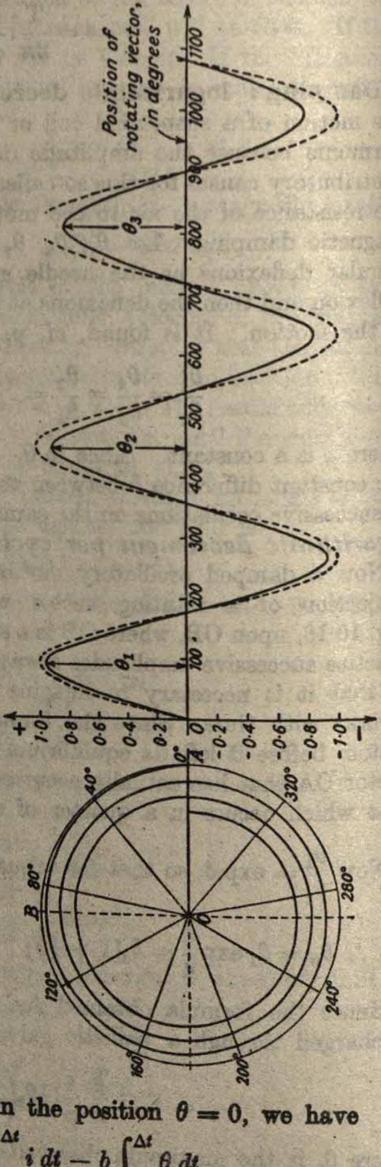


FIG. 10·18.—A damped oscillatory motion. [The slight increase in the period of the damped motion is not indicated.]

If the coil starts from rest in the position $\theta = 0$, we have

$$I_0\Big[\dot{\theta}\Big]_{t=0}^{t=\Delta t} = \kappa\int_0^{\Delta t} i\,dt - b\int_0^{\Delta t}\theta\,dt.$$

Now $\int_0^{\Delta t} i\,dt = q$, the total charge which has passed through the galvanometer in time Δt; if this takes place quickly the coil has no time to move before the discharge is over, so that $\int_0^{\Delta t} \theta\,dt = 0$. Hence our equation shows that the coil acquires a *sudden angular velocity*, $\left[\dot{\theta}\right]_{t=0}^{\Delta t} = \omega_0$, given by

$$\mathrm{I}_0\left[\dot{\theta}\right]_{t=0}^{\Delta t} = \mathrm{I}_0\omega_0 = \kappa q \quad . \quad . \quad . \quad \text{(i)}$$

When the discharge has taken place, the coil swings on open circuit without any current flowing in it. The motion of the coil under these conditions and in the absence of damping is

$$\mathrm{I}_0\ddot{\theta} + b\theta = 0, \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

with the conditions $\theta = 0$ and $\dot{\theta} = \omega_0$ when $t = 0$.

The corresponding energy equation, obtained by multiplying throughout by $2\dot{\theta}$ and integrating is

$$\mathrm{I}_0\dot{\theta}^2 + b\theta^2 = \text{constant} \quad . \quad . \quad . \quad \text{(iii)}$$

Applying this to the instants (a) when $t = 0$, (b) when the coil is first momentarily at rest after having moved, i.e. when $\theta = \theta_0$, say, we find

$$\mathrm{I}_0\omega_0^2 = b\theta_0^2 \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Further, the periodic time T is $2\pi\sqrt{\frac{\mathrm{I}_0}{b}}$.

$$\therefore q = \frac{b}{\kappa}\cdot\frac{\mathrm{T}}{\pi}\cdot\frac{1}{2}\theta_0 = \frac{b}{\mathrm{NAB}}\cdot\frac{\mathrm{T}}{\pi}\cdot\frac{1}{2}\theta_0.$$

As before, this may be written

$$q = \frac{\mathrm{T}}{\pi}\cdot\frac{i}{\phi}\cdot\tfrac{1}{2}\theta_0,$$

where ϕ is the deflexion due to a steady current i.

To allow for small damping assume a frictional couple proportional to the angular velocity. Then the equation to the motion of the coil is

$$\mathrm{I}_0\ddot{\theta} + a\dot{\theta} + b\theta = 0,$$

where a is some small constant specifying the amount of damping. If $\alpha = \frac{a}{\mathrm{I}_0}$ and $\beta = \frac{b}{\mathrm{I}_0}$, the above equation may be written

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\theta = 0.$$

A solution of this differential equation, since the damping is small, is

$$\theta = \exp\left(-\tfrac{1}{2}\alpha t\right)\left[\mathrm{C}\cos\tfrac{1}{2}\sqrt{4\beta-\alpha^2}\,t + \mathrm{D}\sin\tfrac{1}{2}\sqrt{4\beta-\alpha^2}\,t\right],$$

where C and D are integration constants. Since $\theta = 0$ when $t = 0$, $\mathrm{C} = 0$.

$$\therefore\ \theta = \exp\left(-\tfrac{1}{2}\alpha t\right)\mathrm{D}\sin\tfrac{1}{2}\sqrt{4\beta-\alpha^2}\,t.$$

The deflexion is first a maximum at time t_1 when

$$\tfrac{1}{2}\sqrt{4\beta-\alpha^2}\,t_1 = \frac{\pi}{2},\quad \text{i.e. } t_1 = \frac{\pi}{\sqrt{4\beta-\alpha^2}}.$$

The deflexion is next a maximum on the *same* side at time t_2 when

$$\tfrac{1}{2}\sqrt{4\beta-\alpha^2}\,t_2 = \frac{5\pi}{2},\quad \text{i.e. } t_2 = \frac{5\pi}{\sqrt{4\beta-\alpha^2}}.$$

$$\therefore\ \text{Period} = t_2 - t_1 = \frac{4\pi}{\sqrt{4\beta-\alpha^2}} = \mathrm{T}.$$

If $\omega_0 = (\dot{\theta})_{t=0}$,

$$\omega_0 = \tfrac{1}{2}\sqrt{4\beta-\alpha^2}\,\mathrm{D},\quad \text{i.e. } \mathrm{D} = \frac{\omega_0}{\tfrac{1}{2}\sqrt{4\beta-\alpha^2}} = \frac{\omega_0\mathrm{T}}{2\pi}.$$

Also

$$\theta_1 = (\theta)_{t=t_1} = \exp\left(-\tfrac{1}{2}\alpha\frac{\pi}{\sqrt{4\beta-\alpha^2}}\right).\mathrm{D},$$

$$\theta_2 = (\theta)_{t=t_2} = \exp\left(-\tfrac{1}{2}\alpha\frac{5\pi}{\sqrt{4\beta-\alpha^2}}\right).\mathrm{D}.$$

$$\therefore\ \frac{\theta_1}{\theta_2} = \exp\tfrac{1}{2}\alpha\mathrm{T} = \exp\delta,$$

where δ is the logarithmic decrement per cycle. $\therefore\ \delta = \tfrac{1}{2}\alpha\mathrm{T}$ and $\lambda = \tfrac{1}{4}\alpha\mathrm{T}$.

Now in the absence of damping $\theta = \mathrm{D}\sin\tfrac{1}{2}\beta t$, i.e. $\mathrm{D} = \theta_0$.

$$\therefore\ \theta_1 = \theta_0\exp\left(-\tfrac{1}{2}\alpha\frac{\mathrm{T}}{4}\right) = \theta_0\exp\left(-\frac{\delta}{4}\right).$$

$$\therefore\ \theta_0 = \theta_1\exp\frac{\delta}{4} = \theta_1\left(1+\frac{\delta}{4}\right) = \theta_1\left(1+\frac{\lambda}{2}\right).\quad [\because\ \delta \text{ is small.}]$$

Hence the charge which has passed through the ballistic galvanometer is given by

$$q = \frac{\mathrm{T}}{\pi}.\frac{i}{\phi}.\theta_1\left(1+\frac{\lambda}{2}\right).$$

Now $\dfrac{\theta_1}{\theta_0} = 1 - \frac{1}{8}\alpha T$, and $\dfrac{\theta_2}{\theta_0} = 1 - \frac{5}{8}\alpha T$, where θ_1 and θ_2 are successive deflexions on the same side.

$$\therefore\ \frac{\theta_0 - \theta_1}{\theta_0 - \theta_2} = \tfrac{1}{5}, \quad \text{or} \quad \theta_0 = \theta_1 + \tfrac{1}{4}(\theta_1 - \theta_2).$$

This gives an easy method of applying the correction for damping.

(*b*) ***Moving-magnet galvanometer.*** Let G be the magnetic field per unit current at the centre of the coils, and let M be the magnetic moment of the small magnet whose centre coincides with that of the coil system. When at rest, and no current is in the coils, this magnet will point with its axis along the direction of the controlling field, H. When a steady current i passes through the coils the magnet finally assumes a position in which its axis makes an angle ϕ with the controlling field and, since H and Gi are perpendicular to one another, we have

$$G i M \cos\phi = MH \sin\phi,$$

or

$$i = \frac{H}{G}\tan\phi.$$

$\dfrac{H}{G}$ is the reduction factor k, while G is the galvanometer constant, and it should be noted that this equation involves the field strength H and not its associated induction B.

If a discharge is passed through the coils and the moment of inertia of the magnet about its axis of rotation is so large (relatively) that the discharge is completed before the magnet has moved appreciably from its position of static equilibrium, then

$$I_0\ddot{\theta} = \text{couple tending to increase } \theta$$
$$= G i M = G M \dot{q}.$$

$$\therefore\ GMq = I_0\omega_0, \qquad \text{(i)}$$

where ω_0 is the angular velocity with which the coil begins to move.

After the discharge is finished the subsequent motion of the magnet, if damping is neglected, is given by

$$I_0\ddot{\theta} + MH\sin\theta = 0 \qquad \text{(ii)}$$

Multiplying throughout by $2\dot{\theta}$ and integrating, we get

$$I_0\dot{\theta}^2 - 2MH\cos\theta = \text{const.}$$
$$= I_0\omega_0^2 - 2MH,$$

since $\dot{\theta} = \omega_0$ when $\theta = 0$.

If θ_0 is the maximum value of θ, and this occurs when $\dot{\theta} = 0$,

$$0 - 2MH\cos\theta_0 = I_0\omega_0^2 - 2MH,$$

i.e.

$$I_0\omega_0^2 = 4MH\sin^2\tfrac{1}{2}\theta_0.$$

Since the period of the magnet is $\mathrm{T} = 2\pi\sqrt{\dfrac{\mathrm{I}_0}{\mathrm{MH}}}$, we have

$$q = \frac{\mathrm{H}}{\mathrm{G}}\cdot\frac{\mathrm{T}}{\pi}.\sin \tfrac{1}{2}\theta_0,$$

and when $\theta_0 \rightarrow 0$,

$$q = \frac{\mathrm{H}}{\mathrm{G}}\cdot\frac{\mathrm{T}}{\pi}\cdot\tfrac{1}{2}\theta_0.$$

When the swing is small and the motion slightly damped, the equation of motion is, with the usual notation,

$$\mathrm{I}_0\ddot{\theta} + a\dot{\theta} + \mathrm{MH}\theta = 0,$$

or

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\theta = 0,$$

where $\alpha = \dfrac{a}{\mathrm{I}_0}$ and $\beta = \dfrac{\mathrm{MH}}{\mathrm{I}_0}$. Hence, as for a moving-coil galvanometer, the decrement per half-cycle is $\tfrac{1}{4}\alpha\mathrm{T}(= \lambda)$ and if θ_1 is the value of the first angular deflexion,

$$q = \frac{\mathrm{H}}{\mathrm{G}}\cdot\frac{\mathrm{T}}{\pi}\cdot\tfrac{1}{2}\theta_1\left(1 + \frac{\lambda}{2}\right) = \frac{\mathrm{T}}{2\pi}\cdot\frac{i}{\phi}\cdot\theta_1\left(1 + \frac{\lambda}{2}\right),$$

where ϕ is the (small) deflexion due to a steady current i.

The motion of a moving coil galvanometer when a small and steady potential difference is applied across its terminals.—Let us consider a moving coil galvanometer whose coil is not wound on a metal frame, i.e. electromagnetic damping is small. When the steady potential difference v is applied, the linkages associated with the coil begin to change and an induced e.m.f. $-\,\mathrm{NAB}\dot{\theta}$, where the symbols are defined on p. 330, is set up. Assuming that the self inductance of the circuit is negligible, the instantaneous current through the coil is given by

$$i = \frac{v - \mathrm{NAB}\dot{\theta}}{r},$$

where r is the resistance (e.m.u.) of the system. Since the couple on the coil, due to the current and the magnetic field in which it moves, is $\mathrm{NAB}i$, the equation expressing the motion of the coil is

$$\mathrm{I}_0\ddot{\theta} + a\dot{\theta} + b\theta = \mathrm{NAB}\left[\frac{v - \mathrm{NAB}\dot{\theta}}{r}\right],$$

or

$$\ddot{\theta} + \frac{1}{\mathrm{I}_0}\left[a + \frac{(\mathrm{NAB})^2}{r}\right]\dot{\theta} + \beta\theta = \gamma,$$

where γ is the constant $\dfrac{\mathrm{NAB}v}{r\mathrm{I}_0}$. Assuming that the constants are such that the motion is oscillatory, a solution of the above equation is

$$\theta = \Theta \exp\left(-\tfrac{1}{2}\alpha t\right) \cos\left[\tfrac{1}{2}\sqrt{4\beta - \alpha^2}.t + \Psi\right] + \frac{\gamma}{\beta},$$

where $\alpha I_0 = a + \dfrac{(NAB)^2}{r}$, and Θ and γ are constants which may be determined from the conditions that when $t = 0$, $\theta = 0$ and $\dot{\theta} = 0$. The first gives

$$\Theta \cos \Psi = \frac{\gamma}{\beta} = \frac{NAB}{b}.\frac{v}{r},$$

and since

$$\dot{\theta} = \Theta\beta^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\alpha t\right) \cos\left[\tfrac{1}{2}\sqrt{4\beta - \alpha^2}.t + \Psi - \chi\right],$$

where $\tan \chi = \dfrac{\sqrt{4\beta - \alpha^2}}{\alpha}$, the second condition gives

$$\cos(\Psi - \chi) = 0, \text{ i.e. } \Psi - \chi = \tfrac{1}{2}\pi.$$

$$\therefore \cos \Psi = -\sin \chi = -\frac{\sqrt{4\beta - \alpha^2}}{2\beta^{\frac{1}{2}}}.$$

Hence

$$\Theta = -\frac{2\beta^{\frac{1}{2}}}{\sqrt{4\beta - \alpha^2}}.\frac{NAB}{b}.\frac{v}{r},$$

i.e.

$$\theta = \frac{NAB}{b}.\frac{v}{r}\left[1 - 2\sqrt{\frac{\beta}{4\beta - \alpha^2}}.\exp\left(-\tfrac{1}{2}\alpha t\right) \cos\left\{\tfrac{1}{2}\sqrt{4\beta - \alpha^2}.t + \Psi\right\}\right].$$

Finally, when $t \longrightarrow \infty$, we have

$$\phi = [\theta]_{t\to\infty} = \frac{\gamma}{\beta} = \frac{NAB}{b}.\frac{v}{r}.$$

Thus the coil moves with damped harmonic motion whose period is approximately $2\pi\sqrt{\dfrac{I_0}{b}}$. In its motion the coil overshoots its equilibrium ϕ and then oscillates* about this position with diminishing amplitude.

It is left to the reader to show that when $a + \dfrac{(NAB)^2}{r} = 2\sqrt{bI_0}$, the coil approaches its equilibrium position without overshoot, i.e. there is ***critical damping***, and when $a + \dfrac{(NAB)^2}{r} > 2\sqrt{bI_0}$, the coil again approaches its end position without overshoot but much more slowly than when the damping is critical.

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* The curve shown in Fig. 14·12, p. 484, is similar to that which would now indicate how θ varies with time.

The essential features of a ballistic galvanometer.—The theory of a ballistic galvanometer, as just developed, shows that if a charge is to be measured then that charge must have passed through the coil system before the coil has moved from its position of static equilibrium. In practice this is effected by using a coil with a large moment of inertia about its axis of rotation and a suspension with a small torsional constant. Thus the period of oscillation is made large. The use of such a suspension has a further advantage, viz. for a given quantity of electricity a larger throw is obtained, but a limit to the smallness of this constant is set by the fact that the suspension must be sufficiently strong to carry the relatively massive coil; thus, in practice, a compromise between these conflicting ideals must be reached.

Moreover, it has been shown that the throw is affected by damping; since a correction for damping is often involved it is desirable to keep the damping small. Now damping is caused by induced currents set up in the metallic parts of the movable system as it traverses the magnetic field of the instrument, by its motion in the air which is a slightly viscous medium, and by internal friction in the suspension although this is usually negligible. Thus the main factor contributing to the damping of a galvanometer arises from induced currents; these are made vanishingly small if the former on which the coil is wound is non-metallic. In many modern instruments no former is retained, the coil being impregnated with a synthetic resin, such as 'Damarda lacquer' which is hardened by raising the temperature of the coil to about 120° C. for a few hours.

In contrast with this, many direct current instruments are wound on metal formers so that the motion is rather more than critically damped; such instruments can never be used ballistically, but it must be realized that all ballistic galvanometers may be used as ammeters or voltmeters provided the necessary shunt, or shunt and series resistance, is available; usually the shunt is such that the motion of the coil is oscillatory but almost critically damped.

The current sensitivity of a moving-coil galvanometer.—In the first instance the current sensitivity of a moving-coil galvanometer is defined as the deflexion per unit current. Since

$$i = \frac{b}{\mathrm{NAB}}\phi,$$

the current sensitivity, σ_i, is given by

$$\sigma_i = \frac{\phi}{i} = \frac{\mathrm{NAB}}{b}.$$

In practice it is usual to measure the sensitivity, Σ_{I}, as the deflexion in mm. when the current is 10^{-6} amp. and the scale is at a distance of one metre from the mirror attached to the galvanometer coil. Then if ϕ' is the steady deflexion in mm., we have

$$10^{-7}\ (\text{e.m.u.}) = \frac{b}{\text{NAB}}\cdot\frac{\phi'}{2000}.$$

$$\therefore\ \Sigma_{\text{I}} = \frac{\phi'}{10^{-7}} = 2000\frac{\text{NAB}}{b}.$$

The quantity sensitivity of a moving-coil ballistic galvanometer.—Again, in the first instance, the quantity sensitivity of a moving-coil ballistic galvanometer is defined as the (observed) deflexion per unit charge. [N.B. The word 'observed' is inserted into this definition since we do not have to consider the correction for damping, for the sensitivity is governed by the observed value of the deflexion.] Since

$$q = \frac{b}{\text{NAB}}\cdot\frac{\text{T}}{2\pi}\theta_1(1+\tfrac{1}{2}\lambda),$$

where the symbols have their usual meanings, the quantity sensitivity, σ_q, is given by

$$\sigma_q = \frac{\theta_1}{q} = \frac{\text{NAB}}{b}\cdot\frac{2\pi}{\text{T}}\cdot\frac{1}{(1+\frac{1}{2}\lambda)} = \frac{\phi}{i}\cdot\frac{\pi}{\text{T}}\cdot\frac{2}{(1+\frac{1}{2}\lambda)}.$$

Thus
$$\sigma_q = \frac{2\pi}{\text{T}}\cdot\frac{1}{(1+\frac{1}{2}\lambda)}\cdot\sigma_i.$$

If, as usual, λ is small,

$$\sigma_q = \frac{2\pi}{\text{T}}\sigma_i.$$

Now the quantity sensitivity in practice, viz. Σ_{Q}, is usually expressed as the deflexion in mm. produced by an instantaneous discharge of one microcoulomb when the scale distance is one metre. If θ' is the observed deflexion in mm., we have

$$10^{-7}\ (\text{e.m.u.}) = \frac{b}{\text{NAB}}\cdot\frac{\text{T}}{\pi}\cdot\left(\frac{1}{2}\cdot\frac{\theta'}{2000}\right)(1+\tfrac{1}{2}\lambda).$$

$$\therefore\ \Sigma_{\text{Q}} = \frac{\theta'}{10^{-7}} = \frac{\text{NAB}}{b}\cdot\frac{2\pi}{\text{T}}\cdot\frac{2000}{(1+\frac{1}{2}\lambda)}$$

$$= \frac{\phi}{i}\cdot\frac{2\pi}{\text{T}}\cdot\frac{2000}{(1+\frac{1}{2}\lambda)}.$$

If λ is small, we have

$$\Sigma_{\text{Q}} = \frac{2\pi}{\text{T}}\Sigma_{\text{I}}.$$

To calculate the undamped period from the observed period.—It has been shown that the period of a ballistic galvanometer is given by the formula

$$T = \frac{4\pi}{\sqrt{4\beta - \alpha^2}},$$

where α is a damping coefficient; the undamped period would be $T_0 = \frac{2\pi}{\beta^{\frac{1}{2}}}$. Since the logarithmic decrement per half cycle is $\lambda = \frac{1}{4}\alpha T$, we have

$$T^2 = \frac{16\pi^2}{\frac{16\pi^2}{T_0^2} - \frac{16\lambda^2}{T^2}}, \quad \text{or} \quad T_0 = T\sqrt{\frac{\pi^2}{\pi^2 + \lambda^2}}.$$

In practice λ is usually very small so that T_0 may be taken as being equal to T.

Further remarks on the damping of moving-coil galvanometers.—In addition to the mechanical damping investigated on p. 335, we have to consider the fact that the movement of the coil in the permanent field of the galvanometer magnet induces currents in the circuit if this is closed, and these currents tend to bring the coil to rest. Now although the coil moves in a radial field yet, on account of the fact that there is an iron core, there will be a change in flux as the coil moves. This change in flux will be proportional to θ, the angle which the plane of the coil makes with its position of static equilibrium, for, as shown in Fig. 10·19 (*a*), it is the lines of induction in the air-gap defined by θ which pass through the coil. The value of the flux may be calculated in the following way.

Let the coil consist of N turns each with an area A. When the plane of the coil makes an angle θ with the field in the core as in Fig. 10·19 (*b*), the magnetic flux across the cylindrical surface P_0Q_0QP, where PQ is the vertical side of the coil, is

$$B[l(\tfrac{1}{2}w)\theta],$$

where B is the magnetic induction at points along PQ and RS.

The flux through each complete turn of the coil is therefore

$$2B\left(l\frac{w}{2}\theta\right) = BA\theta,$$

where $A = lw$ is the area of the coil.

If the N turns are made of fine wire so that they may be considered to be in a plane, the effective flux for the coil is

$$NAB\theta = \Psi \text{ (say)}.$$

Hence the induced e.m.f. is $-NAB\dot{\theta}$, and the current in the circuit is given by

$$i = -\frac{\mathrm{NAB}\dot{\theta}}{r},$$

where r is the resistance in e.m.u. of the complete circuit.

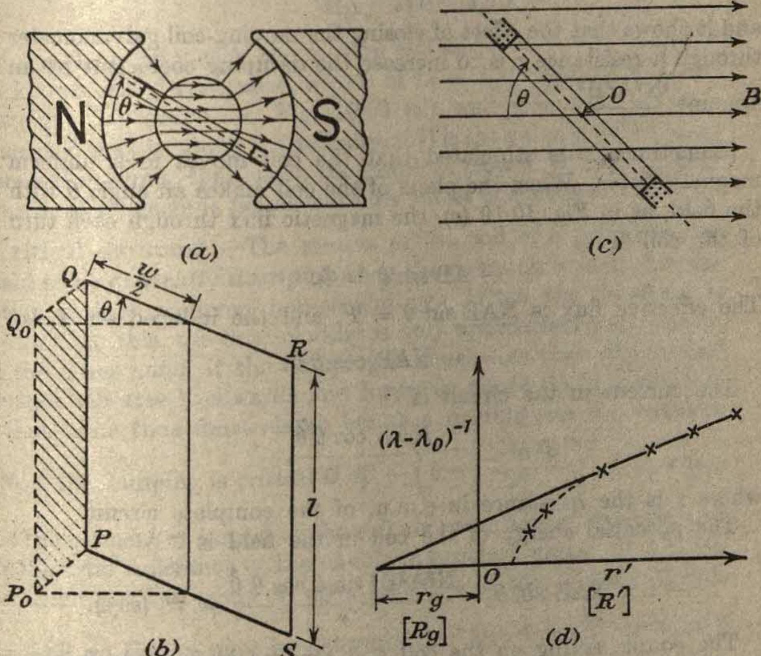


Fig. 10·19.—Experimental study of the damping of a moving-coil galvanometer.

Now the potential energy of the coil in the field may be derived, at once, from that of its equivalent magnetic shell [cf. p. 125] and is

$- i \times$ the flux of magnetic induction in the positive direction $\times$ N

$$= - i \times \mathrm{BA}\theta \times \mathrm{N} = \frac{\mathrm{N}^2\mathrm{A}^2\mathrm{B}^2\theta}{r} \times \dot{\theta} = \mathrm{W} \text{ (say).}$$

The couple acting on the coil and tending to increase θ is

$$-\frac{\partial \mathrm{W}}{\partial \theta} = -\frac{\mathrm{N}^2\mathrm{A}^2\mathrm{B}^2\dot{\theta}}{r}. \qquad [\dot{\theta} \text{ is an independent variable.}]$$

The equation of motion for the coil is

$$\mathrm{I}_0\ddot{\theta} = \text{sum of couples tending to increase } \theta$$

$$= -\left(a + \frac{\mathrm{N}^2\mathrm{A}^2\mathrm{B}^2}{r}\right)\dot{\theta} - b\theta,$$

where I_0 is the moment of inertia of the coil about its axis of rotation, $a\dot{\theta}$ is the couple due to mechanical damping and b is the torsional constant of the suspension. This equation may be written

$$\ddot{\theta}+\left(\alpha+\frac{N^2A^2B^2}{rI_0}\right)\dot{\theta}+\beta\theta=0,$$

and it shows that the effect of closing the moving-coil galvanometer through a resistance r is to increase the damping coefficient by an amount $\dfrac{N^2A^2B^2}{rI_0}$.

[Sometimes it is stipulated that the coil moves in a uniform magnetic field. When the plane of the coil makes an angle θ with the field, as in Fig. 10·19 (*c*), the magnetic flux through each turn of the coil is

$$AB\sin\theta=\Phi.$$

The effective flux is $NAB\sin\theta=\Psi$, and the induced e.m.f. is

$$=-NAB\cos\theta\,\dot{\theta}.$$

The current in the circuit is

$$-\frac{NAB\cos\theta\,\dot{\theta}}{r},$$

where r is the resistance in e.m.u. of the complete circuit.

The potential energy of the coil in the field is

$$-i\times NAB\sin\theta=\frac{N^2A^2B^2\sin\theta\cos\theta\,\dot{\theta}}{r}=W\ \text{(say)}.$$

The couple acting on the coil and tending to increase θ is

$$-\frac{\partial W}{\partial\theta}=-\frac{N^2A^2B^2[\cos^2\theta-\sin^2\theta]\dot{\theta}}{r},$$

since $\dot{\theta}$ is an independent variable.

When $\theta\longrightarrow 0$, $\sin\theta\longrightarrow 0$ and $\cos\theta\longrightarrow 1$, so that the couple becomes

$$-\frac{A^2N^2B^2\dot{\theta}}{r},$$

and the equation of motion for the coil is, as before,

$$\ddot{\theta}+\left(\alpha+\frac{N^2A^2B^2}{rI_0}\right)\dot{\theta}+\beta\theta=0.]$$

Now on open circuit the logarithmic decrement per half-cycle is given by $\lambda_0=\frac{1}{4}\alpha T_0$. When the circuit is closed, its total resistance being r, we have

$$\lambda=\frac{1}{4}\left(\alpha+\frac{N^2A^2B^2}{rI_0}\right)T.$$

Since $T \fallingdotseq T_0$, we get

$$\lambda - \lambda_0 = \frac{1}{4} \cdot \frac{N^2A^2B^2}{rI_0} T_0,$$

or

$$r = \frac{N^2A^2B^2T_0}{4I_0} \cdot \frac{1}{(\lambda - \lambda_0)}.$$

If r_g is the resistance of the moving coil and $r' = r - r_g$, where r' is a known added resistance, a plot of $(\lambda - \lambda_0)^{-1}$ against r' yields a straight line graph, cf. Fig. 10·19 (*d*), and from this graph a value for r_g may be found. [Actually, if R is the value of a resistance expressed in ohms, then when the graph is plotted R′ and R_g may be used, as a constant factor will only affect the slope of the line.]

Critical damping.—The motion of the coil of a galvanometer is said to be ***critically damped*** when the damping is such that the motion is just rendered non-oscillatory. When the damping is greater than this, the final reading is only approached very slowly. On the other hand, if the damping is much less than the critical the spot vibrates backwards and forwards about its final position so that some time must elapse before a reading can be obtained.

Now the damping is critical if $4\beta - \left(\alpha + \dfrac{N^2A^2B^2}{rI_0}\right)^2 = 0$, where I_0 is the moment of inertia of the coil about its axis of rotation and r is the total resistance of the circuit in absolute units. If λ is the decrement per half-cycle on open circuit, and this is easily found, $\lambda = \frac{1}{4}\alpha T$, so that α may be determined. Also $T = \dfrac{2\pi}{\beta^{\frac{1}{2}}}$, so that β may be found. If the resistance of the circuit when the damping is just critical is determined then a value for NAB may be calculated.

Additional notes on galvanometers.—A modern galvanometer should satisfy the following critical demands :—

(*a*) Its zero must be steady.
(*b*) Variations in temperature should have little effect.
(*c*) The construction should be robust and the insulation as nearly ideal as possible.
(*d*) The movable system should be protected effectively from dust and the disturbing effects of air currents.
(*e*) No damage should take place when the instrument is in transit.

In several modern types of moving-coil galvanometer, the coil is not suspended but is stretched between two wires ; in this way a high zero steadiness is attained. The coil is enclosed in an air-tight case. The clamping of the coil secures a safe transport for the instrument and the act of clamping is automatically preceded by a loosening of the stretched wires.

Modern high sensitivity galvanometers usually have quartz suspensions or stretched 'wires'. In such instances the current is conveyed to the coil by means of two short strips of gold foil.

A good galvanometer should have a short time of indication; the precision attainable with a given instrument is found to depend very largely on the quickness of its response. A slow instrument integrates all extraneous disturbances but by working with a galvanometer with a quick response the effect of these disturbances may be eliminated almost completely.

In using a moving-coil galvanometer to measure current it is often an advantage to have slight underdamping rather than critical damping. With slight underdamping the spot of light just overshoots its final position and then returns to it. By such means an observer may easily satisfy himself that the instrument is free from all 'sticking effects' and that the full deflexion has been attained.

When the period of a given coil has been adjusted by the selection of the correct suspension, the condition of near-critical damping has to be secured. When the circuit resistance is small the galvanometer will, in general, be overdamped and critical damping may be attained by adding the necessary resistance to the circuit.

A more satisfactory method for attaining this condition of near-critical damping is to reduce the strength of the controlling field for it is found that in this way the sensitivity is not reduced to the same extent. A convenient method for reducing the field strength makes use of a so-called ***magnetic shunt***. This is a piece of soft iron which bridges across the top of the magnet at a selected distance from the gap. Sometimes the shunt is placed across part of the gap and either in contact or nearly in contact with the pole pieces. When the shunt is close to the gap, it considerably reduces the field strength. For example, it is found that a 12-ohm. galvanometer requiring an external resistance of 100 ohm. for critical damping may be used in a circuit whose total resistance is as low as 20 ohm. if a suitable magnetic shunt is used.

When the external circuit has a large resistance, the magnetic field should be kept a maximum and critical damping attained by shunting the galvanometer with a suitable resistance.

Example.—The coil of a moving-coil galvanometer has 400 turns and is suspended in a uniform magnetic field with induction B = 1000 gauss by a phosphor-bronze strip whose torsion constant is 2·0 erg. radian.$^{-1}$. The coil is 2·0 cm. wide and 3·0 cm. high, and has a moment of inertia of 2·5 gm.cm.^{2} about its axis of rotation.

If the galvanometer resistance is 250 ohm. calculate a value for the resistance which, when connected across the galvanometer terminals, will give critical damping. Assume the damping to be entirely electromagnetic.

The equation of motion is, with the notation previously used,

$$\ddot{\theta} + \left(\alpha + \frac{N^2A^2B^2}{rI_0}\right)\dot{\theta} + \beta\theta = 0.$$

If $\alpha \rightarrow 0$, the motion will be just critical when

$$4\beta - \frac{N^4A^4B^4}{r^2I_0{}^2} = 0.$$

With the data provided $\beta = \frac{2}{2{\cdot}5} = 0{\cdot}8$. Hence the damping is critical when

$$4 \times 0{\cdot}8 = \frac{6^4 \times (400)^4 \times 10^{12}}{r^2 \times 6{\cdot}25}$$

or

$$r = 1{\cdot}29 \times 10^{12} \text{ e.m.u.},$$

i.e.

$$R = 1290 \text{ ohm}.$$

Hence the additional resistance required is 1040 ohm.

Example.—A moving-coil galvanometer has a sensitivity 4·0 cm. per μA. with the scale 1 metre away, and the time of oscillation is 3·14 seconds. If the galvanometer is critically damped when the total circuit resistance is 3000 ohm., calculate a value for the moment of inertia of the moving system about its axis of rotation.

Since 1 μA. = 10^{-7} e.m.u., from the sensitivity data we have

$$10^{-7} = \frac{b}{NAB}\cdot\frac{4}{200} = \frac{4\beta I_0}{200NAB} \quad . \quad . \quad . \quad \text{(i)}$$

Assuming that $\alpha \rightarrow 0$, $\beta = \frac{4\pi^2}{T^2} = 4.$

The condition for critical damping gives

$$4\beta = \left(\alpha + \frac{N^2A^2B^2}{rI_0}\right)^2,$$

and since we have assumed $\alpha = 0$,

$$4 = \frac{N^2A^2B^2}{rI_0} = \frac{N^2A^2B^2}{3 \times 10^{12} \times I_0}.$$

$$\therefore I_0 = \frac{N^2A^2B^2}{12 \times 10^{12}} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

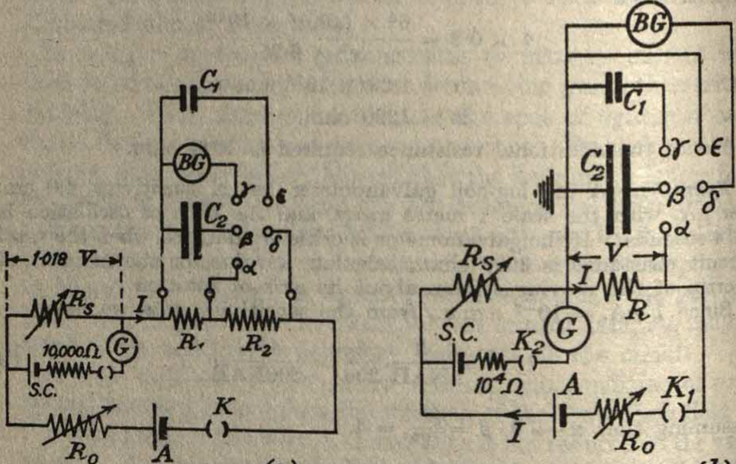
Eliminating NAB from equations (i) and (ii) we have,

$$I_0 = \frac{256I_0{}^2 \times 10^{10}}{4 \times 12 \times 10^{12}}.$$

$$\therefore I_0 = 18{\cdot}8 \text{ gm.cm.}^2.$$

Comparison of a large capacitance with a much smaller one.—The usual methods for the comparison of the capacitances of two condensers are only applicable when the capacitances are of the same order of magnitude. The following two methods may be used to compare capacitances of very different orders of magnitude.

(*a*) Suppose that C_1 and C_2, Fig. 10·20 (*a*), are two condensers and, for example, $100\ C_1 \simeq C_2$. These condensers are arranged in conjunction with two resistances R_1 and R_2, of such values that $99\ R_1 = R_2$. The current through them is supplied by the accumulator A and its magnitude controlled in the manner indicated. Then if V is the potential difference across R_1, that across $R_1 + R_2$ is 100 V. By closing the key K and then using the mercury cups α, β, γ, δ, ε, each capacitor may be charged in turn and then discharged through a ballistic galvanometer BG. This must be a high resistance instrument so that it shall have a high quantity-sensitivity.



(a) (b)

FIG. 10·20.—The comparison of a small capacitance and a large capacitance.

In accurate work the current, I, from the cell A must be kept constant; this is done by adjusting the rheostat R_0 so that the potential difference across the resistor R_s, with its resistance fixed at a suitable value, is equal to the e.m.f. of the standard cell S.C. If θ_1 and θ_2 are the mean values of a series of galvanometer throws for each capacitor C_1 and C_2, then

$$\frac{\theta_1}{\theta_2} = \frac{(R_1 + R_2)IC_1}{R_1 IC_2},$$

or

$$\frac{C_1}{C_2} = \frac{R_1}{R_1 + R_2} \cdot \frac{\theta_1}{\theta_2}.$$

The values of R_1 and R_2 should be such that $\theta_1 \simeq \theta_2$.

(*b*) In this method the two capacitors C_1 and C_2, a ballistic galvanometer BG, a resistor R and a set of mercury cups are arranged as in Fig. 10·20 (*b*). The current I through R is supplied from the accumulator A and is kept constant by adjusting R_0 so that the potential difference across a resistor R_s is equal to the e.m.f. of the standard cell S.C. The condenser C_2 is charged and then its charge is shared with the small condenser C_1. This condenser is discharged and the charge remaining on C_2 is again shared with C_1; C_1 is discharged and the process repeated n times. Finally, the charge left on C_2 is discharged through BG; let the throw be θ_n. If θ_0 is the throw when C_2 is charged and immediately discharged through BG, then

$$\frac{C_1}{C_2} = \left(\frac{\theta_0}{\theta_n}\right)^{\frac{1}{n}} - 1.$$

To prove this let V be the steady potential difference across R. Then the initial charge on C_2 is given by

$$Q_0 = C_2V = k\theta_0, \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where k is a constant which includes the correction for damping. At the first sharing of charges the potential of the system is

$$\frac{Q_0}{C_1 + C_2} = \frac{C_2V}{C_1 + C_2} = V_1, \text{ say.}$$

$$\therefore \text{ Charge on } C_2 = Q_1 = C_2V_1 = \frac{C_2{}^2}{C_1 + C_2}V.$$

After the second sharing of charges

$$Q_2 = C_2V_2, \quad \text{and} \quad V_2 = \frac{Q_1}{C_1 + C_2},$$

so that

$$Q_2 = \frac{C_2{}^3V}{(C_1 + C_2)^2}.$$

After the nth sharing of charges

$$\frac{C_2{}^{n+1}V}{(C_1 + C_2)^n} = Q_n = k\theta_n \quad . \quad . \quad . \quad \text{(ii)}$$

From (i) and (ii) we get

$$\left(\frac{C_1 + C_2}{C_2}\right)^n = \frac{\theta_0}{\theta_n}, \quad \text{or} \quad \frac{C_1}{C_2} = \left(\frac{\theta_0}{\theta_n}\right)^{\frac{1}{n}} - 1.$$

This equation may be written

$$n \log\left(\frac{C_1}{C_2} + 1\right) = \log \theta_0 - \log \theta_n,$$

so that if n is varied a linear graph is obtained by plotting $\log \theta_n$ against n. The slope is $-\log\left(\frac{C_1}{C_2} + 1\right)$.

Determination of a capacitance in terms of resistance and time.—The capacitor C, Fig. 10·21 (*a*), whose capacitance is to be determined, a galvanometer G and a cell B, e.m.f. V, are arranged as shown, suitable keys being provided so that C may be charged and then discharged through G. If θ is the ballistic throw and all quantities concerned are expressed in practical units, we have

$$\mathrm{CV} = \mathrm{Q} = \frac{\mathrm{T}}{2\pi}\cdot\frac{\mathrm{I}}{\phi}\cdot\theta\left(1+\frac{\lambda}{2}\right).$$

To determine $\frac{\mathrm{I}}{\phi}$ and remove V from this equation a small steady current is passed through G. This is done by placing across the cell B a known resistance S (say $0{\cdot}1 - 1{\cdot}0\ \Omega$) in series with a large and known adjustable resistance R (maximum value 10,000 Ω).

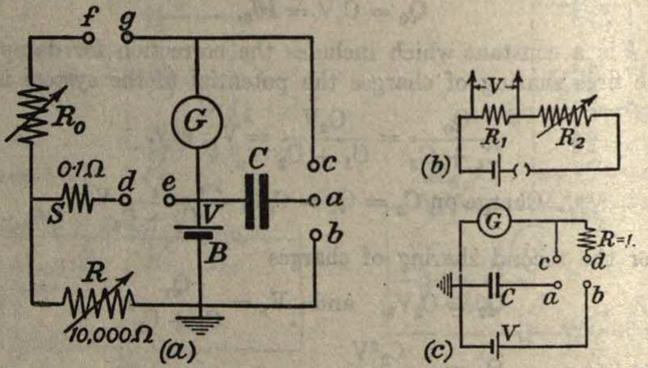


FIG. 10·21.—Determination of a capacitance in terms of resistance and time.

The small potential difference thus developed across S is used to send a steady current, I, through G; the magnitude of the current is controlled by R_0, its value being so large that its shunting effect on S may be neglected. If R_g is the resistance of the galvanometer

$$\mathrm{I} = \frac{\frac{\mathrm{V}}{\mathrm{R}}\cdot\mathrm{S}}{(\mathrm{R}_g + \mathrm{R}_0)},$$

and this produces a steady deflexion ϕ.

Hence
$$\mathrm{CV} = \frac{\mathrm{T}}{2\pi}\cdot\frac{\mathrm{VS}}{\mathrm{R}(\mathrm{R}_g+\mathrm{R}_0)}\cdot\frac{\theta}{\phi}\left(1+\frac{\lambda}{2}\right),$$

or
$$\mathrm{C} = \frac{\mathrm{T}}{2\pi}\cdot\frac{\mathrm{S}}{\mathrm{R}(\mathrm{R}_g+\mathrm{R}_0)}\cdot\frac{\theta}{\phi}\left(1+\frac{\lambda}{2}\right).$$

If C is large and the cell an accumulator then, in general, Q will be so large that the spot of light moves off the galvanometer scale. To overcome this a resistance R_1, Fig. 10·21 (b), (say 100 Ω), is placed in series with a large resistance R_2 and the cell; the potential difference across R_1 is used as V. [In the laboratory it should be verified that the quantity of electricity passing through G cannot be varied by placing a resistor in series with C. If this resistor has a resistance 20 MΩ, the quantity discharged is unchanged but the throw will be reduced. This is because the time constant, cf. p. 474, of the capacitor-resistance combination is now so large that the condition for the discharge to have passed through G before its coil has moved is violated and the equations developed in connexion with the theory of a ballistic galvanometer are no longer applicable.]

Fig. 10·21 (*c*) shows another circuit for carrying out this experiment. R is a large resistance, 1 MΩ, and by connecting *b* and *d* a small steady current may be sent through G. It is easy to show that

$$C = \frac{T}{2\pi}\cdot\frac{1}{R + R_g}\cdot\frac{\theta}{\phi}\cdot\left(1 + \frac{\lambda}{2}\right),$$

and in using this expression R_g may be neglected since $\frac{R_g}{R} \rightarrow 0$. In practice only a small fraction of the voltage available from the cell may be required.

Further note on the sensitivity of a moving-coil galvanometer.—(*a*) ***The current sensitivity.*** The current sensitivity, cf. p. 340, is given by $\sigma_i = \frac{NAB}{b}$, so that by making b small and B large, i.e. by having a thin suspension and a strong permanent magnet, the sensitivity may assume a high value. The sensitivity is also dependent upon the effective area of the coil but since with a circular or square coil of considerable cross-section the gap between the pole pieces becomes wide, with a consequent reduction in the value of the field and a disturbing of its radial features, it is customary to make the coil long and narrow.

It is also important to use wire with an insulating cover which is free from traces of ferromagnetic impurities, for these give rise to an additional restoring couple which reduces the sensitivity.

(*b*) ***The quantity sensitivity.*** When ballistic galvanometers are used to measure quantity they should have a long period, at least ten seconds, if it is to be convenient for the observer to record several successive swings. This is attained in practice by making A large and *b* small. It is, of course, essential to keep the air-damping as small as possible but since the period has to be larger than for a galvanometer designed for current measurement, the coil may be more massive.

The Grassot fluxmeter.—The use of a ballistic galvanometer for the measurement of magnetic flux is restricted by the fact that the flux linkages through a search coil must be varied so quickly that the associated electric charge passes through the galvanometer before the moving part is displaced from its position of static equilibrium. The fluxmeter, designed by GRASSOT in 1904, enables a magnetic field to be explored with the aid of a search coil, but its use is not limited by the above restriction. In a Grassot fluxmeter, which is in good working order, the deflexion of the fluxmeter is the same whether the flux change in the search coil be made quickly or at some slower rate. The deflecting system is at rest both before and after the charge of electricity has passed through its coils. This result is obtained by making the damping due to the electromagnetic effect between a permanent magnet and the coil as it rotates predominant, while at the same time all other sources of control are reduced to a negligible amount. Hence a fluxmeter of this type is really a ballistic galvanometer with a suspension almost free from a torsional couple when it is twisted.

To achieve these remarkable features the coil C, shown in Fig. 10·22 (*a*), is suspended by a single cocoon silk fibre F whose upper end is attached to a flat spiral spring H to minimize damage from small mechanical shocks. The permanent field is supplied by a magnet NS, whose pole pieces are fashioned with concave cylindrical surfaces and between these there is a coaxial soft-iron cylinder, A, so that the coil moves in a radial magnetic field—cf. Fig. 10·22 (*b*). T_1 and T_2 are external terminals which permit the passage of a transient current from a search coil through the coil of the instrument.

To establish a working formula for this ballistic galvanometer with its almost torsionless suspension, let us imagine that a search coil K, Fig. 10·22 (*c*), consisting of N turns each of area A, is connected to the coil C; let r be the resistance and l the self inductance of the complete circuit; these two quantities are expressed in e.m.u. Let the coil C make an angle θ with some reference line fixed in the field and passing through the axis of the core of the fluxmeter. Let Φ be the flux through each turn of the search coil; then Ψ, the number of linkages, is given by $\Psi = \mathrm{N}\Phi$. The e.m.f. induced in the search coil at any instant is $e = -\mathrm{N}\dot{\Phi}$. This will be considered numerically as it causes other effects; among other things, a transient current through C is established and, in consequence, C moves.

Let the effective flux through the fluxmeter coil C, when it is in a position θ, be $k\theta$, where k is a constant, viz. NAB, cf. p. 342. It must be noted that although the magnetic field in which C moves is radial, the field is only radial in the air-gap; the lines of magnetic induction in the core are straight and it is the flux due to these that is represented by the term $k\theta$; it is only as an approximation that k is considered constant. When the coil C moves an e.m.f., opposing that induced in the search coil, is set up; it is $k\dot{\theta}$.

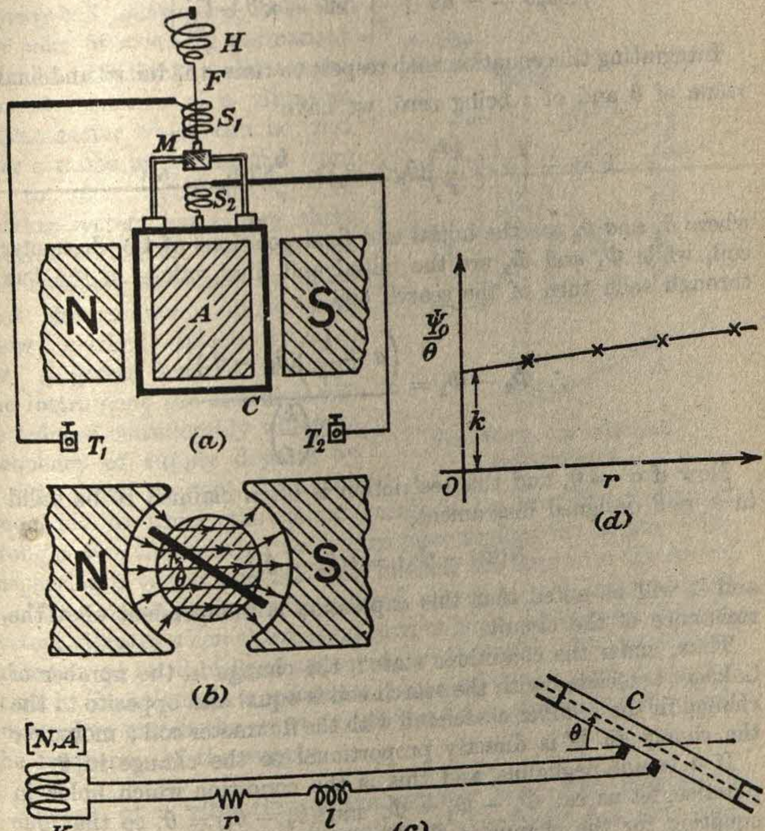


FIG. 10·22.—A Grassot fluxmeter.

Under these circumstances we have

$$[N\dot{\Phi} - k\dot{\theta}] - l\frac{di}{dt} = ri,$$

the e.m.f. $-l\frac{di}{dt}$ being due to the self inductance of the circuit. The equation expressing the motion of the fluxmeter coil is

$$I_0\theta = ki - a\theta,$$

where I_0 is the moment of inertia of the coil about its axis of suspension, ki is the couple due to the current, cf. p. 334, and $a\dot{\theta}$ is the couple due to air-damping.

$$\therefore\ I_0\ddot{\theta} = -a\dot{\theta} + \frac{k}{r}\left[N\dot{\Phi} - k\dot{\theta} - l\frac{di}{dt}\right].$$

Integrating this equation with respect to time, the initial and final value of $\dot{\theta}$ and of i being zero, we have

$$0 = -\left(a + \frac{k^2}{r}\right)(\theta_2 - \theta_1) + \frac{k}{r}N(\Phi_2 - \Phi_1),$$

where θ_1 and θ_2 are the initial and final positions of the fluxmeter coil, while Φ_1 and Φ_2 are the initial and final values of the flux through each turn of the search coil.

$$\therefore\ \Phi_2 - \Phi_1 = \frac{\left(a + \frac{k^2}{r}\right)(\theta_2 - \theta_1)}{N\left(\frac{k}{r}\right)}.$$

Now if $a \longrightarrow 0$, and this restriction is often claimed to be valid in a well designed instrument,

$$N(\Phi_2 - \Phi_1) = k(\theta_2 - \theta_1),$$

and it will be noted that this expression is independent of r, the resistance of the circuit.

Thus, under the conditions stated, the change in the number of linkages associated with the search coil is equal and opposite to the change in the number associated with the fluxmeter coil; moreover the change in Ψ is directly proportional to the change in θ.

If a is not negligible, and this is the condition which holds in practice, let us call $\Phi_2 - \Phi_1 = \Phi$, and $(\theta_2 - \theta_1) = \theta$, so that our equation for the change in flux becomes

$$N\Phi = \left(k + \frac{ar}{k}\right)\dot{\theta}.$$

Suppose that we carry out a series of experiments keeping $N\Phi$ constant, say Ψ_0, and observe values of θ for different values of r, the total resistance of the fluxmeter circuit. Then

$$\frac{\Psi_0}{\theta} = \left(k + \frac{ar}{k}\right).$$

A plot of $\Psi_0\theta^{-1}$ against r will yield a straight line if our theory is correct and enable us to find a value for the constants k and a. A typical graph is shown in Fig. 10·22 (d).

The Campbell moving-coil vibration galvanometer.—These galvanometers are used for the detection of small alternating currents. They are of the d'Arsonval type and have a moving-coil suspended between the poles of a strong permanent magnet. Fig. 10·23 indicates the general structure of a vibration galvanometer which can be used over a range of frequencies from 50 to 1000 cycle.sec.$^{-1}$. The moving system has a very short natural period of vibration and the damping is small in order that the resonance curve shall show marked selectivity, i.e. for a given current through the instrument the deflexion will be reduced considerably when the frequency of supply departs by a very small amount from the frequency to which the galvanometer is tuned. The tuning is brought about by moving the bridge piece so that the length of the suspension is varied, and finally by altering the tension in the spring S_1. The moving coil carries a small mirror (4 × 2 mm.) and the system vibrates when alternating current is passed through the coil, an extended band of light appearing on the scale. The number of turns in the coil and its resistance, plus that of the connexions, are adjusted to match the resistance of the external circuit and to suit the frequency of the supply.

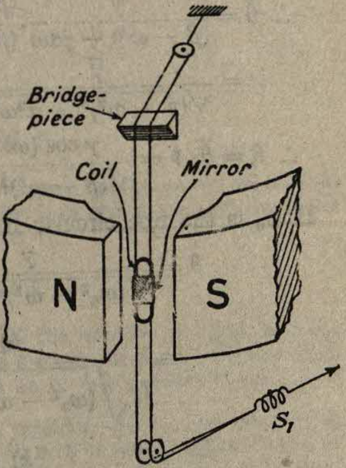


FIG. 10·23.—A vibration galvanometer.

Theory of a vibration galvanometer with one degree of freedom.—The Campbell vibration galvanometer has only one degree of freedom, i.e. the suspended system only rotates about the axis of suspension. The equation of motion is

$$I_0\ddot{\theta} + a\dot{\theta} + b\theta = ki,$$

where the symbols have their usual meanings. If $i = \hat{I}\cos\omega t$,

$$I_0\ddot{\theta} + a\dot{\theta} + b\theta = k\hat{I}\cos\omega t,$$

i.e.

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\theta = \gamma\cos\omega t.$$

The complementary function in the solution is relatively unimportant; the particular integral gives the steady oscillation and is readily obtained by the symbolic method [cf. p. 545]. We have

$$(-\omega^2 + j\alpha\omega + \beta)\vec{\theta} = \vec{\gamma}.$$

$$\therefore\ \vec{\theta} = \frac{\vec{\gamma}}{(\beta - \omega^2) + j\alpha\omega} \cdot \frac{(\beta - \omega^2) - j\alpha\omega}{(\beta - \omega^2) - j\alpha\omega}$$

$$= \frac{\vec{\gamma}}{\sqrt{(\beta - \omega^2)^2 + \alpha^2\omega^2}} \exp(-j\phi), \text{ where } \tan\phi = \frac{\alpha\omega}{\beta - \omega^2}.$$

$$\therefore\ \theta = \vec{\theta} \cdot \vec{\imath} = \frac{\gamma\cos(\omega t - \phi)}{\sqrt{(\beta - \omega^2)^2 + \alpha^2\omega^2}}.$$

If ω_0 is the free circular frequency, $\omega_0{}^2 = \beta$, and we have

$$\theta = \frac{\gamma}{\sqrt{(\omega_0{}^2 - \omega^2)^2 + \alpha^2\omega^2}} \cos(\omega t - \phi)$$

$$= \frac{k\hat{\mathrm{I}}}{\mathrm{I}_0\sqrt{(\omega_0{}^2 - \omega^2)^2 + \dfrac{a^2}{\mathrm{I}_0{}^2}\omega^2}} \cos(\omega t - \phi)$$

$$= \frac{k\hat{\mathrm{I}}}{\sqrt{\mathrm{I}_0{}^2(\omega_0{}^2 - \omega^2)^2 + a^2\omega^2}} \cos(\omega t - \phi).$$

If the 'frequency' becomes $n\omega$, we have

$$\theta_n = \frac{k\hat{\mathrm{I}}_n}{\sqrt{\mathrm{I}_0{}^2\{\omega_0{}^2 - (n\omega)^2\}^2 + a^2n^2\omega^2}} \cos(n\omega t - \phi_n),$$

and this gives the deflexion due to the nth harmonic when this is present.

The maximum current-sensitivity will be obtained when the instrument is exactly tuned to the frequency of the a.c. used, i.e. when $\omega_0 = \omega$, and $n = 1$. Then

$$\hat{\theta}_1 = \frac{k\hat{\mathrm{I}}_1}{a\omega_0} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

[$\hat{\theta}_1$ and $\hat{\mathrm{I}}_1$ refer to the first harmonic or fundamental.]

If the instrument is correctly tuned to the 'frequency' ω_0, the deflexion caused by a component $\hat{\mathrm{I}}_n$ will produce an amplitude

$$\hat{\theta}_n = \frac{k\hat{\mathrm{I}}_n}{\sqrt{\mathrm{I}_0{}^2(\omega_0{}^2 - n^2\omega^2)^2 + a^2\omega^2n^2}}$$

$$\doteqdot \frac{k\hat{\mathrm{I}}_n}{\mathrm{I}_0(n^2\omega^2 - \omega_0{}^2)} \quad \left[\begin{array}{l}\because\ a\omega \rightarrow 0, \text{ and the denominator is ar-}\\ \text{ranged as shown because } n^2\omega^2 > \omega_0{}^2.\end{array}\right]$$

$$\doteqdot \frac{k\hat{\mathrm{I}}_n}{\mathrm{I}_0\omega_0{}^2[n^2 - 1]}, \quad [\omega_0 = \omega, \text{ if tuned.}]$$

so that θ_n is negligibly small.

For high current sensitivity, equation (i) shows that a must be small, the free period low and the coil-constant k large.

EXAMPLES X

10·01. Show that if the coil of a tangent galvanometer is rotated about a vertical axis so that the needle is in the same vertical plane as the coil, then the current i is given by

$$i = \frac{aH_0}{2\pi N} \sin \phi,$$

where ϕ is the deflexion, a the radius, N the number of turns, and H_0 the horizontal component of the earth's magnetic field. [When used in this way the instrument is known as a ***sine galvanometer***.]

10·02. (*a*) The circumference of a circular coil is 100 cm. It is used in a sine galvanometer. If $H_0 = 0{\cdot}18$ oersted., show that the largest current which can be measured by the galvanometer is 4·56 amp.

(*b*) In setting up a sine-galvanometer there is a small error β in estimating the direction of the meridian. If i is the value of the current calculated from the observed deflexion θ, show that the correct value for the current is $i[1 + \beta \cot \theta]$.

10·03. The primary of a standard mutual inductance consists of 500 turns wound uniformly in a single layer of length 100 cm. The secondary, which is inside the primary and at its centre, has 200 turns of mean diameter 4 cm. and is joined in series with a ballistic galvanometer, the total resistance being 100 ohm. On reversing a current of 3 amp. in the primary, a deflexion of 100 scale divisions is obtained. Calculate the sensitivity of the galvanometer in microcoulombs per scale division. [0·096.]

10·04. A suspended-magnet ballistic galvanometer of resistance 25,000 ohm. has a period of 2·5 seconds and logarithmic decrement per half cycle of 0·012 when on open circuit. A condenser of unknown capacitance was charged to a potential of 2·04 volt, and discharged through the galvanometer. The throw was 15·6 cm. When the same p.d. was applied to the galvanometer shunted by a resistance of 200 ohm. and with a resistance of 1000 ohm. in series, the steady deflexion obtained was 8·57 cm. What is the capacitance of the condenser? [4·82 μF.]

10·05. A solenoid A, 200 cm. long and 10 cm. in diameter, is uniformly wound with 1600 turns. S is a search coil, 4 cm. in diameter and wound with 30 turns; it is mounted coaxially midway along A. If S is connected to a ballistic galvanometer, calculate a value for the quantity of electricity which will pass through the galvanometer when a current of 4 amp. in the solenoid is reversed. Assume that the permeability of the cylindrical shell on which the solenoid is wound is unity and that the resistance of the galvanometer circuit is 2000 ohm. [0·152 microcoulomb.]

10·06. A standard condenser of 1 μF. capacitance was charged to a potential difference of 2·2 V. and then discharged through a moving-coil galvanometer. The first two successive deflexions, taken on the same side, were 15 cm. and 14 cm. The galvanometer mirror was at the centre of a curved scale of radius 100 cm. Calculate: (*a*) the quantity of electricity discharged in the galvanometer circuit, (*b*) the decrement per half-cycle, (*c*) the decrement per cycle, (*d*) the quantity sensitivity of the galvanometer, (*e*) the current sensitivity if the period is 5·6 sec.

[2·2 μC., 0·0346, 0·0692, 6·8 div. per μC., 6·1 div. per μA.]

10·07. A moving-coil galvanometer has a period of 6·5 sec. Two successive swings on the same side of the zero were found to be 10·6 cm. and 9·8 cm. respectively. A steady current of 10^{-4} amp. gave a steady deflexion of 10 cm. A condenser charged to a p.d. of 500 V., when discharged through the galvanometer, produced a throw of 10 cm. Calculate a value for the capacitance of the condenser. [0·21 μF.]

10·08. Give the theory of a moving-coil galvanometer used ballistically. Calculate the period and the current required to produce a deflexion of 1 mm. on a scale 100 cm. from the mirror of a ballistic galvanometer given the following data:—

Coil 500 turns, 4 cm. long by 1·5 cm. wide; moment of inertia about its vertical axis 6·0 gm.cm.^{2}. The suspension has a torsional stiffness 1·5 erg. radian.$^{-1}$ and the coil moves in a radial magnetic field, the value of B, the magnetic induction, at points on the vertical sides of the coil may be taken as 750 gauss. Damping is to be neglected.

[12·56 sec., 3·34 × 10^{-9} amp.]

10·09. A coil of wire of three turns, each 1 cm.^{2} in effective area, is connected to a ballistic galvanometer of the moving-magnet type and the total resistance of the circuit is 24 ohms. On introducing the coil into a strong magnetic field, the maximum deflexion recorded by the galvanometer is the same as that recorded when a condenser of capacitance 6 μF. whose plates have a potential difference of one volt between them is discharged through it. Obtain a value for the magnetic induction associated with the field at points on the vertical sides of the coil. Would your result be correct if a moving-coil galvanometer were employed?

[4·8 × 10^3 gauss.]

10·10. A moving-coil ballistic galvanometer is connected in series with a cell, a resistance and a key. Neglecting all damping, show that on closing the key, the initial throw will be twice the final steady deflexion.

10·11. Derive an expression for the strength of the magnetic field at the centre of a straight solenoid, whose length is great in comparison with its diameter, in which a current is flowing.

A secondary coil of 400 turns of mean radius 3 cm. is placed coaxially with and in the middle of the long (primary) solenoid, and is connected to a ballistic galvanometer, the total resistance of the circuit being 1000 ohms. When a current of 5 amperes in the primary coil is reversed quickly the galvanometer registers an initial throw of 40 divisions. What is the magnitude of the current which must be passed through the galvanometer in order to produce a *steady* deflexion of 30 divisions? Assume that there are 20 windings per cm. on the primary coil and that the period of oscillation of the galvanometer is 5 seconds. (G)

[2·68 μA.]

10·12. The coil of a moving-coil galvanometer has 300 turns and is suspended by a phosphor-bronze strip, whose torsion constant is

2 erg. radian.$^{-1}$. The magnetic field is radial and at points on the vertical sides of the coil the magnetic induction has a value of 1000 gauss. The coil is 2 cm. wide and 2·5 cm. long, and has a moment of inertia of 1·5 gm.cm.^{2} about its axis of rotation. If the galvanometer resistance is 120 Ω, calculate a value for the resistance which must be placed in series with the galvanometer to give critical damping. It may be assumed that the damping is entirely electromagnetic. [551 Ω.]

10·13. With the scale at a distance of 1 metre from the coil, a certain moving-coil galvanometer has a current sensitivity of 50 mm. μA.$^{-1}$. The time of free oscillation is 3·162 sec. If the galvanometer is critically damped when the total circuit resistance is 2000 Ω, calculate a value for the moment of inertia of the moving system about its axis of rotation. Assume that the damping is entirely electromagnetic. [8 gm.cm.^{2}]

10·14. A wire, bent into the form of a circle of radius a cm., carries a current i e.m.u. Determine the strength of the magnetic field due to the current at a point on the axis of the circle at a distance x cm. from its centre. Deduce the field strength at the centre of the circle.

A circular galvanometer coil, having a mean cross-sectional area of 10 cm.^{2}, is suspended inside a long straight solenoid in such a way that its plane makes an angle of $\frac{\pi}{4}$ with the axis of the solenoid. If the suspended coil consists of 200 turns of wire and carries a current of 0·5 mA., while the solenoid consists of a winding of 5 turns per centimetre length and carries a current of 6 A., calculate the torque which must be applied to the suspended coil to prevent it from rotating about a vertical axis. [2·67 dyne.cm.]

10·15. A pair of circular coils, each 40 cm. in diameter and with 250 turns of fine wire, are set up as in a Helmholtz galvanometer. A small coil of 100 turns and mean radius 2·00 cm. is suspended with its centre at the centre of the two fixed coils and its plane normal to that of either larger coil. Find the couple on the small coil when a current of 500 mA. passes through all the coils. [353 dyne.cm.]

10·16. A current balance consists of a coil of 500 turns arranged with its plane vertical. The mean radius is 40 cm. At the centre of the above coil there is another small coil of mean radius 3·00 cm. and with 100 turns. This small coil has its plane horizontal and is supported on a knife edge placed normal to the axis of the large coil. When a certain steady current is passed through both coils in series, a mass of 0·50 gm. placed on the small coil at a distance of 6·0 cm. from the common centre and on the axis of the large coil, counterbalances the couple exerted on the small coil by the mutual action of the currents. Obtain a value for the current. [1·15 A.]

10·17. A circular metal hoop of radius a and resistance r is suspended by a fine wire attached to a point on its circumference so that it hangs vertically in a uniform horizontal magnetic field with induction B; when the hoop is at rest its plane is parallel to the field. The moment of inertia of the suspended system about the suspension is I_0 and the torsional constant is b. Neglecting mechanical damping and the self inductance of the hoop, calculate the period and decrement per cycle of small oscillations of the hoop.

$$\left[\text{If } \alpha = \frac{\pi^2 a^4 B^2}{r I_0} \text{ and } \beta = \frac{b}{I_0},\ T = \frac{4\pi}{\sqrt{4\beta - \alpha^2}},\ \delta = \tfrac{1}{2}\alpha T.\right]$$

10·18. A moving-coil galvanometer has the following constants; resistance 100 Ω, moment of inertia of suspended system, 2·0 gm.cm.^{2}: restoring couple due to the suspension, 2·5 gm.cm.^{2}sec.$^{-2}$radian^{-1}. When the terminals are connected to a coil of resistance 1500 Ω, the logarithmic decrement per half-cycle is 36×10^{-2}. On open circuit the decrement per half-cycle is 3×10^{-2}, and in each instance the period may be taken as 6·0 sec. Obtain a value for the external critical damping resistance. [59 ohm.]

10·19. The current sensitivity of a moving-coil galvanometer is 200 scale divisions per μA. and its period is 10 seconds. A 1 μF. condenser is charged to 1 V. and then discharged through the galvanometer. A ballistic throw of 120 scale divisions is observed. The secondary of a mutual inductance of 1 mH. is then connected to the galvanometer terminals when the total resistance of the circuit comprising the secondary and the galvanometer is 1000 Ω. When a current of 1·0 A. is made, or broken, in the primary of the mutual inductance a ballistic throw of 100 scale divisions is observed. Show that in each experiment the same quantity of electricity is passed through the galvanometer and comment on the observations.

[$Q = 10^{-6}$ coul., $\lambda_1 = 0{\cdot}094$, $\lambda_2 \doteqdot 0{\cdot}5$.]

10·20. The arms AB, BC, and DA of a Wheatstone bridge contain resistances of 20, 20 and 20 ohms respectively; CD contains a certain metal rectifier the current through which is found to be directly proportional to the square of the applied voltage. A galvanometer connects B and D and current is supplied to the bridge so that 2·0 V. is the drop in potential across it; the bridge is found to be balanced. The current to the bridge is then raised so that the potential drop across the bridge is 6·0 V. when it is again balanced. If this balance is effected by altering the resistance in BC, obtain a value for this resistance. [10 ohms.]

10·21. A voltmeter of resistance G is connected (*a*) in series with a resistance R_1 and a cell of e.m.f. E and internal resistance B, and (*b*) in parallel with a resistance R_2, which combination is then placed in series with the same cell. If the readings of the voltmeter are identical in the two instances, prove that $R_1R_2 = BG$.

CHAPTER XI

FERROMAGNETISM: A GENERAL SURVEY OF THE FACTS

Whenever an iron core is used in a current-carrying solenoid, it is observed that there is a marked increase in the strength of the magnetic field surrounding the solenoid. To investigate the matter further let A, Fig. 11·01, be a toroidal solenoid supported on a core free from iron impurities. K is a narrow gap in this core. D is a battery supplying current to the solenoid while S is a search coil encircling the solenoid and connected to a ballistic galvanometer G. However S is moved, provided it still encircles A, the galvanometer deflexion remains zero. If however S is withdrawn through the gap K, a large throw of the galvanometer is observed. This proves that the lines of magnetic induction are closed.

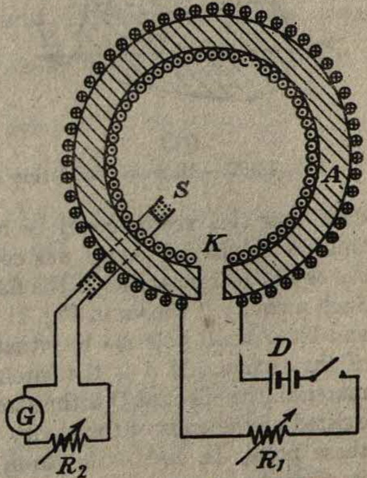


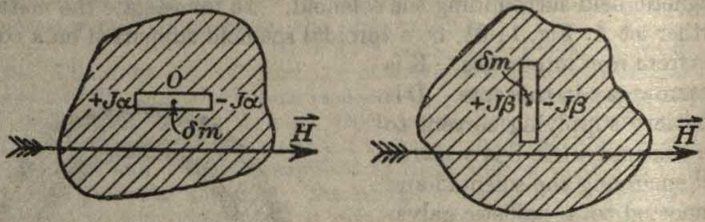
FIG. 11·01.—A fundamental experiment with a ferromagnetic substance.

When the core is made of soft iron the lines of induction are still closed but the throw obtained when S is withdrawn is only maintained of the same order as that before if the resistance of the galvanometer circuit is increased a thousand-fold, i.e. the flux across the gap has been increased a thousand times. With a core made from nickel or cobalt the flux is only increased about a hundred times.

With cores of copper, aluminium, zinc, etc., no observable change can be detected but very exact measurements show that with aluminium there is a very small increase and with bismuth a very small decrease. Substances which increase the flux in the interior of a toroidal solenoid are said to be ***paramagnetic;*** others are ***diamagnetic.*** For such substances the change is a few parts in a hundred thousand; substances which increase the flux many times are termed ***ferromagnetic.***

Kelvin's description of the magnetic state of matter.— Let Fig. 11·01 now represent an iron-cored solenoid with a small gap at K. Experiment shows that as long as the search-coil S encloses the solenoid, there is no ballistic throw produced in G when S is moved. The lines of induction are still closed, i.e. the flux across the air-gap is equal to that in the iron.

The vector $\vec{H}$ is introduced to describe the fact that the strength of the magnetic field within the core is independent of its nature; it depends only on the magnitude of the current in the coil. To show how this vector could be measured Kelvin imagined that a long needle-shaped cavity was constructed in the core, the axis of the cavity being parallel to the field when the specimen was absent. Such a cavity is shown in Fig. 11·02 (*a*). Let α be its cross-section and let a small pole δm be situated at O, a point near the centre of the cavity. If J is the intensity of magnetization within the material, the effect at O is the same as if free poles of strength $\pm$ Jα existed at the ends of the cavity. If r_1 and r_2 are the distances of these poles, Jα and $-$ Jα, from O respectively, the force on δm is given by



(*a*) (*b*)

FIG. 11·02.—Magnetic induction and its relation to field strength.

$$\delta\vec{F} = \vec{H}\,\delta m + \frac{\vec{J}\alpha\,\delta m}{\mu_0 r_1^2} + \frac{\vec{J}\alpha\,\delta m}{\mu_0 r_2^2},$$

where μ_0 is the magnetic permeability of free space. Hence the force per unit pole on the small pole at O is

$$\lim \frac{\delta\vec{F}}{\delta m} = \vec{H} + \frac{\vec{J}}{\mu_0}\left[\frac{\alpha}{r_1^2} + \frac{\alpha}{r_2^2}\right].$$

Now if α is very small compared with r_1^2 and with r_2^2, the last two terms can be neglected so that the force per unit pole at O is $\vec{H}$.

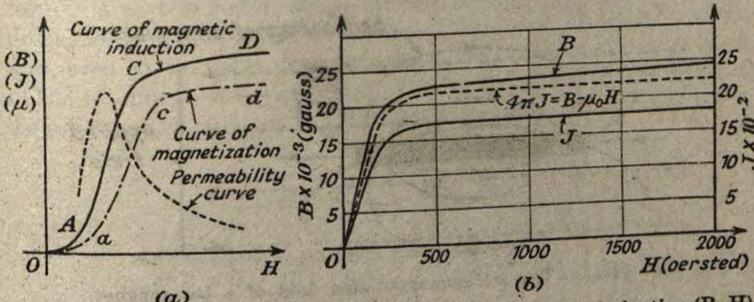
Now suppose the cavity is very flat, i.e. its length in the direction of $\vec{H}$ is very small—cf. Fig. 11·02 (*b*). If β is the cross-sectional area of the cavity, the force per unit pole on a small pole within the cavity and due to the distribution of magnetism on the flat ends of the cavity is, by Gauss' theorem, $\frac{4\pi\vec{J}}{\mu_0}$; the total force per unit pole is therefore $\vec{H} + \frac{4\pi\vec{J}}{\mu_0}$, and this is a measure of the ***magnetic induction***, $\vec{B}$, which, by analogy with the corresponding problem in electrostatics, is defined by

$$\vec{B} = \mu_0\vec{H} + 4\pi\vec{J}.$$

Other shaped cavities may be drawn, but the field within them is a complicated functon of $\vec{H}$ and $\vec{J}$ and cannot be calculated except for spherical* and ellipsoidal cavities.

Magnetic susceptibility and permeability.—The quantity $\frac{J}{H}$ is known as the ***magnetic volume susceptibility*** of the substance and is denoted by the symbol χ; thus $\chi = \frac{J}{H}$. $\frac{B}{H}$ is the ***permeability*** of the medium, μ. Thus $\mu = \mu_0 + 4\pi\chi$.

Curves of magnetization and of magnetic induction.—The curve obtained by plotting J against H in any particular instance, cf. Fig. 11·03 (*a*), is known as a ***magnetization curve***. Such



(*a*) (*b*)

FIG. 11·03.—Curves of magnetization (J, H) and of magnetic induction (B, H). (*a*) At relatively low values of H; (*b*) at higher values of H.

curves differ widely among themselves and yet each one has characteristics common to all. Near to the origin the curve is almost linear and J increases comparatively slowly with the magnetizing field. The slope of the curve here gives a value for the initial susceptibility of the material. As H increases the rate of increase of J with H, i.e. $\frac{dJ}{dH}$, becomes progressively greater, attains a maximum value and then diminishes steadily until $\frac{dJ}{dH} \rightarrow 0$. When this condition is reached the material is said to be saturated ; the intensity of magnetization is then denoted by J_s.

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* Cf. p. 36.

If corresponding values of B are plotted against H, cf. Fig. 11·03 (*b*), a ***curve of magnetic induction*** is obtained. In general, its shape is not markedly different from that of the corresponding J–H curve, the slope near O giving the initial permeability of the material. In one particular, however, the curves do differ ; since $B = \mu_0 H + 4\pi J$, $\left(\frac{\partial B}{\partial H}\right)_{J=\text{const.}} = \mu_0$, i.e. when the saturated condition is reached, B still goes on increasing, so that B never reaches a saturation value. This distinction is emphasized in Fig. 11·03 (*b*), where values of $4\pi J$ as well as of B are plotted against H. The quantity $4\pi J$, which is also $B - \mu_0 H$, is called the ***intrinsic induction*** of the ferromagnetic material. When $J \rightarrow J_s$, the intrinsic induction also reaches a constant value.

The self-demagnetizing field.—In the sequel it will be found that the material to be tested is generally in the form of a wire or strip and in ideal conditions the specimen should be infinitely long, since it is only under such conditions that the magnetizing field H may be put equal to $4\pi ni$, the field due to the current in the solenoid containing the specimen. For all specimens of finite length the true field is the vector sum of the applied field ($4\pi ni$) and the 'self-demagnetizing field' caused by the presence of free poles within the specimen. More accurately, this self-demagnetizing field should be considered as the resultant of the fields due to the electron currents orientated by the action of the applied field. For our

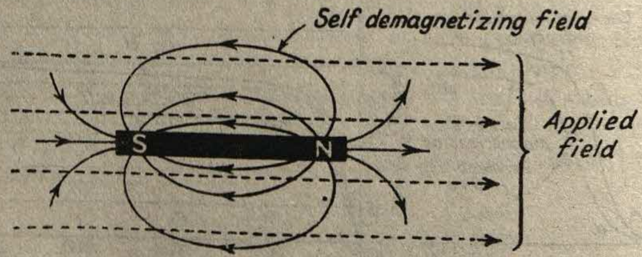

FIG. 11·04.—The self-demagnetizing field of a bar magnet.

present purpose the self-demagnetizing effect will be examined in the light of the older view-point only. Thus, suppose that N and S, Fig. 11·04, are the free poles in a material caused by its presence in an applied field H_a. The lines of force due to these poles are indicated and in the specimen the field due to the free poles opposes the field H_a, i.e. the specimen will not be magnetized so strongly by the applied field as it would be if the demagnetizing effect were zero.

It may be shown theoretically that if the specimen is in the form of an ellipsoid of revolution, the self-demagnetizing field is constant throughout the specimen and for such specimens the true field, H, is given by $\mathrm{H} = \mathrm{H}_a - \mathrm{DJ}$, where H_a is the applied field and D a constant, known as the ***demagnetizing factor.*** The value of D depends only on the dimensions of the specimen and not upon the permeability of its material; D can be calculated from the ratio of the lengths of the axes of the ellipsoid of revolution.

Maxwell showed that for an ellipsoid having semi-axes, a, b and b. where a is the major axis and $b = a\sqrt{1 - e^2}$,

$$\mathrm{D} = 4\pi\left(\frac{1}{e^2} - 1\right)\left[\frac{1}{2e}\ln\left(\frac{1+e}{1-e}\right) - 1\right].$$

Thus when $a = 500b$, $\mathrm{D} = 3 \times 10^{-4}$, so that when $\chi = 250$

$$\mathrm{H} \simeq 0{\cdot}92\mathrm{H}_a,$$

since $(1 + \mathrm{D}\chi)\mathrm{H} = \mathrm{H}_a$, i.e. the field is reduced by about 8 per cent.

Now although the ellipsoid is theoretically the most simple form of specimen to use, yet in practice it is difficult to make and therefore is only used in fundamental researches. In most work with ferromagnetic materials it is usual to find that the specimen is in the form of a narrow rod or strip. Under such circumstances the self-demagnetizing field is not a constant; it varies within the specimen and, even so, its value does not depend only on the external shape of the specimen—it depends upon the susceptibility of the material. To estimate the value of D in such cases is very troublesome so that

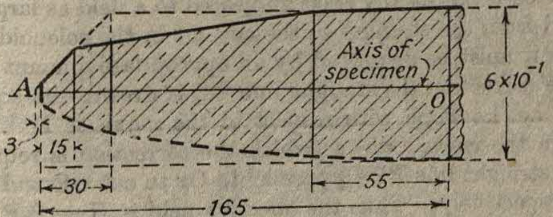

FIG. 11·05.—Ferromagnetic specimen with a known demagnetizing factor. [The upper half is a section showing the shape according to Maurer and Meissner; the lower half is that of an ellipsoid. All distances in mm. and only one half of the specimen is shown.]

in practice the length of the rod is made equal to several hundred times its diameter and D is then treated as a constant.

In 1922 **Maurier** and **Meissner** suggested that if the specimens were made up of a central cylindrical portion with each end portion consisting of two frustums of suitable size and shape, then for all practical purposes the specimen could be regarded as an ellipsoid. Fig. 11·05 shows the dimensions recommended for one half of the specimen ; the shape may be realized if one imagines that the diagram rotates about the axis OA. In the lower half of the diagram a section of a true ellipsoid is shown. It should be noted that all distances are expressed in mm., those at right angles to the axis of the specimen being magnified ten times.

Determination of a true curve of magnetization from one obtained experimentally.—The ferromagnetic specimen being in the form of a wire or rod, let *Oabc*, Fig. 11·06 (*a*), be the curve obtained by plotting values for J against corresponding values of the applied field due to the current in the solenoid. On account of the demagnetization due to the presence of free poles in the specimen, the specimen is not at any stage subjected to a field as large as that calculated from the strength of the current in the solenoid. As we have seen it must be decreased by an appropriate amount DJ. To make this correction the straight line OE is drawn—its equation is $H = DJ$, i.e. its slope with respect to the x-axis is D^{-1}. If P is a point on the curve *Oabc* at which the correction is required, we draw the straight line PP_1P_2 parallel to Ox to cut OE and Oy in P_1 and P_2 respectively. Thus the effective field is $H - P_2P_1$, so that by marking off along PP_2, a length $PP_0 = P_1P_2$, we obtain the point P_0 on the true curve of magnetization. Proceeding in this way the curve OA, which is the true curve of magnetization, is obtained.

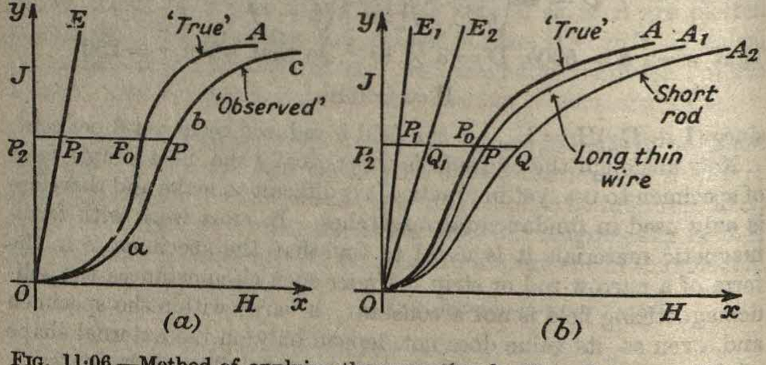

FIG. 11·06.—Method of applying the correction for the self-demagnetizing effect in rods.

At this stage it is interesting to examine the effect of the length of the specimen on the magnetization curve obtained experimentally. In Fig. 11·06 (*b*), let OA be the true J–H curve. If the specimen is long the demagnetization effect is small; let OE_1 be the line $H = D_1J$, where D_1 is the appropriate demagnetizing factor. OA_1 will be the observed curve in this instance, a point P on it corresponding to a point P_0 on OA being obtained by drawing $P_0P_1P_2$, as before, and making $P_0P = P_2P_1$. For a shorter specimen of the same material, the demagnetization factor D_2 will be larger; let OE_2 be the line $H = D_2J$. The point Q on the observed curve in this instance is obtained by making $P_0Q = P_2Q_1$ where Q_1 is the point of intersection of P_0P_2 with OE_2.

A magnetometer method for obtaining a J-H curve.—The magnetometer method for investigating the properties of ferromagnetic materials is due to Ewing; one of his arrangements is as follows. C_1, Fig. 11·07 (*a*), is a long narrow vertical solenoid in which the specimen, A, is arranged to hang vertically. M is a

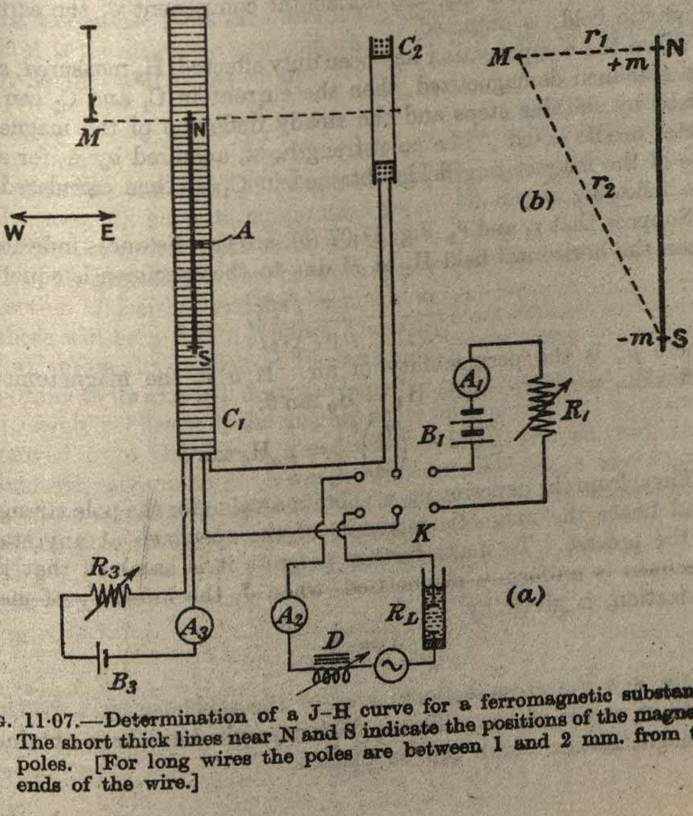

Fig. 11·07.—Determination of a J-H curve for a ferromagnetic substance. The short thick lines near N and S indicate the positions of the magnetic poles. [For long wires the poles are between 1 and 2 mm. from the ends of the wire.]

reflecting magnetometer situated to the east (or west) of the specimen. C_2 is a circular coil arranged in series with C_1 and the usual cells, etc., for supplying and adjusting the current in the coils. A_1 is a d.c. ammeter.

The axis of the coil C_2 passes through the centre of the magnetometer needle and, ideally, through the upper pole of the rod or wire, A. The key, K, permits the current in C_1 to be reversed and also the injection of an alternating current from a suitable supply into C_1. This current is controlled by a choke D and its magnitude indicated by the a.c. ammeter A_2. This is the demagnetizing circuit and the specimen A must be demagnetized before the experiment is begun. The magnitude of the alternating current is at first large and then is gradually reduced to zero by means of D and a liquid resistor R_L.

The coil C_2 serves a dual purpose; firstly, it acts as a compensating coil so that the magnetometer needle remains undeflected when a larger current than is used to magnetize A passes through both coils, the specimen being absent at this stage; secondly, by passing a known current through C_2 only and observing the magnetometer deflexion, a value for H_0, the horizontal component of the earth's magnetic field, is obtained.

When the compensation has been duly effected, H_0 measured, and the specimen demagnetized, then the current in C_1 and C_2 can be raised by suitable steps and the steady deflexion of the magnetometer needle noted. The pole strength, m, acquired by A, for any one of the magnetizing fields obtained in C_1, is then calculated in the following way.

Suppose that r_1 and r_2, Fig. 11·07 (*b*), are the distances indicated. Then the horizontal field H_1 at M due to the specimen is equal to

$$\frac{m}{\mu_a r_1^2} - \frac{m}{\mu_a r_2^2}\left(\frac{r_1}{r_2}\right),$$

where μ_a is the permeability of air. If θ is the magnetometer deflexion, we have, since $H_1 = H_0 \tan\theta$,

$$\frac{m}{r_1^2}\left[1 - \left(\frac{r_1}{r_2}\right)^3\right] = \mu_a H_0 \tan\theta.$$

Thus, from the experiment, we obtain a value for the pole strength (and hence the magnetic moment) of the specimen at any stage in the process. To make further progress it is assumed that the specimen is uniformly magnetized, when J, the intensity of magnetization, is given by

$$J = \frac{m}{\alpha},$$

where α is the cross-sectional area of the specimen.

When values of J at different field strengths have been obtained a magnetization curve can be plotted.

So far only the essential theory of the method has been given. Since the solenoid C_1 is vertical the specimen A will be present in the vertical magnetic field of the earth. To eliminate this field in the neighbourhood of the specimen the coil C_1 is wound with an additional layer of wire which is connected to a battery B_3 and an ammeter A_3, etc. The current in this outer layer is adjusted to a calculated value so that H_V is annulled. In making this calculation a value for H_V is assumed.

To avoid making this assumption the current from B_3 is switched on and alternating current passed through C_1. This alternating current is gradually reduced to zero and if the field due to the current from B_3 is such that H_V is annulled within C_1 then the magnetometer needle will be undeflected when the alternating current is made zero. The method is one of trial and error.

Finally, of course, a correction for the self-demagnetizing effect in the specimen must be made.

The maximum and mean values of the intensity of magnetization in the material of a bar magnet.—Let A be the constant cross-sectional area and L the geometric length of the magnet. If M is its magnetic moment, the mean intensity of magnetization in its material is given by

$$J_{\text{mean}} = \frac{M}{AL}.$$

Now Φ, the flux of magnetic induction, will have its maximum value across the central section of the magnet, cf. Fig. 11·08, for between this section and either end of the magnet there will be a fringing effect, i.e. the flux across any section of the magnet normal to its axis will decrease from the central section outwards. If a search coil lies over the magnet at its central section and is quickly moved from there to a great distance from the magnet, then a ballistic galvanometer to which the search coil is connected may be used to measure Φ and this is usually equated to $4\pi m$, where m is the pole strength of the magnet.

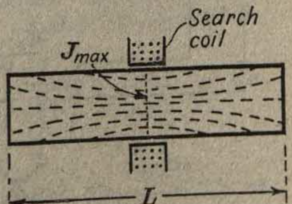

FIG. 11·08.—The maximum and mean values of the intensity of magnetization in a bar magnet.

$$\therefore\ 4\pi m = \Phi = B_{\max}A$$
$$= 4\pi J_{\max}A.$$
$$\therefore\ J_{\max} = \frac{m}{A} = \frac{M}{2lA},$$

where $2l$ is the distance between the poles.

$$\therefore\ \frac{J_{\max}}{J_{\text{mean}}} = \frac{L}{2l}.$$

The ballistic method for testing ferromagnetic materials.—In this method a specimen of the ferromagnetic material on test is placed in a known magnetic field and a value for the change in the magnetic induction B is derived from a knowledge of the quantity of electricity induced in a secondary coil surrounding the specimen when the magnetizing field is suddenly changed from one known value to another.

Early work by the ballistic method was largely carried out on specimens in the form of a small anchor ring but the practical disadvantage that each specimen had to be wound separately with both primary and secondary windings prevented the method from becoming one which could readily be used in industry. The main advantage of the ring method is that no free poles exist so that the self-demagnetizing effect is zero.

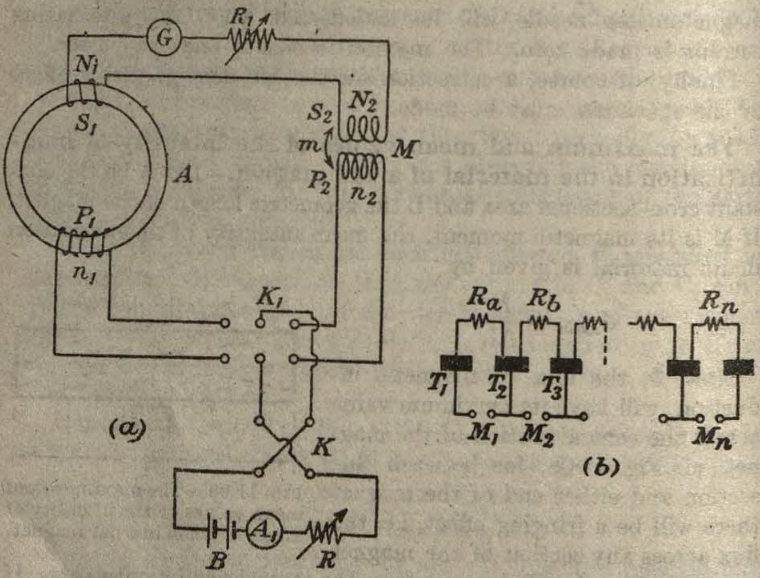

FIG. 11·09.—Ballistic method for the determination of a B–H curve.

Nowadays the method is applied to straight specimens, the secondary coil being wound near its centre. The self-demagnetizing effect is least at the mid-point of the specimen but, nevertheless, in precision work a correction must be applied. On account of its theoretical importance the ring-method will be described.

In Fig. 11·09 (*a*) A is the specimen of ferromagnetic material and it is in the form of an anchor ring. It is wound uniformly with a primary P_1 [for clarity in the diagram this winding is only represented symbolically—cf. p. 403, however]. S_1 is the secondary coil connected to a ballistic galvanometer G, an adjustable resistor R_1, and the secondary S_2, of a standard mutual inductance whose primary is P_2. P_1 and P_2 are connected to mercury cups at K_1, a reversing key K, a battery B, an ammeter A_1 and a variable resistor R. When the current through P_1 is changed, the change must be rapid so that R must be specially designed. Wire resistors R_a, R_b, . . . [Fig. 11·09 (*b*)] are connected to brass blocks, T_1, T_2, . . . and these are connected in turn to mercury cups M_1, M_2, . . . By placing thick copper wires across the cups various resistances may in effect be removed and the current through P_1 increased.

Before commencing the experiment the specimen must be demagnetized by means of an alternating current—in order to avoid complications on the diagram the means of injecting such a current into the circuit are not shown.

When the specimen has been demagnetized completely, the current through P_1 is increased by measured steps and at each change the throw of the galvanometer is noted. When the current in P_1 has reached a value i let the induction within the specimen have a value B. If A is the cross-sectional area of the specimen, the flux through each turn of S_1 will be AB, so that the effective flux through S_1 is N_1AB, where N_1 is the total number of turns in S_1, i.e. $\Psi_1 = N_1AB$.

When the current through P_1 is raised to a value $(i + \Delta i)$, let the number of linkages through S_1 increase by $\Delta\Psi_1$. Then

$$\Delta\Psi_1 = N_1A\Delta B.$$

If θ is the ballistic throw of the galvanometer, the quantity of electricity passing through G is $k\theta(1 + \frac{1}{2}\lambda)$, where k is a constant and λ the logarithmic decrement per half-cycle, so that

$$k\theta(1 + \tfrac{1}{2}\lambda) = \frac{\Delta\Psi_1}{r} = \frac{N_1A\Delta B}{r},$$

where r is the total resistance (e.m.u.) of the galvanometer circuit.

To find ΔB, the ballistic galvanometer must be standardized. Let the current i_s in P_2 be reversed rapidly. Then the flux change in S_2 is $2mi_s$, where m is the mutual inductance of M—it is expressed in e.m.u. If this gives a ballistic throw θ_s, we have

$$\frac{2mi_s}{r} = k\theta_s(1 + \tfrac{1}{2}\lambda).$$

$$\therefore\ \Delta B = \frac{\theta}{\theta_s}\left(\frac{2mi_s}{N_1A}\right).$$

Thus ΔB can be found, and it should be noted that neither the value of the logarithmic decrement nor the resistance of the galvanometer circuit has to be known. The value of B is given by

$$B = \frac{1}{\theta_s}\left(\frac{2mi_s}{N_1A}\right)\Sigma\,(\theta),$$

since B is proportional to the sum of all the throws obtained as the current in P_1 is increased in steps from zero to i.

The applied field, H, is $4\pi n_1 i$, so that a curve of magnetic induction is obtained by this method.

Ferromagnetic materials and hysteresis.—Substances capable of becoming magnetized to a high degree are said to be ferromagnetic, as distinct from all other substances which exhibit exceptionally feeble magnetic properties. Only a few elements such as iron, nickel and cobalt and their alloys are, in general, ferromagnetic. Magnetite (Fe_3O_4) and iron pyrites (FeS_2) are naturally occurring crystals with distinct ferromagnetic properties. Finally we have the so-called Heusler alloys ; these were discovered about 1898 by HEUSLER. These ternary alloys are ferromagnetic and yet each is composed of elements which are not in this class of substance. Heusler discovered that alloys of copper and manganese become ferromagnetic when one of the metals aluminium, antimony, arsenic, bismuth or tin is added in certain proportions. The addition of boron produces a similar effect. The copper-aluminium-manganese alloys have been investigated most fully and in every respect they behave as ferromagnetics. When these alloys are quenched, i.e. quickly cooled in water or a thin oil, from high temperatures, they do not become ferromagnetic. Recently manganese arsenide, MnAs, and silver difluoride, AgF_2, have been shown to be ferromagnetic materials. Their magnetic transformation points, cf. p. 436, are 45° C. and − 110° C. respectively.

The real distinguishing characteristic of a ferromagnetic material is not that it can be strongly magnetized but that the intensity of magnetization is not directly proportional to the magnetizing field; it varies in a most remarkable manner and tends to a limiting value. For all other substances, as we shall see later [cf. p. 436], the relation is strictly linear and there appears to be no limiting value for the intensity of magnetization. In addition, where ferromagnetic materials are concerned, the value of the magnetization is not a definite function of the applied field ; it is found to depend upon the previous magnetic, thermal and mechanical treatment of the specimen. In other words as far as the previous magnetic treatment is concerned, ferromagnetic materials appear 'to have a memory', i.e. they exhibit a phenomenon known as ***hysteresis*** [Greek, ὑστέρησις, a coming late].

In Fig. 11·10 the curve OaA is a typical J–H curve or magnetization curve for a ferromagnetic substance. When the point A corresponding to an applied field H_m has been reached let the applied field be decreased in stages to zero so that points on the portion AR of the complete curve may be obtained. During this stage in the cycle of operations it is proposed to carry out, the value of J lags behind that of H and although at R the applied field is zero, the material is still strongly magnetized.

If now the field is reversed and increased in numerical value the value of J decreases and becomes zero when the magnetizing field has a value represented by the point C on the diagram. If the negative field is increased until the field reaches a value $-H_m$, the portion CP of the complete curve will be obtained. The portions PS, SD, and DA of the complete loop may then be obtained and unless the specimen is demagnetized again completely the portion O*a*A of the cycle cannot again be obtained. The curve ARCPSDA comprises a ***hysteresis loop.***

Such a loop is always traced out when a ferromagnetic material is subjected to a magnetic field which is varied cyclically between limits $+H_m$ and $-H_m$. These loops are of great technical importance since they indicate the behaviour of ferromagnetic materials in relatively weak magnetic fields.

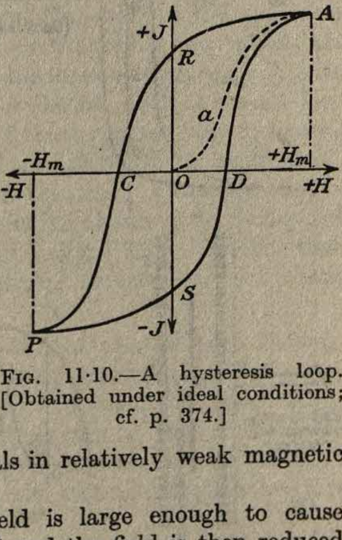

FIG. 11·10.—A hysteresis loop. [Obtained under ideal conditions; cf. p. 374.]

When the maximum applied field is large enough to cause magnetic saturation in the material and the field is then reduced to zero, the residual magnetization, represented by OR on the diagram, is known as the ***remanent magnetization*** or ***retentivity***, J_r, while if C is the point $(-H_C, 0)$, then H_C is known as the ***coercivity*** (coercive field) of the ferromagnetic material to which the loop refers. We may say that the retentivity is a measure of the ability of the substance to retain its state of magnetization when not subjected to adverse treatment, while the coercivity measures the ability to retain that state in spite of adverse treatment.

Sometimes a hysteresis loop is obtained with values of B plotted against H; the two curves are very similar. The intercept on the B-axis is called the ***remanent induction***, B_r. It must be noted however that the numerical value of the applied field to make $B = 0$ is not the coercivity, for $B = \mu_0 H + 4\pi J$, and if $B = 0$, $\mu_0[H]_{B=0} = [-4\pi J]_{B=0}$, so that if $H_{B=0}$ is negative, then the corresponding value for $J_{B=0}$ is positive.

Ferromagnetics are also characterized by a so-called ***magnetic transformation point,**** i.e. a temperature above which they become paramagnetic. It has been shown that at this temperature there is a marked variation in the specific heat of the material.

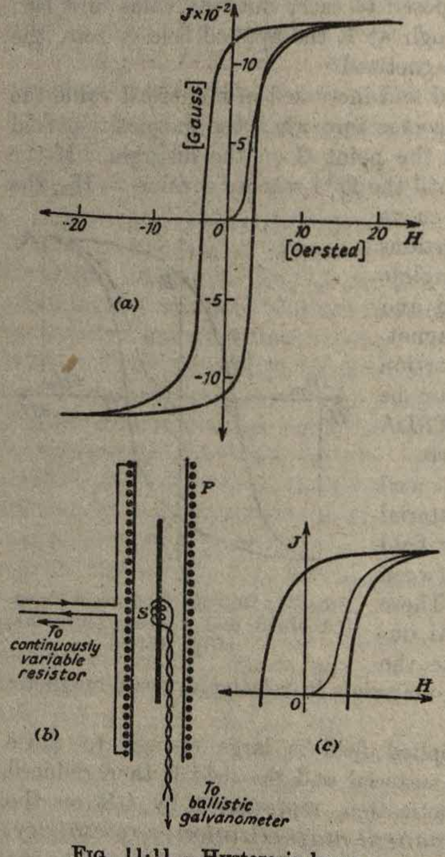

FIG. 11·11.—Hysteresis loops.

Hysteresis loops in practice.—The hysteresis loop shown in Fig. 11·11 (*a*) is drawn from data obtained by EWING about 1893. The specimen was an annealed soft iron wire and the curve was obtained by the magnetometer method. Such a curve is similar to those often obtained today. When it is compared with that shown in Fig. 11·10 it is noted that the initial or virgin curve intersects the outer or main part of the loop. In recent years SNOEK has shown that this departure from the ideal loop of Fig. 11·10 is due to the fact that the magnetizing current is varied too rapidly.

If H_a is the applied magnetic field, D the demagnetizing factor, and J the intensity of magnetization within the specimen, then H, the effective field to which the specimen is subjected, is given by

$$H = H_a - DJ \qquad \text{[cf. p. 365].}$$

Hence

$$\Delta H = \Delta H_a - D\Delta J.$$

Thus when a very rapid change in H_a is made, for a very short time H may differ considerably from its final value due to the fact that DΔJ does not immediately attain its final value. In addition, when H_a is changed, the associated change in J is accompanied by eddy currents and these temporarily shield the interior of the specimen from the applied field. Since for a ring specimen D is zero, such a specimen does not exhibit this effect.

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* Cf. p. 436.

In order to obtain the ideal hysteresis loop when the specimen is in the form of a rod, Snoek adopted the following procedure. The sample is demagnetized and then the magnetizing current is brought very slowly to the required value. This having been done, the sample is withdrawn from the magnetizing solenoid and the change in effective flux associated with a suitably placed solenoid is determined in the usual way. To minimize the effect of the air-gap between the secondary coil and the specimen, the secondary is placed within the primary and just fits over the rod as shown in Fig. 11·11 (*b*). For each point on the curve, the whole above process must be repeated. When the measured curve, i.e. the curve obtained by plotting J against H is sheared back to correct for the demagnetizing effect, the curve obtained is practically independent of the length of the specimen and, in addition, it is found that the virgin curve lies entirely within the main hysteresis loop—cf. Fig. 11·11 (*c*). The fact that the curve obtained by Ewing differs so fundamentally from that shown in Fig. 11·11 (*c*) proves that in Ewing's experiments the magnetizing current was not changed sufficiently slowly.

The energy loss per cycle.—It has already been shown, cf. p. 237, that the energy stored in the surrounding medium when a current has been established in an inductive coil is $\frac{1}{2}li_0^2$ ergs, or $\frac{1}{2}\mathrm{LI}_0^2$ joules. This represents the energy of the magnetic field. The result is true as long as the self inductance is constant, i.e. the flux due to the self induction of the coil is proportional to the current throughout the range considered.

When ferromagnetic materials are present and it is desired to calculate the energy of the magnetic field it is necessary to know the relation between B and H. Two ways of dealing with the problem are as follows:—

(*a*) Suppose that the core of a uniformly wound toroidal solenoid, whose radius of cross-section is not too large compared with its mean radius, is a ferromagnetic substance. If E is the e.m.f. applied to the whole circuit of which the solenoid is part, e the back e.m.f. due to the changing flux in the core of the solenoid, R the resistance of the circuit and i the instantaneous current,

$$\mathrm{E} - e = \mathrm{R}i.$$

$$\therefore\ \mathrm{E}i\,\delta t - ei\,\delta t = \mathrm{R}i^2\,\delta t.$$

Now $\mathrm{E}i\,\delta t$ represents the energy supplied by the battery in time δt, while $i^2\mathrm{R}\,\delta t$ is the energy dissipated as heat on account of the Joule effect. The term $ei\,\delta t$ must therefore be the energy stored in the magnetic field in time δt. [The energy loss due to eddy currents is neglected.]

If the solenoid has N turns and A is the constant cross-section of the core, the number of linkages is given by $\Psi = \mathrm{NAB}$, so that

$$|e| = \dot{\Psi} = \mathrm{NA}\dot{\mathrm{B}}.$$

Hence the energy of the magnetic field increases in time δt, i.e. the time in which the current increases from i to $i + \delta i$, by an amount

$$\begin{aligned} \delta \mathrm{W} = |e|\, i\, \delta t &= \mathrm{NA}i\dot{\mathrm{B}}\, \delta t \\ &= \mathrm{NA}i\, \delta \mathrm{B} \quad . \quad . \quad . \quad \text{(i)} \end{aligned}$$

If λ is the mean length of the solenoid the applied field is given by

$$\mathrm{H} = \frac{4\pi \mathrm{N}i}{\lambda} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Eliminating i from these equations, we have

$$\delta \mathrm{W} = \frac{1}{4\pi}\lambda \mathrm{AH}\, \delta \mathrm{B} \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Since $\lambda \mathrm{A}$ is the volume of the specimen and this is assumed constant, the work done on unit volume of the specimen during the above change in its state of magnetization is $\frac{1}{4\pi}\mathrm{H}\, \delta \mathrm{B}$.

Now let us consider $\frac{1}{4\pi}\oint \mathrm{H}\, d\mathrm{B}$ with reference to the loop shown in Fig. 11·12. In passing from a state represented by the point M to that represented by N, the work done per unit volume in changing the value of the magnetic induction within the material is $\frac{1}{4\pi}\mathrm{H}\,.\,\delta \mathrm{B}$, where $\mathrm{H} = m\mathrm{M}$ and $mn = \delta \mathrm{B}$, m and n being the projections of M and N on the vertical axis through O. Hence from A to P the work done per unit volume is

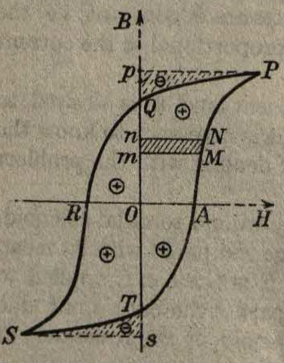

FIG. 11·12.—The irrecoverable energy loss associated with a hysteresis loop.

$$\frac{1}{4\pi} \times \text{area OAP}p,$$

where p is the projection of P on the above vertical axis.

From P to Q the work done is negative for H is here positive while $\delta \mathrm{B}$ is negative, i.e. the area PpQ must be considered as a negative one: during this stage the material gives up energy. Hence from

A to Q via P the work done per unit volume is represented by $\frac{1}{4\pi} \times$ area OAPQ; this energy is irrecoverable.

Continuing this process all round the cycle and observing, for example, that for elements in the area OQR both H and δB are negative so that the work done is positive, we find that the total irrecoverable energy per unit volume is

$$\frac{1}{4\pi} \times [\text{Area of the B–H loop}] = W_h \text{ (say)},$$

i.e.
$$W_h = \frac{1}{4\pi}\oint H\,dB.$$

This irrecoverable energy appears as heat in the specimen and is termed the ***hysteresis loss*** of the material. Hysteresis losses are small for those materials which have very narrow hysteresis loops, e.g. soft iron.

Since
$$B = \mu_0 H + 4\pi J,$$
$$\delta B = \mu_0 \delta H + 4\pi\,\delta J.$$
$$\therefore \quad W_h = \frac{1}{4\pi}\oint H\,dB = \frac{1}{4\pi}\left[\oint \mu_0 H\,dH + 4\pi \oint H\,dJ\right].$$

Now $\oint H\,dH = \left[\frac{1}{2}H^2\right]_a^a = 0.$

Hence $\quad W_h = \oint H\,dJ.$

This is sometimes known as ***Warburg's law*** (1881). The worked example on p. 379 gives an idea of the rise in temperature per cycle, or rather per second, since, in the example, the cycles are continuous.

(*b*) Let M be the constant magnetic moment of one of the molecular magnets which comprise the material and let ϕ be the complement of the angle its axis makes with the applied field, H. We select this angle since when the field is H + δH, the angle will be $\phi + \delta\phi$ and $\delta\phi$ will be positive if δH is positive, cf. Fig. 11·13. The components of M parallel and perpendicular to H are M sin ϕ and M cos ϕ, respectively.

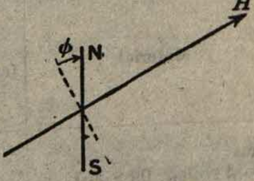

FIG. 11·13.—Couple on a small magnet.

For all the magnetic particles in unit volume

$$\Sigma M \sin\phi = J, \text{ and } \Sigma M \cos\phi = 0,$$

for otherwise there would be a component of the magnetic moment at right angles to the direction of magnetization; such would be contrary to the meaning of the term. Hence

$$\begin{aligned}\delta J &= \delta(\Sigma\, M \sin\phi)\\ &= \Sigma(M\cos\phi\,\delta\phi).\end{aligned}$$

Now the couple acting on a magnetic molecule is $MH\cos\phi$ and for a small rotation $\delta\phi$ the work done by the agent effecting the rotation is $MH\cos\phi\,\delta\phi$. For all the magnetic particles in unit volume the work done is

$$\Sigma(MH\cos\phi\,\delta\phi) = H\,\Sigma\,(M\cos\phi\,\delta\phi) = H\,\delta J.$$

For a complete cycle of magnetization the work done per unit volume is $\oint H\,dJ$. Since $B = \mu_0 H + 4\pi J$, $\delta B = \mu_0\delta H + 4\pi\,\delta J$.

$$\begin{aligned}\therefore\ W_h = \oint H\,dJ &= \frac{1}{4\pi}\oint H\,dB - \frac{1}{4\pi}\mu_0\oint H\,dH\\ &= \frac{1}{4\pi}\oint H\,dB,\end{aligned}$$

since $\oint H\,dH$ is zero. Thus the work expended per cycle per unit volume is $\frac{1}{4\pi}$ times the area of the B–H loop.

Steinmetz formula.—An empirical formula to enable values to be calculated for W_h, the energy loss in erg.cm.$^{-3}$cycle.$^{-1}$, was given by STEINMETZ in 1916. It states that

$$W_h = \eta(B_m)^{1\cdot6},$$

where (B_m) is the maximum numerical value of the induction and η is

HYSTERESIS LOSSES [ERG.CM.$^{-3}$CYCLE.$^{-1}$] IN DIFFERENT MATERIALS AT SPECIFIED VALUES OF THE MAXIMUM INDUCTION [$W_h = \eta B_{\text{max.}}^{1\cdot6}$]

Material	$\eta \times 10^6$ [erg.cm.$^{-3}$cycle.$^{-1}$]	$B_{\text{max.}}$ [gauss]	W_h [Energy loss in erg.cm.$^{-3}$cycle.$^{-1}$ for the given value of $B_{\text{max.}}$]
Iron [Cold rolled, 99·9 Fe]	4000	10^4	10^4
Silicon iron [Cold rolled, 3 Si, 97 Fe]	120	10^4	300
Silicon iron [4 Si, 96 Fe]	500	10^4	1300
Mumetal [76 Ni, 3 Cu, 2 Cr, 19 Fe]	60	5000	50
Alnico V [24 Co, 14 Ni, 8 Al, 3 Cu, 51 Fe]	6×10^5	$1\cdot4 \times 10^4$	$2\cdot6 \times 10^6$
Cobalt	8300	5000	6900
Nickel	1500	5000	1240

a constant, the so-called ***Steinmetz coefficient***, for any particular material in a given physical state. The table opposite indicates how W_h varies with the particular ferromagnetic material selected.

Example.—A particular ferromagnetic material (stressed permalloy) has a hysteresis loop which may be taken as a rectangle between the limits ± 2 oersted for the magnetizing field and ± 1000 gauss for the magnetic induction, together with two limbs of zero width and almost parallel to the H-axis. Calculate the rates of rise in temperature due to hysteresis losses when a specimen of this material is placed in a 50 cycle.sec.$^{-1}$ alternating magnetic field $\hat{H}\sin\omega t$ in which $\hat{H}$ is in turn 1, 3 and 20 oersteds respectively. [Assume the specific heat of the material to be 0·42 joule.gm.$^{-1}$ deg.$^{-1}$ C. and its density 8·0 gm.cm.$^{-3}$.]

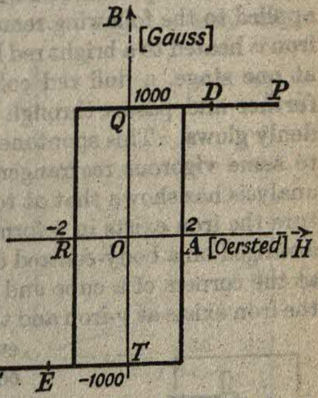

FIG. 11·14.—The hysteresis loop for stressed permalloy.

The hysteresis loop is shown in Fig. 11·14.

(*a*) When $\hat{H} = 1$ oersted, the extreme values of the field to which the specimen is subjected will be ± 1 oersted, i.e. the specimen will never be demagnetized and the 'operating point' on the B–H diagram will move along a line parallel to the H-axis. Since the irrecoverable heat loss due to hysteresis per unit volume per cycle is $\frac{1}{4\pi}\oint H\,dB$, when $\hat{H} = 1$ oersted, this loss is zero.

(*b*) When $\hat{H} = 3$ oersteds, the field varies between +3 and −3 oersteds, i.e. the hysteresis loop is from D to E on the diagram; its area is that of the rectangle. Since unit volume has a mass of 8·0 gm., we have, if $\frac{d\theta}{dt}$ is the rate at which the temperature rises,

$$8 \times 0{\cdot}42 \times 10^7 \times \frac{d\theta}{dt} = \left[\frac{1}{4\pi} \times (4 \times 2000)\right] \times 50,$$

i.e.
$$\frac{d\theta}{dt} = 9{\cdot}5 \times 10^{-4} \text{ deg. C. sec.}^{-1}.$$

(*c*) When the field lies between +20 and −20 oersteds, the area of the loop remains as in (*b*), i.e. the rate of rise in temperature is unaltered.

The effect of temperature on the magnetization of ferromagnetics.—The magnetic properties of all ferromagnetic materials show very marked changes with temperature. When a piece of iron is heated above 760° C. it ceases to be ferromagnetic, but when the temperature falls below this temperature, the ferromagnetic properties are regained. The temperature at which the change takes place is termed the ***magnetic transformation point.*** For low carbon steels this temperature is about 730° C., while for high carbon steels it is about 690° C.

The sudden alteration in the magnetic properties of iron when passing its magnetic transformation point is also marked by other physical changes in the material. Tait showed that the thermoelectric power is one property to show such a change ; the resistivity of iron also shows a very marked change at this temperature, which is also the temperature at which ***recalescence*** occurs ; this term is applied to the following remarkable phenomenon. When a piece of iron is heated to a bright red heat and then allowed to cool it assumes, at one stage, a dull red colour. As the temperature falls a little further and passes through the critical temperature the iron suddenly glows. This spontaneous heating effect is due most probably to some vigorous rearrangement of the atoms in the iron. X-ray analysis has shown that at temperatures below the critical temperature the iron exists in a form known as α-iron ; in it the atoms are arranged on a body-centred cubic lattice, i.e. the atoms are arranged at the corners of a cube and at its centre. At temperatures above, the iron exists as γ-iron and the atoms are arranged on a face-centred cubic lattice, i.e. atoms appear at the corners of the cubes and also at the centres of the faces of the cubes. The rearrangement of the atoms which takes place quite suddenly at the recalescent temperature is accompanied by a change in the length of the specimen which may be demonstrated as follows :—AB, Fig. 11·15, is an iron wire surrounded by a glass tube to protect it from air currents. It is supported from an insulated clamp and its lower end is attached to a pointer moving about a pivot C. A mass of lead, D, serves to keep the wire stretched. When a current is sent through the wire it is heated and its expansion is indicated by the downward motion of the pointer. When the recalescence temperature is reached there is a sudden contraction in length and the pointer moves upwards. As the temperature is increased beyond this point the length again increases. Opposite effects are noticed when the wire is allowed to cool. In addition to these opposite effects, when the recalescence point is attained—as revealed by the jerk in the motion of the pointer—the wire glows brightly for a few seconds. This is due

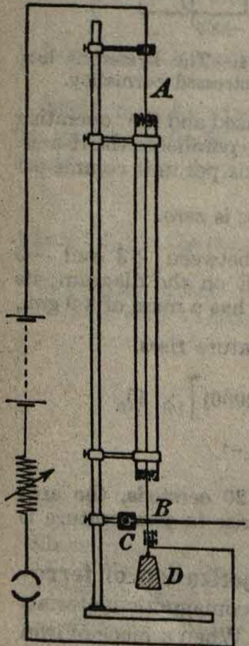

Fig. 11·15.—Experiment on recalescence.

to the large amount of energy liberated when the iron changes from γ-iron to α-iron.

In Fig. 11·16 (*a*) the magnetic induction curves for 'best quality transformer' iron at different temperatures are shown, and in Fig. 11·16 (*b*) the variation of the permeability of iron with

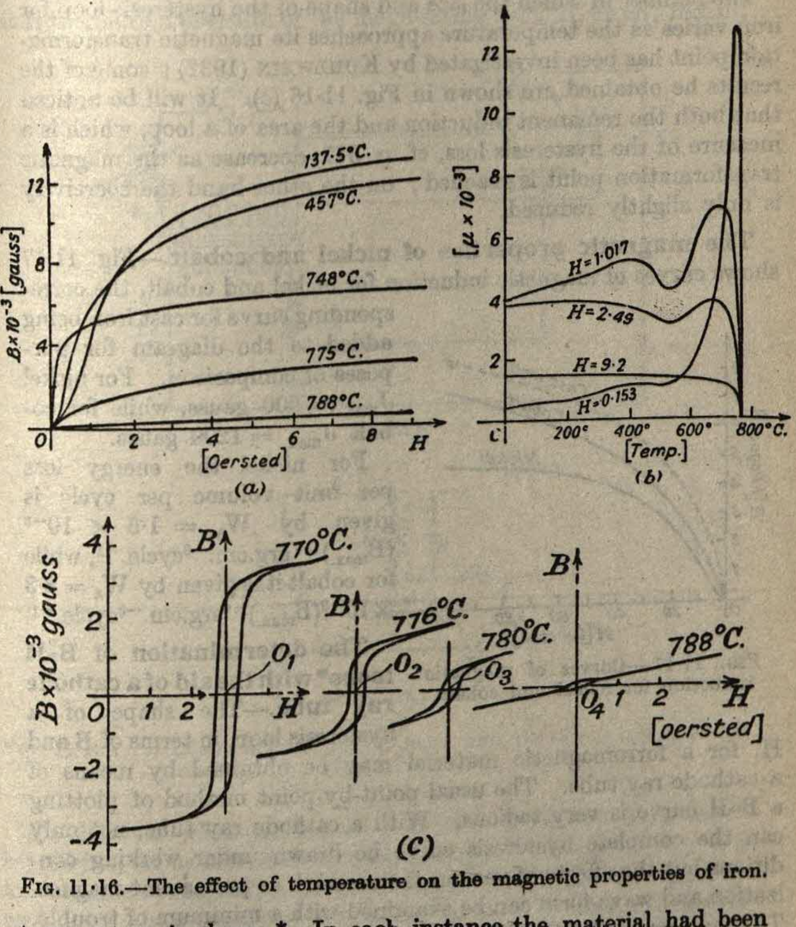

FIG. 11·16.—The effect of temperature on the magnetic properties of iron.

temperature is shown.* In each instance the material had been annealed at 1150° C. In the latter diagram each curve is drawn for a constant value (as shown) of the applied field.

The curves indicate that for small values of the magnetizing field, the permeability increases somewhat slowly with rise of temperature until the magnetic transformation point is approached when the rate of increase of permeability with temperature becomes very high and then diminishes at an even greater rate until at about 780° C. the value of the permeability cannot easily be distinguished from unity.

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* Morris, *Phil. Mag.*, **44**, 213, 1897.

In stronger fields the permeability shows very little change with temperature until at about 650° C. it begins to decrease to become unity at about 780° C.

The manner in which the size and shape of the hysteresis loop for iron varies as the temperature approaches its magnetic transformation point has been investigated by KUHLWEIN (1932); some of the results he obtained are shown in Fig. 11·16 (*c*). It will be noticed that both the remanent induction and the area of a loop, which is a measure of the hysteresis loss, cf. p. 377, decrease as the magnetic transformation point is reached; on the other hand the coercivity is only slightly reduced.

The magnetic properties of nickel and cobalt.—Fig. 11·17 shows curves of magnetic induction for nickel and cobalt, the corresponding curve for cast iron being added to the diagram for purposes of comparison. For nickel $J_{\text{max.}} \doteqdot 500$ gauss, while for cobalt $J_{\text{max.}} \doteqdot 1200$ gauss.

For nickel the energy loss per unit volume per cycle is given by $W_h = 1{\cdot}5 \times 10^{-3} (B_{\text{max.}})^{1{\cdot}6}$ erg.cm.$^{-3}$cycle.$^{-1}$, while for cobalt it is given by $W_h = 8{\cdot}3 \times 10^{-3}(B_{\text{max.}})^{1{\cdot}6}$erg.cm.$^{-3}$cycle.$^{-1}$.

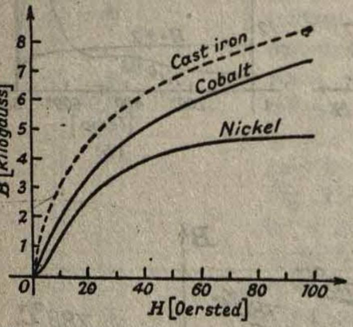

FIG. 11·17.—Curves of magnetic induction for nickel and cobalt.

The determination of B-H loops* with the aid of a cathode ray tube.—The shape of a hysteresis loop, in terms of B and H, for a ferromagnetic material may be obtained by means of a cathode ray tube. The usual point-by-point method of plotting a B–H curve is very tedious. With a cathode ray tube, not only can the complete hysteresis curve be drawn under working conditions but the effects of various factors such as permanent magnetization and wave-form can be examined with a minimum of trouble. The material under test has primary and secondary windings which are shown in Fig. 11·18 (*a*) as P and S respectively.

R_1 is a non-inductive resistor in series with the primary and the current is controlled by means of a choke. The secondary is in series with a non-inductive resistor R_2 and a capacitor C. R_2 is made high to avoid appreciable reaction between primary and secondary circuits, and also that the R_2–C circuit may serve as an integrating circuit. The voltages developed across R_1 and C are applied respectively to the X and Y plates of the cathode ray tube. Now the voltage across R_1 is in phase with the primary current; it is therefore a measure of the magnetizing field, H, at any instant.

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* The type of loop obtained under a.c. conditions is generally known as a ***total loss curve*** because it includes eddy-current losses as well as hysteresis losses.

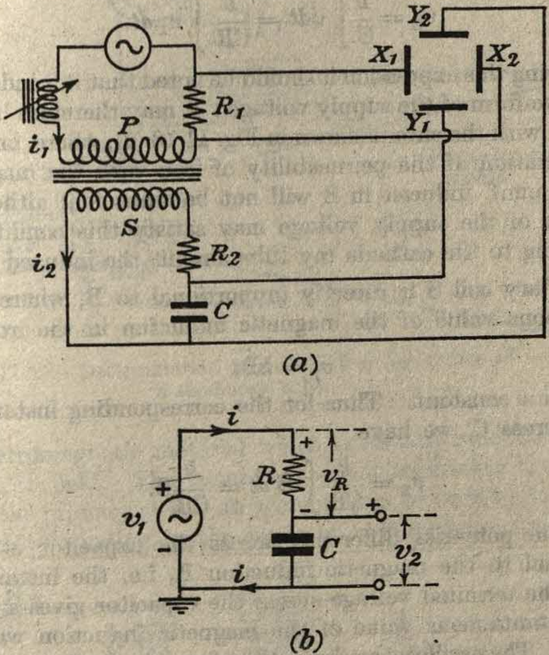

FIG. 11·18.—(a) Determination of B–H loops with the aid of a cathode-ray oscillograph; (b) an integrating circuit.

To discover what the fall in potential across the capacitor C represents, let us consider the circuit shown in Fig. 11·18 (b); it is known as an ***integrating circuit*** and its action may be explained as follows; in proceeding to do this some of the results obtained in Chapters XV and XVI will be anticipated. Consider therefore the instantaneous values of the supply voltage v_1', that across the resistor R, viz. v_R, and finally that across the capacitor C, viz. v_2. If the resistance R is very large compared with the reactance of the condenser, then the instantaneous current will be given by

$$i = \frac{v_1}{R},$$

since $v_R \simeq v_1$. [For example, if $R = 0{\cdot}5\ M\Omega$, $C = 2\mu F$ and $f = 50$ cycle.sec.$^{-1}$, i.e. $\omega = 2\pi f = 100\pi$ radian.sec.$^{-1}$, then the reactance of the capacitor is

$$\frac{1}{\omega C} = \frac{10^6}{100\pi \times 2} \simeq 1600\ \Omega,$$

and this is negligible compared with R.]

Then

$$v_2 = \frac{1}{C}\int i\,dt = \frac{1}{CR}\int v_1\,dt.$$

In deriving this expression it should be noted that it is independent of the wave-form of the supply voltage ; it may therefore be used in connexion with the circuit shown in Fig. 11·18 (*a*), where, on account of the variation of the permeability of iron with the magnetizing field, the e.m.f. induced in S will not be sinusoidal although the wave-form of the supply voltage may satisfy this condition.

Returning to the cathode ray tube circuit, the induced e.m.f. in the secondary coil S is directly proportional to $\dot{B}$, where B is an instantaneous value of the magnetic induction in the iron, i.e.

$$v_1 = k\dot{B},$$

where *k* is a constant. Thus for the corresponding instantaneous voltage across C, we have

$$v_2 = \frac{k}{CR}\int \dot{B}\,dt = \frac{k}{CR}B.$$

Thus the potential difference across the capacitor is directly proportional to the magnetic induction B, i.e. the instantaneous value of the terminal voltage across the capacitor gives a measure of the instantaneous value of the magnetic induction within the iron core. The oscilloscope figure therefore has the form of a closed B–H loop. In order to obtain a loop in the two usual quadrants it may be necessary to reverse the connexions to one set of plates. Occasionally, when large currents must flow in the primary a capacitor is introduced into the primary and the circuit tuned to resonance with the supply.

Since the voltages across R_1 and C are rather small, amplifiers are used to raise their values before they are applied to the deflector plates of the oscillograph.

The circuit shown in Fig. 11·18 is considered and discussed in order that the principles involved may be understood. In practice it is more usual to dispense with R_1 and the X-plates and insert in the primary two coils, C_1 and C_2, Fig. 11·19, arranged symmetrically with respect to the tube. The magnetic field due to the current in these coils is parallel to the electric field due to the Y-plates, so that the component deflexions of the electron beam are mutually perpendicular.

It is sometimes necessary to examine by means of a cathode ray oscillograph the effect of superimposing an alternating magnetic field on a ferromagnetic material which is already subjected to a steady magnetic field. This latter field cannot be obtained by introducing into the primary circuit an accumulator, or even by connecting it to a third winding on the specimen. The reason for this is that the additional circuit thus formed would act as a closed secondary and considerably alter the characteristics of the circuit.

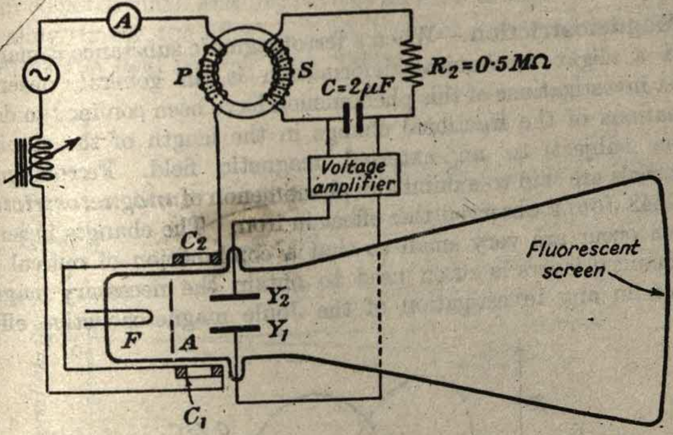

FIG. 11·19.—Determination of a B–H loop by means of a cathode-ray tube.

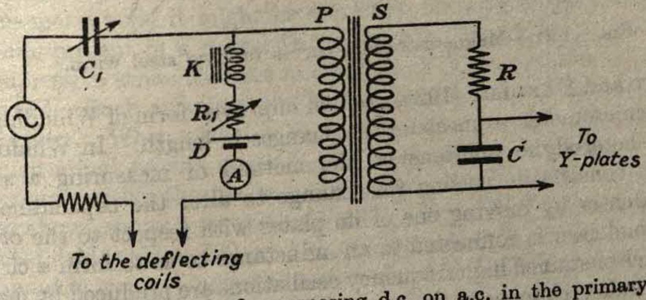

FIG. 11·20.—Method of superposing d.c. on a.c. in the primary of a coil on test.

In order to supply d.c. to the primary a battery of accumulators, D, Fig. 11·20, in series with an ammeter, A, a variable rheostat, R_1, and a high reactance choke, K, is placed across P. This allows direct current to flow but the reactance is so large that no alternating current flows in D ; the d.c. is however effectively superimposed on the a.c. in P. The condenser C_1 isolates D from the alternating current supply and, at the same time, it may be used, for reasons already given, to tune the primary circuit to resonance with the supply.

Magnetostriction.—When a ferromagnetic substance is magnetized a slight mechanical deformation is, in general, observed. Most investigations of this phenomenon have been confined to determinations of the fractional change in the length of the specimen when subject to an external magnetic field. Ferromagnetic materials are said to exhibit the phenomenon of ***magnetostriction.*** In 1842 JOULE observed this effect in iron. The changes in length which occur are very small so that a combination of optical and mechanical levers is often used to obtain the necessary magnification in any investigation of the Joule magnetostrictive effect.

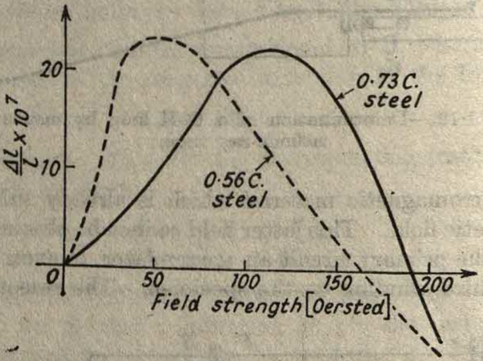

FIG. 11·21.—Magnetostriction curves for two steel wires.

LLOYD and HANDLEY (1929) used an improved form of Whiddington ultra-micrometer to measure the changes in length. In Whiddington's heterodyne micrometer, the method of measuring a small length consists in causing this change to alter the capacitance of a condenser by moving one of its plates with respect to the other. The condenser is connected to an inductance so as to form a circuit in which sustained high frequency oscillations are produced by means of a triode ; changes in the capacitance then produce corresponding changes in the frequency of the oscillations. The frequency changes are detected and measured with great sensitivity by means of a heterodyne method. The results obtained for two carbon steels are indicated in Fig. 11·21.

The curves in Fig. 11·22 show the magnitude and nature of the effect in several ferromagnetic materials. With increasing field strengths a rod of nickel decreases in length but rapidly assumes a steady state in which $\Delta l \rightarrow 0$. For iron Δl is at first positive but reaches a maximum when $H \fallingdotseq 30$ oersted. and then decreases until when H exceeds a value of 150 oersted. Δl becomes negative. Iron is said to show a Joule reversal effect. The magnetostrictive effect in cobalt is small and negative at all field strengths. For the Ni-Fe alloy shown, the effect is a positive one but in all instances $\Delta l \rightarrow 0$ as the field reaches high values.

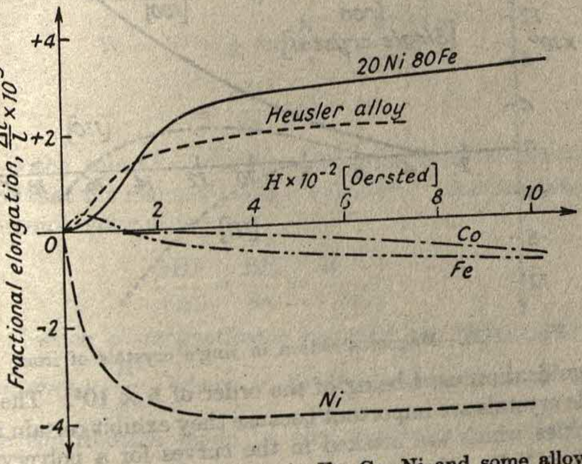

FIG. 11·22.—Magnetostriction in Fe, Co, Ni and some alloys.

Since magnetization is accompanied by a change in the length of a ferromagnetic body it might be expected, conversely, that if the body while present in a magnetic field were subjected to a tensile or a compressive stress changes in its state of magnetization would be observed. Such an effect has been observed; it is termed the ***Villari effect***. When nickel is in a magnetic field and subjected to a longitudinal tensile stress it is found that the intensity of magnetization within the specimen decreases, and vice versa. Iron, as stated above, shows a reversal of the magnetostrictive effect when the applied field is about 150 oersted.; a reversal of the Villari effect is also found in iron. Under the action of a tensile stress the degree of magnetization in iron increases as long as the applied field does not exceed the reversal field strength; in stronger fields the degree of magnetization decreases as the stress increases.

The causes to which these effects must ultimately be attributed are somewhat obscure and it is only since 1930 that a formal theory has been able to elucidate the more important facts as revealed by experiment.

Magnetostriction effects in single crystals of iron.—The Joule magnetostrictive effect in single crystals of iron is found to depend upon the direction along which the magnetic field is applied and the effect measured. WEBSTER and others have investigated the problem and some of the results obtained are shown in Fig. 11·23, the magnification used being of the order of 5×10^4. The curves for single crystals are important because they exhibit certain marked peculiarities which are masked in the curves for a polycrystalline material. [N.B. Numbers in square brackets define directions within the crystal, cf. p. 452.]

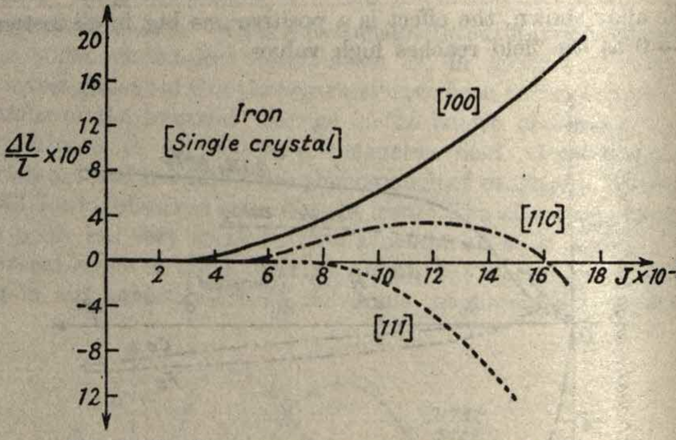

FIG. 11·23.—Magnetostriction in single crystals of iron.

Magnetic energy.—The work required to establish the magnetic field due to a current i in a coil of self inductance l has been shown [cf. p. 237] to be $\frac{1}{2}li^2$. When the current is steady this energy is considered to be stored in the medium ; when the current decreases the energy lost by the medium is dissipated as electromagnetic radiation, due to changes in the magnetic induction at any point, and possibly as heat, due to eddy currents in neighbouring conductors. Let us therefore use this expression, viz. $\frac{1}{2}li^2$, for the energy stored in the medium to calculate the energy per unit volume in a region where there are no free magnetic poles.

For an anchor ring in which A is the constant cross-sectional area, s the mean length of path within the ring and N the total number of turns in the coil with which the ring is wound, we have, if i is the current and μ the permeability of the material of the ring,

$$4\pi\mu\frac{\mathrm{N}}{s}i\mathrm{A} = \text{flux of magnetic induction in the ring.}$$

$$\therefore\ 4\pi\mu\frac{\mathrm{N}^2}{s}i\mathrm{A} = \text{number of linkages associated with the coil.}$$

Hence, provided μ is constant, the self inductance, which is the number of linkages per unit current, is given by

$$l = 4\pi\mu\frac{N^2}{s}A,$$

while the field due to the current is given by

$$H = 4\pi\frac{N}{s}i.$$

The energy stored in the magnetic field is given by

$$W = li^2 = \tfrac{1}{2}.4\pi\mu\frac{N^2}{s}A\left(\frac{sH}{4\pi N}\right)^2$$
$$= \frac{\mu H^2}{8\pi}(sA).$$

But sA is the volume of the ferromagnetic material and it is in this volume that the magnetic field is localized. Thus the magnetic energy per unit volume is

$$\frac{\mu H^2}{8\pi} = \frac{BH}{8\pi} = \frac{\vec{B}\cdot\vec{H}}{8\pi}.$$

In the absence of magnetizable material the energy per unit volume becomes $\mu_0\frac{H^2}{8\pi}$, where μ_0 is the permeability of space. Hence the contribution made by the magnetizable material per unit volume is $\frac{(\mu - \mu_0)H^2}{8\pi}$. Now in general $\mu = \mu_0 + 4\pi\chi$, where χ is the volume susceptibility, so that we may write the expression for the above contribution as $\frac{1}{2}\chi H^2$.

It must be noted that this expression for the contribution made by the magnetizable material to the energy density of the field is only applicable *when no free poles are present.* To proceed, let us therefore consider the potential energy of a permanent magnet when this is brought into a magnetic field, which is assumed to have a constant value over any small cross-section drawn at right angles to a line of force. Also, for simplicity, it will be assumed that the magnet is in the form of a short rod whose axis is always directed along a line of force. It will also be assumed that the poles are situated at the ends of the magnet. Fig. 11·24 (*a*) shows the position of the magnet with respect to the field. If Ω is the magnetic potential at S, at N it is $\Omega + \frac{\partial\Omega}{\partial s}\Delta l$, where Δl is the length of the magnet and δs is an element of length measured along a line of force. The work done against the field in bringing up the done per unit volume may be written

$$-\left[\mathrm{JH} - \int \mathrm{H}\, d\mathrm{J}\right]$$

$$= -[\chi \mathrm{H}^2 - \tfrac{1}{2}\chi \mathrm{H}^2] \quad [\because\ \chi\mathrm{H} = \mathrm{J}, \text{ i.e. } \chi\,\delta\mathrm{H} = \delta\mathrm{J}, \text{ if } \chi \text{ is constant.}]$$

$$= -\tfrac{1}{2}\chi \mathrm{H}^2$$

$$= -\frac{(\mu - \mu_0)}{8\pi}\mathrm{H}^2, \quad \because\ \mu = \mu_0 + 4\pi\chi,$$

and this is the potential energy per unit volume of the substance, where μ_0 is the permeability of empty space.

When the shape of the small element of magnetizable substance is arbitrary and it is brought into a magnetic field given by

$$\vec{\mathrm{H}} = \hat{\imath}\mathrm{H}_x + \hat{\jmath}\mathrm{H}_y + \hat{k}\mathrm{H}_z,$$

the potential energy per unit volume may be calculated by replacing the actual magnetic moment $\overrightarrow{\delta\mathrm{M}}$ of the element by

$$\hat{\imath}\,\delta\mathrm{M}_x + \hat{\jmath}\,\delta\mathrm{M}_y + \hat{k}\,\delta\mathrm{M}_z.$$

Thus $\delta\mathrm{M}_x \equiv m_x\,\delta x$, etc. and the potential energy per unit volume is

$$-\left(\frac{\mu - \mu_0}{8\pi}\right)[\mathrm{H}_x{}^2 + \mathrm{H}_y{}^2 + \mathrm{H}_z{}^2] = -\left(\frac{\mu - \mu_0}{8\pi}\right)\mathrm{H}^2.$$

If the specimen is surrounded by air, magnetic permeability μ_a, the energy density becomes

$$-\left(\frac{\mu - \mu_0}{8\pi}\right)\mathrm{H}^2 - \left[-\left(\frac{\mu_a - \mu_0}{8\pi}\right)\mathrm{H}^2\right]$$

$$= -\left(\frac{\mu - \mu_a}{8\pi}\right)\mathrm{H}^2.$$

Magnetostatic pressure difference in a fluid in a non-uniform magnetic field.—Let us consider a volume $\delta\mathrm{V}$ of a fluid with magnetic permeability μ and surrounded by a medium with permeability μ_a. If H is the strength of the field, and this may have any direction whatsoever, in which the specimen lies, cf. Fig. 11·25 (*a*), the potential energy associated with this element of volume is given by

$$\delta\mathrm{W} = -\left(\frac{\mu - \mu_a}{8\pi}\right)\mathrm{H}^2.\delta\mathrm{V}.$$

Now let the element, cf. Fig. 11·25 (*b*), have a constant cross-sectional area α and a length δs. Then

$$\delta\mathrm{W} = -\left(\frac{\mu - \mu_a}{8\pi}\right)\mathrm{H}^2\alpha.\delta s.$$

Hence the force δF, in the direction of s increasing, is given by

$$\delta F = -\frac{\partial W}{\partial s} = \left(\frac{\mu - \mu_a}{8\pi}\right) H^2 \alpha,$$

and since this force acts across an area α, it will give rise to a pressure p, where

$$p = \left(\frac{\mu - \mu_a}{8\pi}\right) H^2.$$

This pressure is due to the magnetic field and is superimposed on any hydrostatic pressure which may exist in the fluid ; it is termed the ***magnetostatic pressure*** and, being a pressure, will act in all directions.

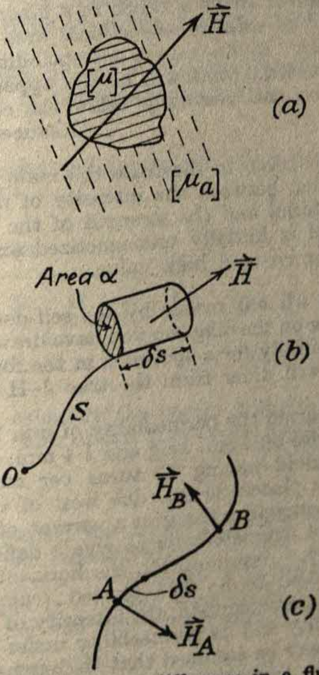

FIG. 11·25.—Magnetostatic pressure difference in a fluid in a non-uniform magnetic field.

If A and B are two points in a fluid and H_A and H_B the corresponding field strengths, we have, cf. Fig. 11·25 (c),

$$p_B - p_A = \left(\frac{\mu - \mu_a}{8\pi}\right)(H_B^2 - H_A^2),$$

and this is independent of the directions of the fields at A and B and of the field strength at any intermediate point on the path. In the sequel, cf. Chapter XIII, use will be made of this theorem.

EXAMPLES XI

11·01. Define *intensity of magnetization, susceptibility.* How do these quantities vary with the strength of the magnetizing field when the material concerned is soft iron?

A long soft iron rod, 0·8 cm. in diameter, is placed in the magnetic meridian and tapped sharply with a hammer. A deflexion of 8° is produced in a magnetometer placed with its centre 6 cm. due E. of one pole of the rod. Assuming H_0 to be 0·18 oersted. calculate a value for the susceptibility of the material of the rod. [10·1 e.m.u.]

11·02. Four metres above a bench in a laboratory is a steel girder 5 metres long and of total cross-sectional area 80 cm.^{2} This girder lies in the meridian. If the volume susceptibility of steel is 40 e.m.u., and $H_0 = 0{\cdot}18$ oersted., what will be the apparent value of H_0 at a point on the bench and vertically below the centre of the girder?

[Reduced by 0·0027 oersted.]

11·03. Draw a sketch to illustrate the main features of the curve showing the relation between the intensity of magnetization, J, in a ferromagnetic material and the strength of the magnetizing field, H, when the material is initially unmagnetized and the magnetic field is increased from zero to a high value.

11·04. Explain what is meant by the self-demagnetizing factor of an iron rod. Show on the diagram you have drawn how the J–H curve obtained experimentally for a specimen in the form of (*a*) a thin wire, (*b*) a thick wire, will differ from the true J–H curve.

11·05. In measuring the permeability of iron by the magnetometer method, an iron wire 60·4 cm. long and 1·4 mm. in diameter was used in a vertical solenoid having 20 turns per cm. length. A mirror magnetometer was placed 25 cm. due west of the lower pole of the wire. After demagnetizing the iron a current of 0·12 A. through the solenoid caused the magnetometer to give a deflexion of 15 cm. on a scale 1 metre away. Assuming H_0, the horizontal component of the earth's magnetic field to be 0·18 oersted., calculate values for the strength of the magnetizing field, the intensity of magnetization in the iron, its susceptibility and its permeability under the conditions of the experiment. [It may be assumed that H_v has been compensated and each pole is 2 mm. from the appropriate end of the wire.]

[3·02 oersted., 580 erg.oersted.$^{-1}$cm.$^{-3}$, 192, $2{\cdot}41 \times 10^3$.]

11·06. To investigate the magnetic properties of a certain steel, a straight steel wire 100·4 cm. long and 2·0 mm. diameter is placed along the axis of a vertical solenoid which has 40 turns per cm. A mirror magnetometer is placed 25 cm. due east of the lower pole of the magnet. The specimen is demagnetized and when a current of 1 amp. is passed through the solenoid the deflexion of the spot of light on a scale 100 cm. from the magnetometer is found to be 18 cm. Deduce values for J, χ and μ, where the symbols have their usual meanings. [Assume that $H_0 = 0{\cdot}18$ oersted., H_v has been compensated and the poles are at points 2 mm. from the ends of the wire.]

[322 e.m.u., 6·40 e.m.u., 81·4.]

11·07. Explain, in detail, what is represented by the area of a complete B–H loop.

Calculate a value for the loss of energy due to hysteresis when 5 kgm. of iron are subjected for 1 hour to cyclic magnetic changes of frequency 25 cycle.sec.$^{-1}$, if the area of the hysteresis loop represents a loss of $2{\cdot}5 \times 10^3$ erg.cm.$^{-3}$cycle.$^{-1}$. The density of iron may be taken as 7·8 gm.cm.$^{-3}$. [$1{\cdot}44 \times 10^3$ joule.]

11·08. Give a survey of methods used in the investigation of the magnetic properties of ferromagnetic samples and treat in detail one method which offers some advantage over the other methods, explaining why it is chosen.

Discuss the magnetic properties of importance in a material suitable for the construction of permanent magnets. (S)

11·09. The ascending and descending values for B and H for a half cycle are as follows :

	Ascending					Descending					
H (oersted.).	16	24	32	40	56	24	16	0	−8	−12	−16
$B \times 10^{-3}$ (gauss.)	0	8	10·4	12·1	14·0	13·6	13·1	11·9	10	8	0

Plot the curve and calculate the energy due to hysteresis dissipated per cm.^{3} per cycle. Also obtain a value for the Steinmetz coefficient. [$7{\cdot}23 \times 10^4$ erg., 0·0168.]

11·10. A long solenoid has n turns per cm. length and its cross-section is πa^2. If the secondary has N turns and the permeability of the core is μ, where $\mu^{-1} = \alpha + \beta\mathrm{H}$ [α and β are constants], show that the effective flux is a simple function of the primary current i. Obtain expressions for the mutual inductance of the coils.

$$\left[\frac{4\pi^2 n\mathrm{N}a^2 i}{\alpha + \beta(4\pi n i)} = \frac{pi}{\alpha + \gamma i}\ \text{(say)}\ ;\ \frac{4\pi^2 n\mathrm{N}a^2}{\alpha + 4\pi n\beta i},\ \frac{p\alpha}{(\alpha + \gamma i)^2}.\right]$$

11·11. What is a *magnetic hysteresis loop* ? How can it be obtained experimentally for a ferromagnetic material in the form of a ring ? Discuss briefly the information of practical importance which can be deduced from the loop.

11·12. A sample of a transformer steel is subjected to a cycle of magnetic changes in which $B_{\max.} = 14{,}000$ gauss. A hysteresis loop is plotted using scales for which 1 cm. ≡ 2 ampere-turn.cm.$^{-1}$ and 1 cm. ≡ 2000 gauss. The area of the loop is 9·28 cm.^{2} Find the loss in watts due to hysteresis when 1000 cm.^{3} of this material is subjected to a magnetizing field at 25 cycle.sec.$^{-1}$. [9·28 watt

11·13. [*Note on the theoretical lifting power of a magnet.* Let J be the intensity of magnetization in the material of the magnet and let us assume that the magnetization is uniform in the magnet and there is no leakage between the magnet and its attached keeper. Then the surface density of the magnetism on the opposing surfaces of the magnet and its keeper will be J and $-$ J respectively. The stress is therefore $\frac{2\pi J^2}{\mu_a}$ so that the force of attraction will be given by $F = \frac{2\pi A J^2}{\mu_a}$, where A is the area of contact between the magnet and its keeper, and μ_a is the permeability of air. Since $B = \mu_0 H + 4\pi J$ and H is small, J may be taken as $\frac{B}{4\pi}$. Thus

$$F = \frac{B^2 A}{8\pi\mu_a} = \frac{BHA}{8\pi} \text{ dyne.]}$$

A long solenoid with 10 turns per cm. length contains an iron rod 2 cm. in diameter ; the length of this rod is much less than that of the solenoid. The rod lies symmetrically within the solenoid and is cut into halves. Calculate a value for the force necessary to separate them when a current of 4 A. is flowing in the solenoid. It is given that on reversing the primary current 70 μC. flow in the secondary circuit of 8 turns wound round the iron rod, the total resistance of this circuit being 200 Ω. [$9{\cdot}7 \times 10^7$ dyne.]

11.14. A certain ferromagnetic material (stressed permalloy) has a rectangular hysteresis loop lying between $-2{\cdot}0 < H < 2{\cdot}0$ oersted for the magnetizing field and $-1200 < B < 1200$ gauss for the magnetic induction. Outside this region the arms of the loop are slightly inclined to the H-axis. Calculate values for the rates of rise in temperature due to hysteresis of a specimen when placed in a 100 cycle.sec.$^{-1}$ alternating magnetic field $\hat{H} \sin \omega t$ for the cases in which $\hat{H}$ is 1, 2 and 10 oersteds respectively. [Assume the thermal capacity of permalloy to be 3·34 joule.deg.$^{-1}$ C. per unit volume.]

[0, $1{\cdot}4 \times 10^{-2}$ deg.C.sec.$^{-1}$, $1{\cdot}4 \times 10^{-2}$ deg.C.sec.$^{-1}$.]

CHAPTER XII

THE MAGNETIC CIRCUIT AND THE DESIGN OF MAGNETS

The magnetic circuit and magnetomotive force.—In the preceding chapter methods for investigating the properties of ferromagnetic materials have been outlined and the more important properties discussed. It is now necessary to show how this information may be utilized in the production of large magnetic fields which are a feature of much electromagnetic apparatus and machinery. The fact that lines of magnetic induction are confined to definite circuital paths lying mainly within a ferromagnetic material suggests the idea of a ***magnetic circuit;*** it is based on an analogy with an electric circuit.

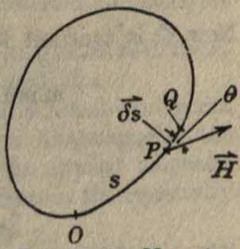

FIG. 12·01.—Magnetomotive force.

Let us therefore consider a closed path as indicated in Fig. 12·01. The path lies in a medium of permeability μ. O is a fixed point on this path from which the distance of a point P on the path may be measured; let us write OP $= s$. Let $\vec{H}$ be the magnetic field at P. If Q is a neighbouring point on the closed path and such that PQ $= \delta s$, then $\vec{H}\cdot\vec{\delta s}$, or $H\,\delta s\cos\theta$, where θ is the angle indicated, is $-\,\delta\Omega$, the drop in magnetic potential from P to Q. Thus

$$\oint \vec{H}\cdot\vec{ds} = -\oint d\Omega$$

= fall in magnetic potential round the circuit.

By analogy with an electrical circuit, the quantity $\oint \vec{H}\cdot\vec{ds}$ is called the ***magnetomotive force*** (m.m.f.) for the closed circuit considered. The unit is the ***gilbert.***

If the path is linked N times with a current i (e.m.u.), then by Ampère's circuital relation, cf. p. 142, we have

$$\oint \vec{H}\cdot\vec{ds} = 4\pi Ni = \frac{4\pi NI}{10},$$

where I is the current in amperes. The product NI is known as the number of ampere-turns.

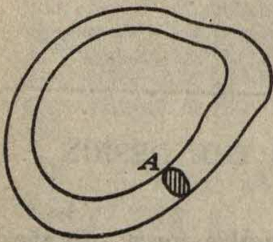

FIG. 12·02.—A narrow tube of magnetic induction; its reluctance.

Now suppose that a tube of magnetic induction has a cross-sectional area A at some section—cf. Fig. 12·02. The width of the tube must be small so that the length of any path taken all round the tube may be considered constant. Then Φ, the flux of magnetic induction across A, is given by

$$\Phi = \vec{B}\cdot\hat{n}A = BA.$$

$$\therefore\ H = \frac{\Phi}{\mu A}.$$

Now Φ is constant for a tube of induction, so that

$$\text{m.m.f.} = \oint \vec{H}\cdot\vec{ds} = \oint \frac{\Phi}{\mu A}ds$$

$$= \Phi \oint \frac{1}{\mu A}ds.$$

$$\therefore\ \Phi = \frac{\text{m.m.f.}}{\oint \frac{1}{\mu A}ds}.$$

By analogy with Ohm's law for an electrical circuit, the quantity $\oint \frac{ds}{\mu A}$ is known as the ***reluctance*** of the magnetic circuit. The unit is the gilbert.maxwell.$^{-1}$.)

To illustrate the above method of arriving at the definition of magnetic reluctance, let us consider an anchor ring made from a substance of permeability μ and upon this ring let there be a coil of N turns carrying a current I amperes. Then in the ring the magnetic induction will have a constant value B. Now the magnetomotive force is the work done per unit pole in taking a small pole round the circuit, viz. $4\pi\frac{NI}{10}$.

Hence

$$\oint \vec{H}\cdot\vec{ds}$$

$$= \oint \frac{B\,ds}{\mu} \quad [\because\ \vec{B} \text{ and } \vec{ds} \text{ are parallel.}]$$

$$= \frac{Bl}{\mu}. \quad \left[\because\ l = \oint ds.\right]$$

But $$B = \frac{\Phi}{A}.$$

$$\therefore\ \Phi = \text{magnetic flux} = \frac{\text{magnetomotive force}}{\text{reluctance}} = \frac{4\pi NI}{10\left(\frac{l}{\mu A}\right)}.$$

The reluctance of the magnetic circuit in this instance is therefore $\frac{l}{\mu A}$.

The reluctance is sometimes known as the ***magnetic resistance;*** this is from analogy with the term electrical resistance which is $\int \frac{\chi\, ds}{A}$, where χ is the resistivity of the material. $\frac{1}{\mu}$ therefore corresponds to χ and is sometimes termed the ***magnetic resistivity*** or ***reluctivity*** of the medium in which the path lies.

The analogy between electrical resistance and reluctance is not complete, however, for when the medium is ferromagnetic the permeability depends upon the strength of the applied magnetic field and hence the reluctivity is variable, whereas the resistivity is a constant if the temperature is constant.)

In practice a circuit may consist of several distinct parts in any one of which μ and A are constants. Then

$$\oint \frac{ds}{\mu A} = \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} = \ldots + \frac{l_n}{\mu_n A_n} = \sum \left(\frac{l}{\mu A}\right).$$

Very frequently $A_1 = A_2 = \ldots = A_n$, at least approximately.

More about magnetomotive force. When introducing the idea of a magnetic circuit it was stated that the tube of induction must be narrow. In practice the magnetic resistance of a circuit with a large cross-section has to be calculated so that it becomes necessary for us to examine the conditions under which the calculations may be made.

Consider a tube of induction linked with a coil of NI ampere-turns as in Fig. 12·03. The tube is finite in section but this is sufficiently small for the magnetic induction B to be normal to the cross-section at the position concerned. Let A be the cross-section of this tube and let us imagine that the tube we are considering is divided up into n elementary tubes; in the diagram only one of these is shown. Let s be the length of this elementary tube and δA its cross-sectional area; δA will not, in general, be constant.

The flux, $\delta\Phi$, across δA is given by $\delta\Phi = B(\delta A)$.

$$\therefore\ \text{m.m.f.} = \oint \vec{H}\cdot\vec{ds} = \oint H\, ds = \delta\Phi \oint \frac{ds}{\mu(\delta A)} = \frac{4\pi NI}{10}.$$

$\therefore\ \Phi$ = flux across any section of the tube of induction considered.

$= \Sigma(\delta\Phi)$

$$= (\text{m.m.f.}) \left[\left\{ \frac{1}{\oint \frac{ds}{\mu(\delta A)_1}} \right\} + \left\{ \frac{1}{\oint \frac{ds}{\mu(\delta A)_2}} \right\} + \cdots + \left\{ \frac{1}{\oint \frac{ds}{\mu(\delta A)_n}} \right\} \right]$$

where $(\delta A)_n$ is the cross-sectional area of the nth elementary tube.

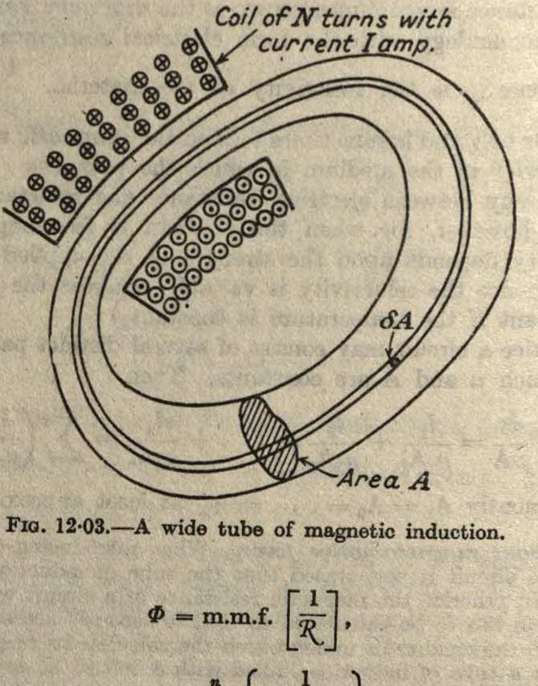

FIG. 12·03.—A wide tube of magnetic induction.

Thus
$$\Phi = \text{m.m.f.} \left[\frac{1}{\mathcal{R}}\right],$$

where
$$\frac{1}{\mathcal{R}} = \sum_{n=1}^{n} \left\{ \frac{1}{\oint \frac{ds}{\mu(\delta A)_n}} \right\}.$$

$\mathcal{R}$ is the reluctance of the tube.

Now if the tube of induction is such that each elementary tube has the same cross-sectional area and length, and if there are N_0 such tubes,

$$\frac{1}{\mathcal{R}} = \frac{N_0}{\oint \frac{ds}{\mu(\delta A)}} = \frac{1}{\oint \frac{ds}{\mu A}},$$

since $N_0 \delta A = A$, and it is only to such paths that the conceptions of reluctance, etc., can be readily applied.

Reluctances in parallel. Suppose that a tube of induction of cross-sectional area A, cf. Fig. 12·04, is divided into two parts of areas A_1 and A_2. Consider that portion of the tube extending from P to Q, the corresponding values of the magnetic potential being Ω_1 and Ω_2.

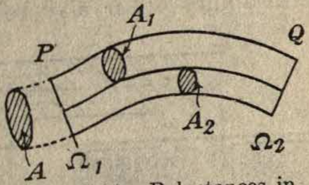

FIG. 12·04.—Reluctances in parallel.

Then

$$\left[\int_P^Q H\,ds\right]_1 = \Omega_1 - \Omega_2,$$

which is the drop in magnetic potential. The suffix 1 denotes that the integral refers to the tube of cross-section A_1. But

$$\Omega_1 - \Omega_2 = \int \frac{\Phi_1\,ds}{\mu A_1},$$

or

$$\Phi_1 = \frac{\Omega_1 - \Omega_2}{\int \frac{ds}{\mu A_1}} = \frac{\Omega_1 - \Omega_2}{\mathcal{R}_1},$$

where $\mathcal{R}_1$ is the reluctance of this portion of the tube. Similarly,

$$\Phi_2 = \frac{\Omega_1 - \Omega_2}{\mathcal{R}_2},$$

and since $\Phi = \Phi_1 + \Phi_2$, we have $\frac{1}{\mathcal{R}} = \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}$.

An iron ring with a small air-gap. It will be assumed that the diameter of the cross-section of the ring is small compared with the diameter of the ring itself. If l is the length of the complete ring and A its cross-sectional area, the flux, Φ_0, across any section is given by

$$\Phi_0 = \frac{4\pi NI}{10\left[\frac{l}{\mu A}\right]},$$

where NI is the number of ampere-turns on the ring.

When a small air-gap of length l_1 is made in the ring the flux is reduced to a value Φ given by

$$\Phi = \frac{4\pi NI}{10\left[\frac{l_1}{\mu_a A} + \frac{l - l_1}{\mu A}\right]},$$

where μ_a is the permeability of air and, for simplicity, it has been assumed that the cross-sectional area of the tubes of induction in air-gap is the same as in the iron; in numerical calculations the permeability of air may be taken as unity.

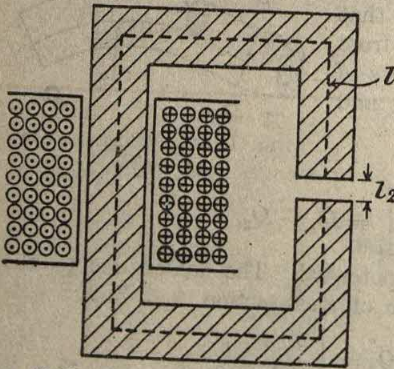

FIG. 12·05.—The core of an electromagnet.

The great effect of an air-gap is realized if the gap is made equal to $l \times 10^{-3}$ and the permeability of iron is taken as 1000. Then $\Phi = 0{\cdot}5\Phi_0$, for the reluctance has been practically doubled.

The core of an electromagnet.—The diagram given in Fig. 12·05 represents an electromagnet with NI ampere turns. Let A be the mean cross-sectional area of the core. Then if l_1 is the length of the magnetic circuit in the iron and l_2 the length of the air-gap, we have

$$\Phi = \frac{4\pi NI}{10\left[\frac{l_1}{\mu A} + \frac{l_2}{\mu_a A}\right]}.$$

For stalloy, if B = 15 kilogauss, $\mu = 1200$, so that for all practical electromagnets $\frac{l_1}{\mu A}$ will be negligibly small compared with $\frac{l_2}{\mu_a A}$ and hence the major portion of the magnetomotive force occurs across the air-gap.

Example.—Find the number of ampere-turns required to produce a flux of 60,000 maxwells in a closed ring of ferromagnetic material if the mean diameter of the ring is 20 cm. and the cross-sectional area is 3 cm.^{2}. Also find the number of ampere-turns when the above ring is cut into two equal portions, each separated from the other by two gaps of width 0·05 cm. Assume that $\mu = 1500$ for the value of B required.

(*a*) With the usual notation, we have

$$6 \times 10^4 \text{ maxwell.} = \Phi = \frac{\text{m.m.f.}}{\text{reluctance}} = \frac{4\pi NI}{10} \cdot \frac{1}{\left(\frac{2\pi . 10}{1500 \times 3}\right)} \text{ maxwell.}$$

$\therefore$ NI = 667 ampere-turns.

(*b*) For the air-gaps, A = 3 cm.^{2}, $l = 0{\cdot}1$ cm., $\mu_a = 1$.

$$\therefore \; 6 \times 10^4 = \frac{\frac{4\pi NI}{10}}{\left(\frac{20\pi - 0{\cdot}1}{1500 \times 3}\right) + \left(\frac{0{\cdot}1}{1 \times 3}\right)}.$$

$\therefore$ NI = 2·26 × 10^3 ampere-turns.

Magnetic flux leakage.—In all magnetic circuits the larger portion of the flux is confined to the circuit but inevitably there must be some loss of flux due to leakage. Thus, wherever there is an air-gap there will be a *fringing effect*, i.e. the lines of induction in air will cross an area greater than that of the ferromagnetic material.

An empirical rule for making a correction for this fringing effect states that to each of the two dimensions defining the actual cross-sectional area of the gap, a length equal to the length of the gap should be added.

An alternative method is to make use of the concept of a leakage factor, which is defined as the ratio of the total flux to the useful flux, i.e. to the flux in the air-gap. This factor is generally taken to be 1·3.

An iron ring with a localized exciting coil.—Let us suppose that a ring of ferromagnetic material is wound with a coil, C, having N turns and covering only a portion of the ring as shown in Fig. 12·06. The lines of induction in the air are mainly confined to the volume encircled by the ring and some of these are shown conventionally in the diagram. By connecting a search coil S to a ballistic galvanometer G the distribution of flux round the ring may be shown to be almost constant; to do this the current I in C is reversed and the galvanometer throw recorded for different positions of S. The flux is greatest when S is at C and least at a section diametrically opposite.

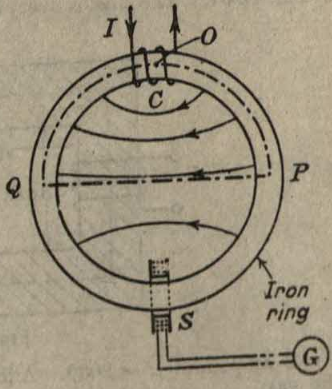

FIG. 12·06.—An iron ring with a localized exciting coil.

Now, in general, the leakage flux cannot be calculated but an approximate calculation serves to discover a value for the magnetic induction B_a along the diameter PQ, if we assume that this path is entirely in air (permeability μ_a).

Let $B = \mu H$ be the induction in the iron (assumed constant) and let r be the mean radius of the ring. Then, applying the circuital relation to the path OPQ, we have

$$\frac{B_a}{\mu_a}(PQ) + \frac{B}{\mu}(QOP) = \frac{4\pi NI}{10},$$

i.e.

$$\frac{B_a(2r)}{\mu_a} + \frac{B}{\mu}(\pi r) = \frac{4\pi NI}{10}.$$

But by applying the same relation to a path of radius r in the iron

$$\frac{B}{\mu}(2\pi r) = \frac{4\pi NI}{10}.$$

$$\therefore\ B_a \fallingdotseq \frac{\pi}{2}\left(\frac{\mu_a}{\mu}\right)B.$$

For transformer steel, with μ as low as 5000, B_a is only $3 \times 10^{-4}B$ and with still higher values of μ, B_a is still further reduced. The fact that the flux in a ring of ferromagnetic material when the m.m.f. is localized is the same as if it were uniformly distributed is of great practical importance, for it enables us to obtain solutions to many problems for which a complete mathematical treatment is well-nigh impossible.

Example.—An electromagnet has the form and dimensions shown in Fig. 12·07. The steel is everywhere (4×4) cm.^{2} and $N = 400$ turns, $I = 2$ amp. and $\mu = 2000$. Allowing for 'fringing' find the flux in the gap.

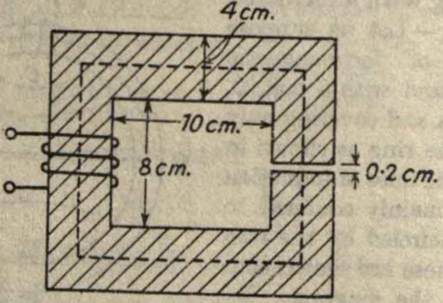

FIG. 12·07.

The mean length of the path in the steel is 51·8 cm.; in air it is 0·2 cm.

$$\therefore\ \Phi = \frac{4 \times 3{\cdot}14 \times 400 \times 2}{10\left[\dfrac{51{\cdot}8}{2000 \times 16} + \dfrac{0{\cdot}2}{16}\right]} = 7{\cdot}12 \times 10^4 \text{ maxwell.}$$

In the air-gap, allowing for fringing, by the first empirical rule,

$$B = \frac{\Phi}{(4{\cdot}2)^2} = 4{\cdot}04 \times 10^3 \text{ gauss.}$$

The second rule gives

$$B = \frac{\Phi}{1{\cdot}3 \times 16} = 3{\cdot}42 \times 10^3 \text{ gauss.}$$

Example.—A U-shaped magnet has N turns of wire in the magnetizing coil. Suppose it has a square cross-section of area A, mean length l, and that r is the distance between its poles. If μ is the permeability of the core, show that the force with which the magnet holds a bar of the same material and cross-section against its poles is approximately

$$\frac{\pi}{25}\cdot\frac{\mu N^2I^2A}{\mu_a(l + r)^2}\text{ dyne.}$$

Let ξ be the width of each air-gap. Then with the usual notation

$$\Phi = \frac{4\pi NI}{10\,\Sigma\left(\frac{l}{\mu A}\right)},$$

and $B = \frac{\Phi}{A}$, if there is no leakage. The force required is $2.\frac{B_aH_a}{8\pi}A$ dyne., where B_a and H_a are the values of the induction and field strength in air; we have at once, since the normal component of the induction at an interface is continuous, $B = B_a$, and $H_a = \frac{B_a}{\mu_a} = \frac{B}{\mu_a}$. Now

$$\Phi = \frac{4\pi NI}{10\left[\frac{l}{\mu A} + \frac{2\xi}{\mu_a A} + \frac{r}{\mu A}\right]}$$

$$= \frac{4\pi NI}{10}\left(\frac{\mu A}{l + r}\right), \qquad \text{if } \xi \longrightarrow 0.$$

$$\therefore \text{ Force} = 2.\frac{B_aH_a}{8\pi}A = \frac{2}{8\pi}.\left[\frac{4\pi NI}{10(l + r)}.\frac{\mu A}{A}\right]^2.\frac{1}{\mu_a}.A$$

$$= \frac{\pi}{25}\frac{\mu^2}{\mu_a}.\frac{N^2I^2A}{(l + r)^2} \text{ dyne.,}$$

and using the results given on pp. 691–2 it is easy to show that this expression has the dimensions of a force.

Example.—A ring of ferromagnetic material $[\mu = 200]$ has a mean radius of 10 cm. and a cross-sectional area $\frac{\pi}{4}$ cm.^{2}. The ring is wound uniformly with 500 turns of wire carrying a steady current of 1 A. Find values for the magnetic flux when (*a*) the ring is complete, (*b*) a saw-cut 0·1 cm. wide extends half-way across the specimen and (*c*) the saw-cut is completed.

$$(a)\ \Phi_{\xi=0} = \frac{4\pi \times 500 \times 1}{10\left[\frac{20\pi \times 4}{200\pi}\right]} = 1570 \text{ maxwell.}$$

$$(c)\ \Phi_{\xi=0\cdot1 \text{ cm.}} = \frac{4\pi \times 500 \times 1}{10\left[\frac{(62\cdot8 - 0\cdot1) \times 4}{200\pi} + \frac{0\cdot1 \times 4}{\pi}\right]} = 1190 \text{ maxwell.}$$

(*b*) When the saw-cut is only half completed, we have an air-gap in parallel with an equal thickness of metal and the cross-sectional area of each is $\frac{\pi}{8}$.

The reluctance of the air-gap when completed is $\frac{0\cdot1 \times 4}{1 \times \pi} = 0\cdot127$. The reluctance of the air-gap when the saw-cut is one-half completed is 0·254 and the reluctance of the piece of metal remaining is $\frac{0\cdot254}{200} = 1\cdot27 \times 10^{-3}$. The effective reluctance of the half-completed gap is $\mathcal{R}$ where

$$\frac{1}{\mathcal{R}} = \frac{1}{0\cdot254} + \frac{1}{1\cdot27 \times 10^{-3}} = 791.$$

$$\therefore\ \mathcal{R} = 0\cdot0013.$$

But the reluctance of the rest of the iron ring is $\frac{62{\cdot}7 \times 4}{200\pi}$, i.e. the total reluctance is practically unaffected by the half-completed gap.

$$\therefore\ \Phi = 1570 \text{ maxwell.}$$

Example.—A long solenoid has n turns per cm. length and its cross-sectional area is πa^2. If a secondary coil wound closely on the solenoid has N turns and the permeability of the core is μ, where $\mu^{-1} = \alpha + \beta\mathrm{H}$, [$\alpha$ and β are constants], obtain expressions for the mutual inductance of the coils.

Since $\mathrm{H} = 4\pi ni$, and $\mathrm{B} = \mu\mathrm{H}$, the flux Φ is given by

$$\Phi = \pi a^2\mathrm{B} = \pi a^2\mu\mathrm{H} = \frac{\pi a^2(4\pi ni)}{\alpha + \beta(4\pi ni)}.$$

$$\therefore\ \Psi = \mathrm{N}\Phi = \frac{4\pi^2 na^2\mathrm{N}i}{\alpha + \beta(4\pi ni)}.$$

If Ψ changes with time the e.m.f. induced in the secondary is

$$e = -\dot{\Psi} = -\frac{d\Psi}{di}\frac{di}{dt}.$$

But
$$e = -m\frac{di}{dt}, \quad \text{i.e. } m = \frac{d\Psi}{di}.$$

$$\therefore\ m = \frac{d\Psi}{di} = \frac{4\pi^2 na^2\mathrm{N}\alpha}{(\alpha + 4\pi\beta ni)^2}.$$

If we write $m_0 = \frac{\Psi}{i}$, which is the flux per unit current, we have

$$m_0 = \frac{4\pi^2 na^2\mathrm{N}}{\alpha + 4\pi\beta ni}.$$

Thus $m < m_0$ and becomes very small when i is large.

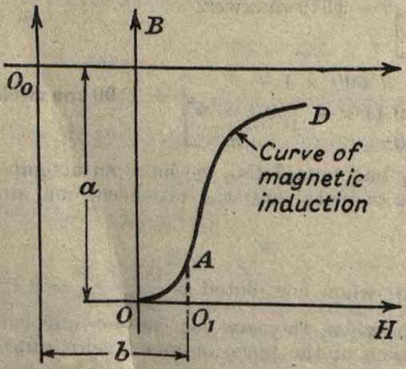

FIG. 12·08.—Fröhlich's equation for a curve of magnetic induction.

Fröhlich's equation for a curve of magnetic induction.—From some of the above examples it would appear that it is very desirable to find, if possible, a simple relation connecting corresponding values of B and H at points on a curve of magnetic induction. As a first approximation it may be assumed that the upper portion of the curve of magnetic induction shown in Fig. 12·08 is part of a rectangular hyperbola.

The lower portion OA of the experimental curve is not part of the hyperbola. Let O_0 be the point of intersection of the asymptotes to the hyperbola. If H_0, B_0 are the co-ordinates of a point on the hyperbola and referred to O_0 as origin, the equation to the hyperbola takes the form $B_0H_0 = k$, where k is a constant. Let the hyperbola cut the H axis in O_1 and let this be the point $(b, -a)$ referred to O_0. The equation to the hyperbola referred to O_1 is

$$(B_1 - a)(H_1 + b) = k,$$

where H_1, B_1 are coordinates of a point referred to O_1. Now at O_1, $H_1 = 0$, $B_1 = 0$, so that $k = -ab$. Hence

$$B_1 = a - \frac{ab}{H_1 + b} = \frac{aH_1}{H_1 + b}.$$

If OO_1 is small, we may write $H_1 = H$, $B_1 = B$ and so have as the required equation

$$B = \frac{aH}{H + b}.$$

By writing this equation in the form

$$H + b = a\left(\frac{H}{B}\right),$$

we see that the constants a and b may be obtained by plotting H against $\left(\frac{H}{B}\right)$.

To show how this approximate equation fits an actual curve, let us consider the following values of H and B which refer to a Heusler alloy [61 per cent Cu, 27 per cent Mn, 13 per cent Al].

H (oersted)	5	10	20	50	100
B (gauss)	800	2300	3200	3800	4100

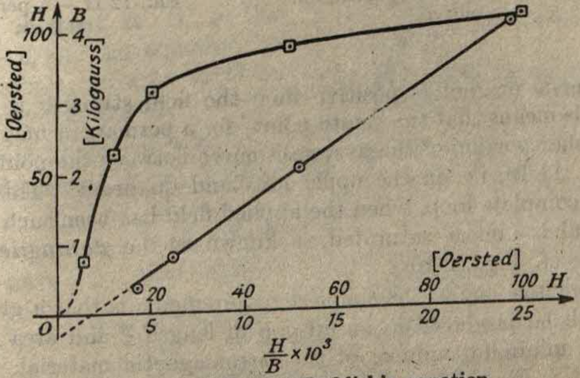

FIG. 12·09.—Test of Fröhlich's equation.

The B–H curve is shown in Fig. 12·09. Superposed on this diagram is the straight line obtained by plotting H against $\frac{H}{B}$. From it we find $a = 4{\cdot}41 \times 10^3$ gauss and $b = 8{\cdot}0$ oersted. Hence for this material we have, approximately,

$$B = \frac{4{\cdot}41 \times 10^3 H}{H + 8{\cdot}0}.$$

Permanent magnets.—Permanent magnets are designed to provide a definite magnetic flux in an air-gap and such magnets must be constructed from materials which retain a high degree of magnetization when the exciting field is made zero and for which the coercivity is high. For transformer steels an essential characteristic is that the hysteresis loop should be narrow but this is not of great importance in connexion with permanent magnets.

The presence of an air-gap in a permanent magnet means that a self-demagnetizing field will exist. If the induction, B, within the permanent magnet is positive then the field strength is negative and this means that the 'state-point' for a permanent magnet must be on that portion of the hysteresis curve between the points R and C, Fig. 11·10, i.e. in the upper left-hand quadrant. This portion of the complete loop, when the applied field has been such that the material is almost saturated, is known as the ***demagnetization curve***—cf. Fig. 12·10.

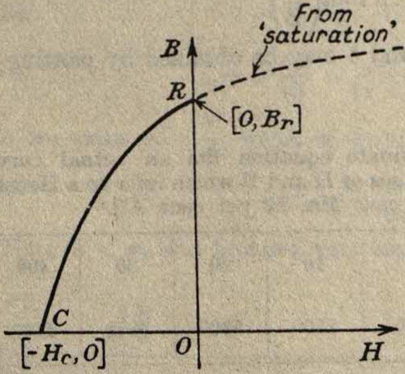

FIG. 12·10.—A demagnetization curve.

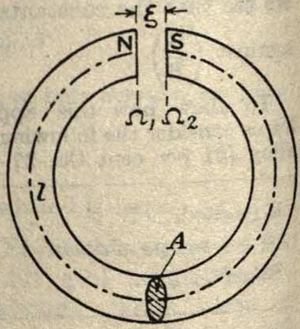

FIG. 12·11.—A permanent magnet.

In practice one of the simplest requirements is that a given flux, Φ, shall be produced in an air-gap of length ξ and area A, using only a minimum volume of the ferromagnetic material.

Let NS, Fig. 12·11, be the gap in a ring magnet. If $\mathcal{R}_g$ is the reluctance of the gap and Ω_1 and Ω_2 the magnetic potentials at N and S, we have

$$\Omega_1 - \Omega_2 = \Phi\mathcal{R}_g.$$

If K is the leakage factor, the flux in the material is KΦ, so that if A is the cross-sectional area of the ring,

$$B = \frac{K\Phi}{A}.$$

If $|H|$ is the numerical value of the field in the ring (assumed constant) and l the mean length of path in it,

$$|H|\,l = \Omega_1 - \Omega_2 = \Phi\mathcal{R}_g.$$

The volume of the ring is $Al = \dfrac{K\Phi}{B}\,\dfrac{\Phi\mathcal{R}_g}{|H|}$

$$= \frac{K\Phi^2\mathcal{R}_g}{B|H|}.$$

The volume of the ring is a minimum when $B|H|$ is a maximum and since the specimen is in a negative field H, the condition is, strictly speaking, that BH shall be a minimum.

The approximate shape of a demagnetization curve.—Let the points R and C on a hysteresis loop be the points $(0, B_r)$ and $(-H_c, 0)$ referred to O the point of intersection of the H and B-axes —cf. Fig. 12·12. If we regard the demagnetization curve as a portion of a rectangular hyperbola let its asymptotes intersect at O_0. If (H_0, B_0) are the coordinates of a point referred to O_0, then the equation to the hyperbola takes the form

$$B_0H_0 = \text{constant} = k \text{ (say)}.$$

If O_0, referred to O, is the point $(-c, a)$, the equation to the hyperbola, referred to O as origin, may be written

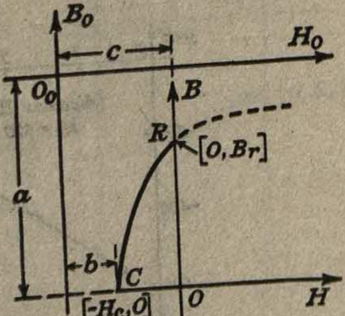

FIG. 12·12.—The equation to a demagnetization curve.

$$(B - a)(H + c) = k,$$

and to determine k we assume that $B = 0$ when $H = -H_c$.* Thus

$$-a(-H_c + c) = k,$$

and we have

$$(B - a)(H + c) = a(H_c - c),$$

or

$$B = a\left[1 + \frac{H_c - c}{H + c}\right] = a\left[\frac{H + H_c}{H + c}\right].$$

---

* Strictly speaking, $J = 0$ when $H = -H_c$, but the error is small.

Now when $H = 0$, $B = B_r$, so that $B_r = \frac{a}{c}H_c$, a relation which will prove useful later.

For the present, the equation connecting B and H may be written

$$H + c = a\left(\frac{H + H_c}{B}\right),$$

which is of the form $y + c = ax$, so that c and a may be readily determined graphically.

Example.—For Alcomax II, a modern ferromagnetic material with an exceptionally high degree of magnetic stability, the demagnetization curve is given by the following values of B and H.

H (oersted) . .	0	− 100	− 200	− 300	− 400	− 500	− 550
$B \times 10^{-2}$ (gauss)	124	122	118·5	114	104·5	80	40

Thus $B_r = 12{,}400$ gauss; by plotting the B–H curve we find $H_c = 570$ oersted, i.e. $OC = -570$ oersted.

Corresponding values of $y = H$ and $x = \frac{H + H_c}{B}$ are plotted in Fig. 12·13. From it we find $c = 610$ oersted, $a = 1{\cdot}35 \times 10^4$ gauss.$^{-1}$

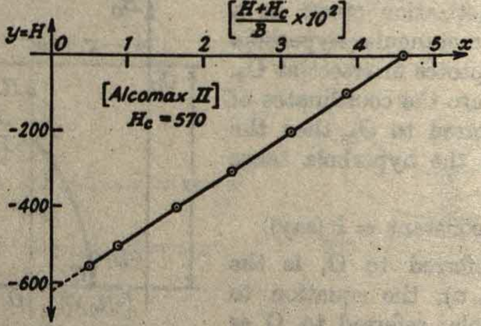

FIG. 12·13.—The demagnetization equation for Alcomax II.

The excellent linearity of the graph justifies the assumptions we have made.

The minimum value of BH for points on a demagnetization curve.—Since

$$B = a\left(\frac{H + H_c}{H + c}\right),$$

we have

$$\frac{BH}{a} = \xi \text{ (say)} = \frac{H^2 + HH_c}{H + c}.$$

Now $$\frac{d\xi}{dH} = \frac{H^2 + 2cH + cH_c}{(H + c)^2},$$

and this is zero if $H = -c \pm \sqrt{c(c - H_c)}$.

Since $-c < H < 0$, the stationary value of H is determined by

$$H = -c + \sqrt{c(c - H_c)}.$$

This corresponds to a minimum of BH since the product is zero at $H = 0$ and $H = -H_c$, and is negative at all other values of H within this range—of course a second differentiation confirms this but it is unnecessary.

The corresponding value of B is given by

$$B = a\left[\frac{-c + \sqrt{c(c - H_c)} + H_c}{\sqrt{c(c - H_c)}}\right]$$

$$= a\left[1 - \sqrt{\frac{c - H_c}{c}}\right] = \frac{a}{c}[c - \sqrt{c(c - H_c)}] = -\frac{a}{c}H.$$

At this stationary value of BH, $\frac{B}{H} = -\frac{a}{c}$.

Thus the working point, P, is given by the intersection of the straight line OO_0 with the magnetization curve.

It should also be noted that

$$\frac{B_r}{H_c} = \frac{a}{c},$$

so that the working point P is most rapidly obtained by drawing the diagonal OO_1, Fig. 12·14, of the rectangle bounded by the axes and the lines $H = -H_c$, $B = B_r$. Hence it is not essential to find values for a and c when only the position of P is required.

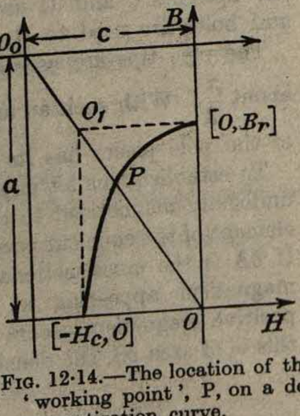

FIG. 12·14.—The location of the 'working point', P, on a demagnetization curve.

Electromagnet design.—A simple form of electromagnet capable of producing a field of 5000 oersteds between the pole pieces is shown in Fig. 12·15 (*a*). The yoke CD is made of dead mild steel [carbon, 0·19 per cent] which is magnetically very soft. The pole pieces AA_1 and BB_1 are made of a soft magnetic material such as very high grade Swedish iron, which is selected on account of its high permeability. The pole tips are made of the same material as the pole pieces and can be screwed into position. The energizing coils S_1 and S_2 each consist of 2400 turns of No. 16 s.w.g.

silk-covered copper wire wound on brass formers held at a fixed distance apart by spacers R_1, R_2 and R_3. The brass cases rest on the base of the yoke while the pole pieces pass easily through them. This enables the pole tips to be fixed at the desired distance apart; the uprights C and D are slotted at the top, cf. Fig. 12·15 (*b*), and bolts are used to clamp the pole pieces rigidly in position.

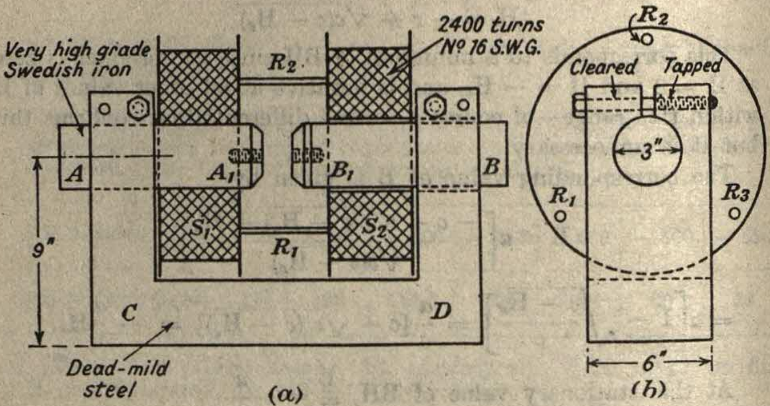

FIG. 12·15.—An electromagnet.

The pole tips are usually truncated cones with a semi-angle of about $\frac{\pi}{3}$. With such an angle the field in the gap and on the axis of the pole pieces has its maximum value.

To establish this let us assume that the material of the cones is uniformly magnetized to the saturation value J_0, and consider an element of the cone cut parallel to its axis as shown in Fig. 12·16 (*a*). If δA is the cross-sectional area of this element, the amounts of magnetism appearing at its ends are $\pm J_0\,\delta A$. Suppose the positive magnetism occurs on the sloping face of the element. If this is of area δS, the density of the free magnetism on the surface

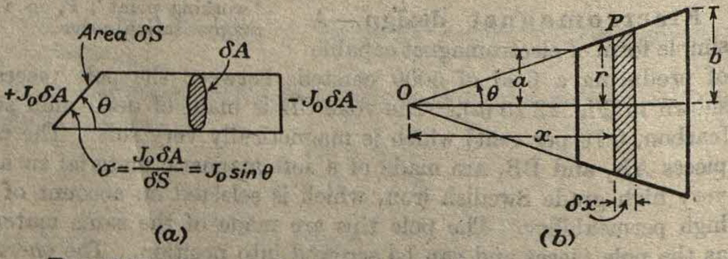

FIG. 12·16.—Magnetic field due to a conical pole piece whose material is saturated.

of the cone is $J_0 \dfrac{\delta A}{\delta S} = J_0 \sin\theta$, where θ is the semi-angle of the cone.

Consider now the section of the pole piece defined by the planes x and $x + \delta x$, Fig. 12·16 (*b*); the radii of the ends of this section are r and $r + \delta r$. The magnetic potential at O, due to the free magnetism which appears on the curved surface of this section, is

$$\delta\Omega = [(J_0 \sin\theta).2\pi r.\delta r \operatorname{cosec}\theta] \div OP = \frac{2\pi r J_0\,\delta r}{(r^2 + x^2)^{\frac{1}{2}}}.$$

$$\therefore\ \delta H = -\frac{\partial}{\partial x}(\delta\Omega) = -(2\pi r J_0\,\delta r)\left[-\frac{1}{2}.\frac{2x}{(r^2 + x^2)^{\frac{3}{2}}}\right] = \frac{2\pi r J_0 x\,\delta r}{(r^2 + x^2)^{\frac{3}{2}}}.$$

$$\therefore\ \frac{\partial}{\partial x}(\delta H) = 2\pi r J_0\,\delta r\left[\frac{1}{(r^2 + x^2)^{\frac{3}{2}}} - \frac{3x^2}{(r^2 + x^2)^{\frac{5}{2}}}\right].$$

Now δH is a maximum when $\dfrac{\partial}{\partial x}(\delta H) = 0$,

i.e. $$r^2 + x^2 - 3x^2 = 0,$$

which gives $x = \dfrac{r}{\sqrt{2}}$, or $\theta = \tan^{-1}\sqrt{2} = 54°\,44'$.

Now $$\delta H = \frac{2\pi r J_0 r \cot\theta\,\delta r}{r^3(1 + \cot^2\theta)^{\frac{3}{2}}} = 2\pi J_0 \sin^2\theta\cos\theta\left(\frac{\delta r}{r}\right).$$

$$\therefore\ H = 2\pi J_0 \sin^2\theta\cos\theta\int_a^b \frac{dr}{r},$$

so that for the two pole pieces the field at O is

$$4\pi J_0 \sin^2\theta\cos\theta \ln\left(\frac{b}{a}\right),$$

where a and b are the dimensions indicated.

In practice the pole pieces are not completely saturated and it is necessary to make θ somewhat greater; θ is usually made $\dfrac{\pi}{3}$.

The most uniform field is given by the condition $\dfrac{\partial^3\Omega}{\partial x^3} = 0$,

i.e. $$\frac{9x}{(r^2 + x^2)^{\frac{5}{2}}} - \frac{15x^3}{(r^2 + x^2)^{\frac{7}{2}}} = 0,$$

or $x = r\sqrt{\dfrac{3}{2}}$. This corresponds to $\theta = 39°\,14'$.

EXAMPLES XII

12·01. Explain what is meant by a *magnetic circuit.* In what respects does a magnetic circuit (*a*) resemble, and (*b*) differ from, an electric circuit ?

An iron rod 50 cm. long and 2·0 cm.^{2} in cross-section is bent to form an anchor ring having an air-gap of 0·10 cm. The ring is then wound uniformly with 500 turns of wire, through which a current of 1·5 A. is passed. If the permeability of iron under these conditions is 850, estimate the strength of the magnetic field in the air-gap.

[5·9 × 10^3 oersted.]

12·02. Calculate the number of ampere-turns required to produce a flux of 50,000 maxwells in a closed ring of ferromagnetic material if the mean diameter of the ring is 20 cm. and the cross-sectional area 4·0 cm.^{2}. Find also the number of ampere-turns to give the same flux when the above ring is cut into two equal portions, each separated from the other by an air-gap of width 0·050 cm. [Neglect magnetic leakage and assume that the permeability of the material is 1250 for the value of the induction required.] [1·5 × 10^4 ampere-turns.]

12·03. An electromagnet consists of a semicircular ring of iron of permeability 1000, wound with 500 turns of wire. The radius of the ring is 15 cm. Two horizontal bars of the same area of cross-section as the ring, viz. 8·0 cm.^{2}, but of magnetic permeability 800 are attached to the pole pieces of the electromagnet and are adjusted to be (*a*) 0·20 cm., (*b*) 0·50 cm. apart. If a current of 10 amp. excites the electromagnet determine values for the strength of the magnetic field in the gap in each instance. [Neglect leakage.]

[21 × 10^3 oersted, 10·5 × 10^3 oersted.]

12·04. A ring of ferromagnetic material [$\mu = 200$] has a mean radius of 20 cm. and a cross-sectional area 2·5π cm.^{2}. The ring is wound uniformly with 1000 turns of wire carrying a steady current of 2·0 amp. Find values for the magnetic flux when (*a*) the ring is complete, (*b*) a saw-cut 2 mm. wide extends half-way across the specimen and (*c*) the saw-cut is completed.

[(*a*) 31,400 maxwell., (*b*) very little change, (*c*) 23,900 maxwell.]

12·05. An electromagnet with a wrought iron core is required to exert a pull of 10 kgm.-wt. upon its armature when this is 0·5 cm. from each pole. The length of the magnet core is 50 cm., length of armature 12 cm. and the cross-section of both core and armature is 6 cm.^{2}. If the coefficient of magnetic leakage is 1·3 [i.e. total flux ÷ useful flux], obtain a value for the required number of ampere turns. [Assume $\mu = 1500$.] [4·88 × 10^3.]

12·06. A rectangular iron bar is bent into the shape of a ring, with an air-gap of 0·2 cm. between the parallel ends of the bar. The mean length of the bar is 40 cm. and its cross-section is 2·0 cm. by 1·5 cm. A winding, consisting of 1200 turns of insulated wire, is wound on to the iron and carries a steady current of 0·25 A. If the permeability of iron is 3000, find the self inductance of the coil, on the assumption that magnetic leakage may be neglected. If a wire carrying a current of 4 A. were placed in the air-gap parallel to the 2 cm. sides of the ends of the iron, calculate the force which would act upon it.

[25·4mH., 1·41 × 10^3 dyne.]

12·07. An iron ring, 100 cm. mean circumference, is made from round iron of cross-section 10 cm.^{2}; a saw-cut 0·20 cm. wide is made in it. It is wound with 200 turns. Find the current required to produce a flux of 10^5 lines across the air-gap, given that the leakage factor is 1·3 and the iron is such that when B = 13 × 10^3 gauss, $\mu = 300$.
[27·6 amp.]

12·08. Determine the number of ampere turns necessary to produce a field with induction 7·2 kilogauss in the air-gap of the iron circuit shown in Fig. Ex. 12·08. [Assume $\mu = 2000$ and that the leakage factor is 1·3; the material is everywhere 4·0 cm. × 4·0 cm., and B(iron) = B(air) × (leakage factor).]
[1680]

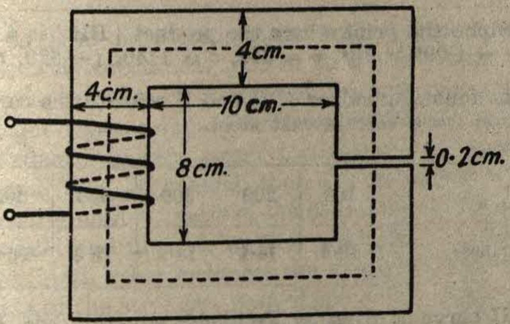

FIG. Ex. 12·08.

12·09. (*a*) Calculate the number of ampere-turns necessary to produce a flux of 1·2 × 10^5 maxwells in a closed ring of silicon steel (4·3 per cent Si) made up of two sections, (i) one of length 80 cm. and cross-sectional area 8·0 cm.^{2} and (ii) one of length 50 cm. and cross-sectional area 12 cm.^{2}.

(*b*) Also calculate the number of ampere-turns when the ring is cut diametrically across at two sections in (i) and air-gaps each 0·50 cm. long are introduced into the circuit. Corresponding values of B and H for this silicon steel are as follows:

H (oersted.) .	0·5	1·0	2·0	5·0	10·0	25	50
B (gauss) . .	4500	9000	10000	10500	11200	12800	15300

[Neglect fringing effects.]

[(*a*) 3100 amp.-turns, (*b*) 15,000 amp.-turns.]

12·10. The following data refer to the demagnetization curve of a certain ferromagnetic alloy. The curve cuts the axes in the points

$(B_r, 0)$, $(0, -H_c)$ and the curve itself may be represented by an equation of the type

$$B = \frac{a(H + H_c)}{b + H + H_c} = a\left[1 - \frac{b}{c + H}\right].$$

By means of a graphical method determine values for a and b; hence find c.

H (oersted.)	0	− 200	− 400	− 600	− 800
B × 10^{-3} (gauss.) . .	7·75	7·05	5·94	4·12	0

Also determine the point where the product $\mid BH \mid$ is a maximum. [$a = 1{\cdot}095 \times 10^4$, $b = 340$, $c = 1140$, $(-520, 5{\cdot}0 \times 10^3)$.]

12·11. The following values of H and B refer to the curve of magnetic induction for a certain cast steel.

H (oersted.) . . .	100	200	300	400	500	600
B × 10^{-3} (gauss) . .	10·6	14·1	14·9	15·6	16·0	16·4

If the B–H curve is given by Fröhlich's equation, viz. $B = \dfrac{aH}{b + H}$ where a and b are constants, obtain numerical values for a and b. [$a = 17{,}600$, $b = 50$.]

12·12. A ring of ferromagnetic material, of uniform cross-section, has an air-gap λ and the mean length of the ring is l. It is known that $B = \dfrac{aH}{b + H}$, where the symbols are as defined in the previous example. Neglecting all fringing and leakage effects, prove that the value of B due to a m.m.f. m, is given by the smaller root of the equation

$$B^2 - \left\{\left(\frac{m + bl}{\lambda}\right) + a\right\}B + \frac{am}{\lambda} = 0.$$

If the ring is of cast steel [cf. Ex. 12·11], with $l = 60$ cm. and $\lambda = 0{\cdot}1$ cm., find the flux-density in the gap when a steady current of 3 A. flows in the windings (4000 turns). [$1{\cdot}45 \times 10^4$ gauss.]

CHAPTER XIII

DIAMAGNETISM AND PARAMAGNETISM

Up to 1845 only iron, nickel and cobalt were known to be magnetizable substances, but in that year FARADAY proved that other substances were capable of being magnetized but only to a very slight degree compared with iron. He suspended rods of different substances in very strong but *non-uniform* magnetic fields and discovered that, in general, the rods orientated themselves in the more nearly uniform parts of the field with their longer axes parallel to the field and in so doing they resemble iron but to a much less extent; in all other instances the axes were normal to the general direction of the strongest part of the field. Substances in the first class were termed by Faraday ***paramagnetics***, those in the second class were ***diamagnetics***. By experiment he showed that paramagnetics have a magnetic volume susceptibility $\left(\chi = \frac{J}{H}\right)$ which is always slightly in excess of zero, whereas diamagnetics have a magnetic susceptibility which is also small but negative.

It must be noted, however, that when rods of any material are placed in a *uniform* magnetic field then the rods always set with their long axes parallel to the field. To explain this, it must be recalled that if H_a is the applied magnetic field, the effective field within the specimen, in consequence of the poles induced in it, is given by

$$H = H_a - DJ,$$

where D is a positive quantity known, in the case of paramagnetics, as the demagnetizing factor; it is only for rods of ferromagnetic material that D is large. Now for paramagnetic materials J is positive and hence $H < H_a$. For diamagnetic materials J is negative with respect to the applied field, so that $H > H_a$.

Since $J = \chi H$, where χ is the volume susceptibility of the specimen, the above equation may be written

$$H = H_a - D\chi H$$

or

$$H = \frac{H_a}{1 + D\chi}.$$

Hence the potential energy per unit volume of the specimen, when this has been brought into the field H_a, is

$$-\frac{(\mu-\mu_0)}{8\pi}H_a{}^2\left(\frac{1}{1+D\chi}\right)^2, \qquad \text{[cf. p. 392].}$$

For paramagnetic materials $\mu > \mu_0$, so that the energy density will be a minimum when $\frac{1}{1+D\chi}$ is a maximum, i.e. when $(1 + D\chi)$ is a minimum. Since both D and χ are positive for paramagnetic materials, this last condition is fulfilled when D is as small as possible. This occurs when the long axis of the rod is parallel to the field, because the degree of magnetization decreases as the axis becomes normal to the field.

For diamagnetic substances $\mu < \mu_0$, so that the energy density is positive. It will be least when $(1 + D\chi)$ is as large as possible. Now χ is negative for diamagnetics, so that $(1 + D\chi)$ is largest when D is as small as possible. As before, this occurs when the axis of the rod is parallel to the field.

Previous to Faraday's work on the subject, magnetization had often been detected apparently in many materials, but such magnetization was probably due to the presence of ferromagnetic impurities.

Towards the end of the eighteenth century it had been noted that antimony and bismuth were repelled from the poles of a powerful magnet and at that time it was assumed that other substances possessed no magnetic properties at all.

In order to examine the magnetic properties of liquids Faraday enclosed them in glass tubes. He discovered that aqueous solutions of iron salts are paramagnetic; water and alcohol are diamagnetic.

Faraday also succeeded in several ways in determining the magnetic character of gases. For instance, if a stream of gas, rendered evident by traces of ammonia and hydrogen chloride, was allowed to ascend between the poles of an electromagnet, it divided into two streams pushed out from the central region, if diamagnetic. Bubbles filled with various gases were attracted into or repelled from the central space between the poles. Oxygen is strongly paramagnetic, while nitrogen and hydrogen are diamagnetic.

Faraday was able to state the following general law:—'***Paramagnetic substances are those which tend to move from positions where the magnetic action is weaker to where it is stronger, i.e. from weaker to stronger parts of the field; diamagnetic substances tend to move from stronger to weaker parts of the field.***'

This law is further illustrated by an experiment devised by Plücker; it distinguishes paramagnetic liquids from those which are diamagnetic. A watch glass containing some liquid is placed on the poles of a strong magnet, the distance between the poles being about 2 cm.; if paramagnetic the liquid is attracted to, and if diamagnetic is repelled from regions where ∇H^2 is a maximum, cf. p. 430. Thus the free surface of the liquid is disturbed, as indicated in Fig. 13·01 (*a*) and (*b*). When the distance between the poles is decreased to about 2 mm. the field outside the region between the poles is relatively weak, and little effect, if any, is seen.

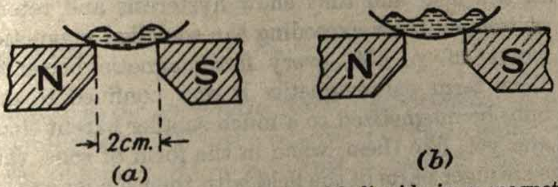

FIG. 13·01.—Plücker's experiments with liquids in a magnetic field.

Experiment.—The fact that liquid oxygen is very strongly paramagnetic is shown in a most spectacular and convincing manner in the following way. A gentle stream of liquid oxygen is poured between the horizontal pole pieces of an excited electromagnet, a field of about 8000 oersteds being required. After a few seconds some of the liquid oxygen fills part of the space between the poles and remains there in a very ebullient state; as the liquid evaporates a narrow column is left with its longer axis in the direction of the magnetic field.

If at any stage in the experiment the field is made negligibly small, the liquid falls rapidly and evaporates almost instantaneously when it comes into contact with a solid object.

Effect of the surrounding medium on the nature of a magnetizable material.—Faraday showed that a paramagnetic solution enclosed in a glass tube behaves as a paramagnetic if suspended in a weaker solution of the same kind and as a diamagnetic if suspended in a more concentrated solution. This is readily shown by filling a glass tube about 3 cm. long and 0·2 cm. internal diameter with a fairly strong aqueous solution of ferric chloride. When, by means of a long silk thread, this tube is suspended between the poles of an electromagnet so that it lies in a horizontal plane, then, in air, it sets with the axis of the tube parallel to the magnetic field. This direction is not changed when the tube is surrounded by a small beaker containing a weak aqueous ferric chloride solution; but if the beaker contains a highly concentrated solution of the same salt then the tube sets with its axis normal to the field.

Ferromagnetics, paramagnetics and diamagnetics.—FARADAY classified all materials as paramagnetics or as diamagnetics but modern practice classifies iron, nickel and cobalt and their alloys as ferromagnetics. To this latter list there may be added manganese and the so-called Heusler alloys which, while they contain no ferromagnetic element, yet have all the properties of a ferromagnetic body. Several other ferromagnetic materials, cf. p. 372, are also known. Ferromagnetics are to-day regarded as 'freak' substances—for their magnetization (J) does not increase linearly with the field and they show hysteresis and retentivity; moreover, at temperatures exceeding the so-called magnetic transformation point, cf. p. 436, every ferromagnetic becomes paramagnetic. The term paramagnetics is now confined to materials which can only be magnetized to a much smaller extent than ferromagnetics and yet, like them, when in the form of rods, they tend to set in the stronger parts of the field with their longer axes parallel to that part of the field. In paramagnetics the intensity of magnetization (J) is proportional to the strength of the field (H), i.e. J is a linear function of H. In 1895 CURIE showed that for a given field the mass susceptibility [cf. p. 421] for paramagnetics is very nearly inversely proportional to the absolute temperature; this statement is now known as ***Curie's law.***

For diamagnetics, Curie showed that the intensity of magnetization is very nearly independent of temperature. Bismuth, alone among the diamagnetic materials, appears to be an exception.

This marked distinction in the behaviour of paramagnetics and diamagnetics with respect to temperature suggests that paramagnetism is concerned mainly with molecular structure which varies with temperature whereas diamagnetism is concerned with atomic structure, i.e. with the configuration of the orbital electrons in an atom, and this is independent of temperature.

Magnetic susceptibility.—The equation expressing the relation between the three magnetic quantities, B, H and J, is

$$B = \mu_0 H + 4\pi J, \qquad \text{[cf. p. 363]}$$

and this on dividing throughout by H gives

$$\frac{B}{H} = \mu_0 + 4\pi\frac{J}{H},$$

$$\text{or} \quad \mu = \mu_0 + 4\pi\chi,$$

where μ is the permeability and χ the magnetic susceptibility of the material under consideration. Since J is the intensity of magnetization or the magnetic moment per unit volume, strictly speaking χ is the magnetic volume susceptibility. This is the quantity usually determined experimentally. In theoretical investigations the quantity χ_m appears; this is the magnetic moment per unit mass of substance divided by the strength of the field. If ρ is the density of the material, the mass of unit volume is ρ (numerically), so that if J_m is the magnetic moment per unit mass of substance,

$$J_m = \frac{J}{\rho},$$

and this gives

$$\chi_m = \frac{\chi}{\rho}.$$

χ_m is known as the ***magnetic mass susceptibility*** of the material. If χ_m is multiplied by M^*, the relative molecular mass† of the substance we obtain $\chi_m M^* = \chi_M$; this is termed the ***molar susceptibility*** of the material.

Experimental determination of the (volume) susceptibilities of paramagnetic and diamagnetic substances.—The experimental investigation of the magnetic susceptibilities of paramagnetic and diamagnetic substances is one of difficulty in comparison with similar measurements on ferromagnetic substances. The principal reason for this is that paramagnetic and diamagnetic substances are characterized by their feeble response even to high values of the magnetizing field; on the other hand, however, the demagnetizing factor, so marked with ferromagnetic materials, is of negligible account. For, in general,

$$\begin{aligned} B &= \mu_0 H + 4\pi J \\ &= \mu_0(H_a - DJ) + 4\pi J, \end{aligned}$$

where H_a is the applied field and D is a factor depending upon the shape of the body. When J is small in comparison with H_a we can neglect the demagnetizing factor D and use $H = H_a$.

If a small volume δv of a substance is placed in a non-uniform magnetic field, such as that existing between the poles of an electromagnet, then the specimen, if paramagnetic, will tend to move into the strongest part of the field and, if diamagnetic, into the weakest part. In deriving an expression for the force on the material let us assume that the magnetic field is everywhere parallel to a given direction in a fixed plane but possesses a gradient in a direction at right angles to the given direction. At a distance z from the given plane let H be the field strength parallel to the given direction; at a distance $z + \delta z$ let the field be $H + \delta H$. Then the force in the z-direction is given by

$$\delta F = \tfrac{1}{2}\,\delta v(\chi - \chi_a)\frac{\partial}{\partial z}(H^2),$$

where χ_a is the volume susceptibility of air.

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† M^* is used to avoid confusion with M, the moment of a magnet.

To prove this we proceed as follows. Suppose that a small specimen of volume δv is at a height z above a datum level, as shown in Fig. 13·02 (*a*), and it is moved to a height $z + \delta z$, the magnetic field changing from H to H + δH. Let δF be the force on the specimen and in the direction of z increasing due to its presence in the non-uniform field. Then since an equal volume of air replaces the specimen when its position is changed, the energy of the specimen in field H and of the air in field H + δH, which it will replace, is

$$-\frac{(\mu-\mu_0)\mathrm{H}^2}{8\pi}\delta v-\frac{(\mu_a-\mu_0)(\mathrm{H}+\delta\mathrm{H})^2}{8\pi}\delta v,$$

where μ is the permeability of the specimen, μ_a that of air and μ_0 that of free space—cf. p. 392.

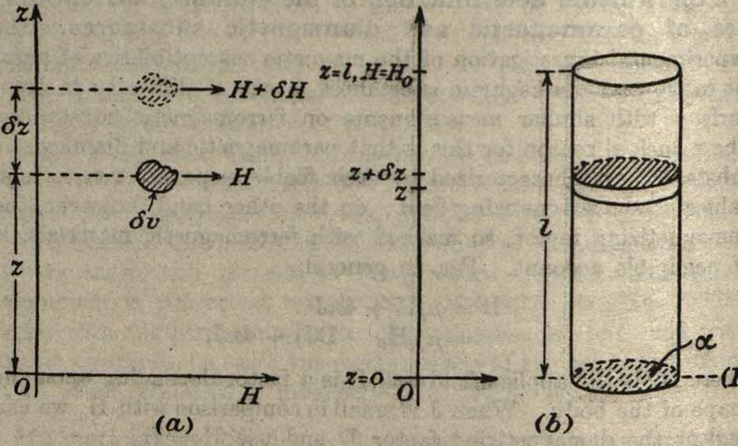

FIG. 13·02.—Forces on a body in a magnetic field.

In the final position, the energy of the specimen in field H + δH and of the air, in field H, which replaces it is

$$-\frac{(\mu-\mu_0)(\mathrm{H}+\delta\mathrm{H})^2}{8\pi}\delta v-\frac{(\mu_a-\mu_0)\mathrm{H}^2}{8\pi}\delta v.$$

Now $\delta\mathrm{F}(\delta z)$ = work done by the magnetic field in moving the specimen

= decrease in potential energy of the system

$$=(\mu-\mu_a)\frac{2\mathrm{H}\,\delta\mathrm{H}}{8\pi}\delta v$$

$$=\tfrac{1}{2}(\chi-\chi_a)\delta(\mathrm{H}^2)\delta v,$$

since $\mu = \mu_0 + 4\pi\chi$, etc. $\quad\therefore\ \delta\mathrm{F} = \delta v.\tfrac{1}{2}(\chi-\chi_a)\frac{\partial}{\partial z}(\mathrm{H}^2).$

Consider now a specimen in the form of a rod or bar of constant cross-sectional area α and length l hanging vertically downwards in a magnetic field, the axis of the rod being parallel to the z-axis and the field varying in the manner indicated in Fig. 13·02 (*b*). Then for the portion of the specimen lying between the planes z and $z + \delta z$, so that its volume is $\alpha\,\delta z$, we have

$$\delta F = \alpha\,\delta z.\tfrac{1}{2}(\chi - \chi_a)\frac{\partial}{\partial z}(H^2),$$

where δF is the force in the direction of z increasing. Thus F, the force on the complete rod (and, of course, in the direction of z increasing), is given by

$$\begin{aligned} F &= \tfrac{1}{2}\alpha\int_0^l(\chi - \chi_a)\frac{\partial}{\partial z}(H^2)dz \\ &= \tfrac{1}{2}\alpha\int_H^{H_0}(\chi - \chi_a)d(H^2) \\ &= \tfrac{1}{2}\alpha(\chi - \chi_a)(H_0^2 - H^2). \end{aligned}$$

If H_0^2 is negligible compared with H^2, as is usual in practice,

$$F = -\tfrac{1}{2}\alpha(\chi - \chi_a)H^2.$$

If $\chi > \chi_a$, the force is directed *downwards* when the lower end of the specimen lies in the strongest part of the field, and we have

$$|F| = \tfrac{1}{2}\alpha(\chi - \chi_a)H^2.$$

The above formula has been widely applied in the determination of the susceptibilities of paramagnetic and diamagnetic solids, liquids and gases; the accuracy of the investigation is mainly determined by the precision with which the force F can be measured.

Gouy's method for determining the susceptibility of paramagnetics and diamagnetics.—In this method a precision balance is used in conjunction with the magnetic field excited by a powerful electromagnet. The specimen, in the form of a cylinder of constant cross-sectional area, is suspended so that it hangs from beneath the pan of a sensitive balance. The axis of the cylinder, cf. Fig. 13·03 (*a*), is vertical and the lower end of the specimen lies in a strong homogeneous magnetic field while the upper end is in a field of negligible strength.

When the force on the specimen and due to the magnetic field is directed downwards it may be counterpoised directly by adding standard masses to the other pan; these masses will have to be removed if the material under investigation is diamagnetic.

Let m be the additional mass (considered algebraically) required to equilibrate the balance when the magnet is excited. Then if g is the intensity of gravity

$$|F| = mg = \tfrac{1}{2}\alpha(\chi - \chi_a)H^2.$$

Since the magnetic field in which the lower end of the specimen is situated is fairly uniform, its magnitude is most easily measured by means of a search coil in conjunction with a fluxmeter.

Liquids are placed in tubes as shown in Fig. 13·03 (*b*), and if the length of tube occupied is one half its total length and the space above the liquid is free from air, the term χ_a vanishes from the equation.

To prove this it is at once clear that the total pull on the glass container is zero provided it is placed symmetrically with respect to the magnetic field. To proceed let us suppose that a mixture of gas and vapour is present in the space above the solution ; let χ_g be its susceptibility. Then the force acting upwards on the solution is

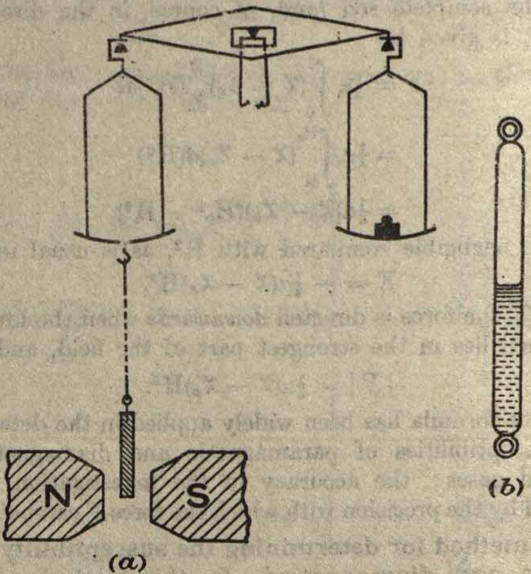

Fig. 13·03.—Gouy's method for measuring the magnetic susceptibility of a solid.

$$\tfrac{1}{2}\alpha(\chi - \chi_a)\mathrm{H}^2, \quad \uparrow$$

while that on the gaseous mixture is

$$\tfrac{1}{2}\alpha(\chi_g - \chi_a)\mathrm{H}^2, \quad \downarrow$$

and this acts in a downwards direction. The resultant force upwards is therefore only $\frac{1}{2}\alpha\chi\mathrm{H}^2$ provided χ_g is zero and this implies that the amount of gas and vapour above the liquid must be small, i.e. no gas must be present and the equilibrium pressure of the vapour must be low. For aqueous solutions, at room temperature, this latter condition is always fulfilled.

In connexion with this method it must be noted that the specimen should never have a rectangular cross-section for if the specimen is slightly disturbed it tends to touch the pole pieces of the magnet and, of necessity, these must be close together in order to obtain a strong magnetic field. Also if the specimen is metallic, the swing of the balance must be considerably restricted on account of the damping effect caused by the motion of a conductor in a magnetic field.

Quincke's method.—This method is applicable to the determination of the susceptibilities of aqueous solutions, many liquids and liquefied gases. It is particularly useful in investigating the variation in the susceptibility of aqueous solutions of iron salts with concentration. In such solutions the susceptibility is found to be independent of the concentration of the acid radicle but it is directly proportional to the iron content of the solution. The method is as follows:—A U-tube with one limb of wide bore and the other narrow is used. The axis of the U-tube is vertical and its narrow limb passes between the poles of an electromagnet as shown in Fig. 13·04. It is partly filled with the solution as indicated. To diminish any effects in the small tube due to a possible variation of the angle of contact between the solution and the glass about 1 cm.^{3} of 'wetting agent' per litre should be added; this does not alter the susceptibility appreciably. The initial position (no field) of the meniscus in the narrow limb must be adjusted so that, when the field is on, the meniscus lies in the region where the field is strongest. The field should be uniform throughout a small volume containing the meniscus so that its value may be determined with the aid of a search coil.

To discover how, when the magnetic field is applied, the difference in level of the two menisci is related to the field strengths at the menisci, it is convenient to make use of the conception of magnetostatic pressure as defined on p. 392 and of the theorem which follows. Thus from Fig. 13·04 (*b*), in which the variation in the shape and size of the cross-sectional area of the connecting tube is immaterial, we have, if P_0 is atmospheric pressure and surface tension effects are negligible, and if Z is the vertical distance between the levels of the menisci when the field is applied,

P_0 + pressure due to a column of liquid, length Z, in air
+ the increase in magnetostatic pressure from A to B
= the pressure at B, viz. P_0.

If ρ is the density of the solution, σ that of air and g the intensity of gravity, we have

$$g(\rho - \sigma)Z + \left[\frac{\mu - \mu_a}{8\pi}\right](H_B^2 - H_A^2) = 0,$$

so that if $\left(\frac{H_B}{H_A}\right)^2 \rightarrow 0$, and σ is small in comparison with ρ, then

$$g\rho Z = \frac{\mu - \mu_a}{8\pi} . H^2,$$

where H is written for H_A and μ_a may nearly always be taken as unity. From this equation we have

$$g\rho Z = \tfrac{1}{2}(\chi - \chi_a)H^2.$$

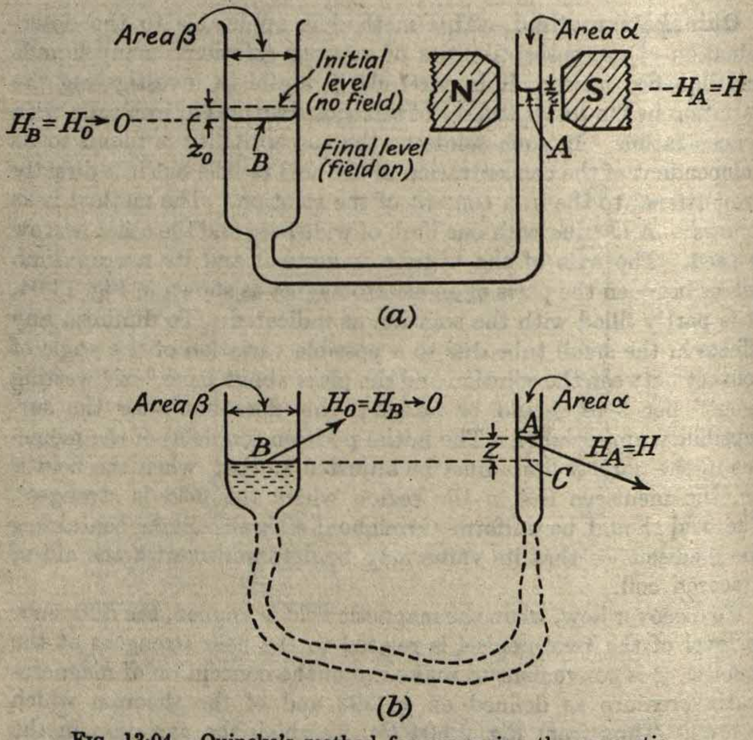

FIG. 13·04.—Quincke's method for measuring the magnetic susceptibility of a liquid.

Unfortunately Z cannot be measured directly but its value may be calculated in terms of observable quantities as follows.

If α is the cross-sectional area of the narrow limb and β that of the wide one, z the observed rise of the meniscus in the narrow tube and z_0 the fall (not observed) in the wide tube

$$\beta z_0 = \alpha z,$$

since the total volume of liquid is constant.

$$\therefore \; Z = (z + z_0) = z\left(1 + \frac{\alpha}{\beta}\right).$$

Hence $$\tfrac{1}{2}(\chi - \chi_a)H^2 = g\rho z\left(1 + \frac{\alpha}{\beta}\right).$$

Quincke's method has the advantage that no correction is necessary for the presence of dust particles even if their susceptibility is high, and no blank experiments are necessary to correct for the magnetic effect of the glass container.

If a solution of known susceptibility is available, Quincke's method, in reverse, provides a convenient means of measuring an unknown field in a narrow gap.

The magnetic susceptibility of an aqueous solution of nickel chloride.—In 1939 NETTLETON and SUGDEN made accurate determinations of the magnetic susceptibilities of various aqueous solutions of nickel chloride of known concentration in order that any one may be used as a standard. The reason for using such solutions is that the nickel content may be estimated very accurately by precipitation with dimethyl glyoxime and the molar susceptibility of nickel chloride is independent of the concentration of the solution used to measure it. The Gouy method was used but the magnetic field was measured by a null method; Fig. 13·05 (*a*) shows the essential features of the electrical circuit. By means of an ebonite plate E, resting on the pole-pieces of the electromagnet N–S, the search coil C is supported in a fixed position in the field, and connected in series with a Grassot fluxmeter F and the secondary coil of the standard mutual inductance M. The primary circuit is as shown; it contains a standard 1 *Ω* resistor so that the current in the primary circuit can be measured with the aid of a potentiometer. The mercury cup D is included in the circuit so that when the search coil C is suddenly raised it lifts the stiff wire W and breaks the circuit which includes the primary of the standard mutual inductance M; this produces a change in the linkages associated with the secondary of M and if the current in the primary of M is suitably adjusted the integrated effect of the two changes in magnetic flux is zero, i.e. the fluxmeter reading is not altered. The fluxmeter is therefore used as a null instrument; it is fitted with a mirror and used with a lamp and scale so that ample sensitivity is obtained.

If S is the effective area of the search coil in cm.^{2}, I amp. the balancing current, and M the mutual inductance in microhenries, the magnetic induction B associated with the field to be measured is given by

$$(M \times 10^{-6}) \times 10^9 \times \frac{I}{10} = BS,$$

i.e. $$B = \frac{100MI}{S}.$$

To determine the effective area of the search coil with as high an accuracy as possible, the circuit shown in Fig. 13·05 (*b*) is used. In principle the method involves the measurement of the mutual inductance between the search coil C and a pair of Helmholtz coils A, B. This is done by reversing, with the aid of the key K, the current which flows through the above coils and the primary

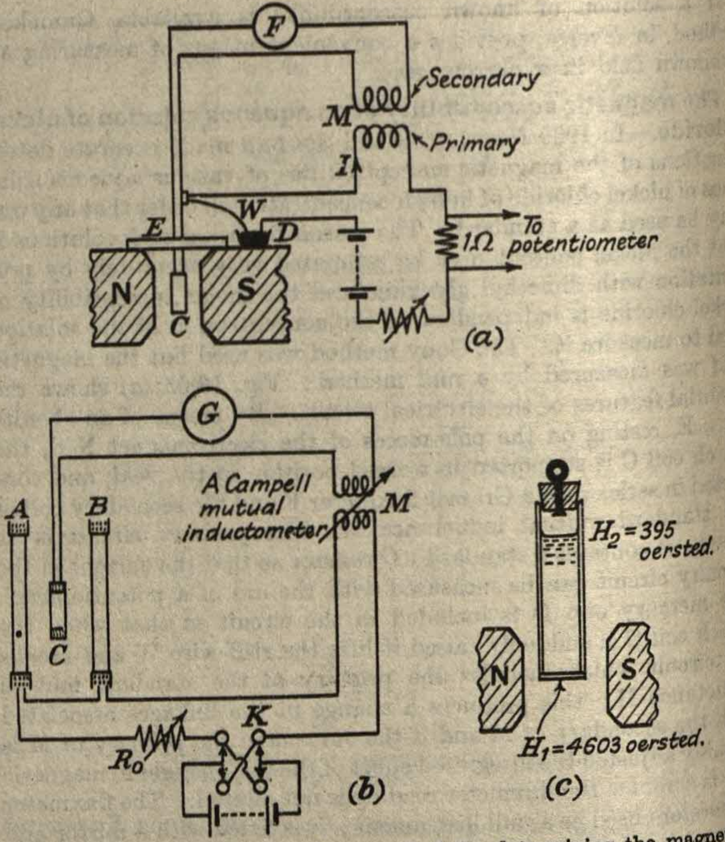

FIG. 13·05.—Nettleton and Sugden's method for determining the magnetic susceptibility of an aqueous solution.

of the Campbell mutual inductometer which is in series with them. The mutual inductance between the primary and secondary of this instrument is adjusted until reversal of the current gives no deflexion of the sensitive galvanometer G. A coil of measured area is then substituted for the search coil and the measurement repeated. Provided the field is uniform over the volume occupied by the search coil and by the 'area-coil', the areas will be proportional to the measu[illegible] mutual inductances.

To measure the susceptibility, the liquid is contained in a glass tube about 20 cm. long. Its lower end, cf. Fig. 13·05 (*c*), is ground flat and closed by a glass disc held in position by a trace of Apiezon wax. The upper end is closed by a rubber stopper provided with a glass hook to enable the tube to be suspended from one arm of a long-beam balance, standard masses being used to measure the pull due to the magnetic field ; the error is less than 10^{-5} gm.-wt. The pull on the empty tube is measured to correct the readings for the nickel chloride solution. With the usual notation

$$F = mg = \tfrac{1}{2}\alpha(\chi - \chi_a)(H_1^2 - H_2^2),$$

where H_1 is the field at the lower end of the liquid column. [In this work H and B were apparently taken to have the same numerical values.]

To determine the force on a small body situated in a non-uniform magnetic field.—In an earlier part of this chapter, cf. p. 422, an expression for the force in the *z*-direction on a small body in a magnetic field which was everywhere parallel to the *y*-direction was obtained. Before giving an account of Curie's classical experiments on magnetic susceptibility, it is necessary to generalize the above formula.

Let δF_y be the force in the *y*-direction on an element of volume δv of a specimen and suppose that it lies in air, of susceptibility χ_a and permeability μ_a. Then the potential energy of the element is

$$-\frac{(\mu - \mu_0)}{8\pi}(H_x^2 + H_y^2 + H_z^2)\delta v,$$

where μ is the permeability of the material of the body, μ_0 that of a vacuum, and $\vec{H} = \hat{\imath}H_x + \hat{\jmath}H_y + \hat{k}H_z$ is the magnetic field. If the element moves under the action of the force δF_y through a distance δy to a point where the field is $\hat{\imath}(H_x + \delta H_x) + \hat{\jmath}(H_y + \delta H_y) + \hat{k}(H_z + \delta H_z)$, then the energy of the specimen is

$$-\frac{(\mu - \mu_0)}{8\pi}[(H_x + \delta H_x)^2 + (H_y + \delta H_y)^2 + (H_z + \delta H_z)^2]\delta v.$$

Remembering that the initial and final values of the potential energy associated with the air which replaces the specimen are

$$-\frac{(\mu_a - \mu_0)}{8\pi}[(H_x + \delta H_x)^2 + (H_y + \delta H_y)^2 + (H_z + \delta H_z)^2]\delta v,$$

and $-\dfrac{(\mu_a - \mu_0)}{8\pi}[H_x^2 + H_y^2 + H_z^2]\delta v,$

Usually H_z and $\frac{\partial H_z}{\partial y}$ are of the order 10^4 oersteds and $0{\cdot}5 \times 10^4$ oersted.cm.$^{-1}$, so that any method making use of the above equation might appear to be less precise than the Gouy method since δF_y is small. Fortunately, however, this is not true since δF_y may be measured with a sensitive torsion balance, i.e. one for which the torsional constant is less than 1 erg. radian.$^{-1}$.

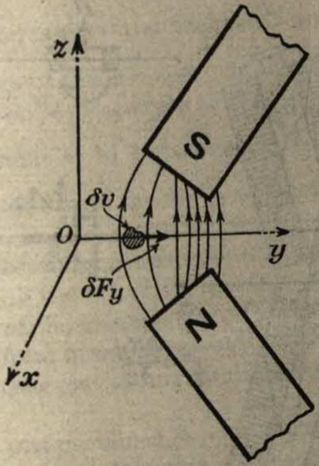

FIG. 13·06.—Force on a small body in a magnetic field whose configuration is as shown.

Curie's work on paramagnetics and diamagnetics from 25° C. to 1370° C.—CURIE's work on the 'Magnetic properties of materials at different temperatures' was carried out in 1895 and is worthy to take rank among the great classical experimental researches. The whole investigation* may be said to form the experimental foundation for much modern theoretical work on the phenomenon of magnetization.

Curie wished to determine whether dia-, para- and ferromagnetism were fundamentally distinct and this he did primarily by studying the effect of varying temperatures on the susceptibilities of the various types of magnetic material. Previous observations were insufficient to enable any generalizations to be made because only relatively small temperature ranges had been used. As being most convenient when high accuracy is not the paramount aim of the investigation, the Faraday method was used to measure the susceptibility of a material under given conditions. In this method a small specimen is suspended from a torsion arm at a point in a non-uniform magnetic field. The external couple required to hold the specimen at the point selected is then measured.

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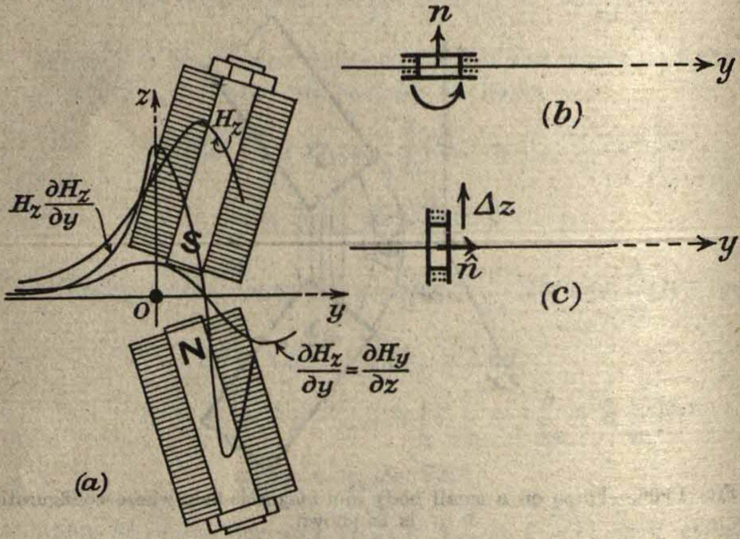
* *Ann. de Chim. et Physique*, 5, 289, 1895.

A diagrammatic plan of the magnets is shown in Fig. 13·07 (*a*). The specimen hangs at O. Since the pole pieces are flat and their axes inclined to one another, the lines of induction (force) are curved. A small body, of volume Δv, placed at O will tend to move in the y-direction under a force



n
y
(b)
Δz
y
$\hat{n}$
(c)
z
H_z
$H_z \frac{\partial H_z}{\partial y}$
S
O
y
N
$\frac{\partial H_z}{\partial y} = \frac{\partial H_y}{\partial z}$
(a)

FIG. 13·07.—Curie's apparatus for investigating paramagnetics and diamagnetics.

$$\Delta F_y = (\chi - \chi_a)\,\Delta v . H_z \frac{\partial H_z}{\partial y}.$$

To find H_z a small coil wound on a wooden former and connected to a ballistic galvanometer was used. This coil was placed at any point on Oy with its plane, i.e. its windings, parallel to that axis, cf. Fig. 13·07 (*b*), and rotated through an angle π; the throw gave H_z.

The direct measurement of $\frac{\partial H_z}{\partial y}$ by moving a search coil along Oy would involve measuring a small change in a large quantity; Curie tried this method but soon abandoned it. Instead, use was made of the fact that $\frac{\partial H_z}{\partial y} = \frac{\partial H_y}{\partial z}$, since each is equal to $-\frac{\partial^2 \Omega}{\partial y\, \partial z}$, where Ω is the magnetic potential. The coil was placed at a point on Oy, with its windings normal to Oy, cf. Fig. 13·07 (*c*), then moved a short distance Δz, i.e. parallel to Oz, thus giving $\frac{\partial H_y}{\partial z}$.

After this procedure had been carried out at many points along Oy, curves were drawn, as in Fig. 13·07 (a), to show how H_z, $\frac{\partial H_z}{\partial y}$ and $H_z\frac{\partial H_z}{\partial y}$ varied along Oy. The specimen was placed in the region where $H_z\frac{\partial H_z}{\partial y}$ was a maximum, so that small errors in the adjustment of its position were of less account. It should be noted, however, that owing to the finite dimensions of the search coils it was difficult to be exactly certain as to corresponding points for H_z and $\frac{\partial H_z}{\partial y}$.

The displacement due to the forces acting on the specimen never exceeded 0·15 cm. so that it was not necessary to restore the specimen to its zero position; it was sufficient to observe the deflexion. This was done with the aid of a microscope and a pointer attached to the other end of the torsion arm. The couple per unit deflexion was determined from observations on the time of vibration of the system when its moment of inertia about the axis of rotation was known.

The specimen was contained in a small glass bulb surrounded by an electric furnace and this was water-jacketed externally. To correct for the force on the bulb and supporting rod, the force was measured when the bulb was empty and also when it contained the substance under investigation.

The Sucksmith ring balance for the measurement of susceptibility.—The method most generally useful for the measurement of magnetic susceptibility depends upon that of Faraday; in this the force on a small specimen placed in a non-uniform magnetic field is measured. If an isotropic substance of volume Δv and volume susceptibility χ is placed in a non-uniform magnetic field $\vec{H}$ which is everywhere parallel to the xOy plane, i.e.

$$\vec{H} = \hat{\imath}(0) + \hat{\jmath}(H_y) + \hat{k}(0),$$

but has a finite gradient $\frac{\partial H_y}{\partial z}$ in a direction parallel to Oz, the force on the specimen is given by

$$\Delta F_z = (\chi - \chi_a)H_y\frac{\partial H_y}{\partial z}.\Delta v,$$

where χ_a is the volume susceptibility for air.

In the early experiments of Curie the substance was placed on the arm of a torsion balance and the force determined by the method already described on p. 431. The best position for the specimen tends to move towards a pole-piece. To eliminate this a single cocoon fibre is fixed between two adjustable rods R. These enable the fibre to be displaced until it touches the suspension ; it is fixed in position by a speck of shellac and does not alter the sensitivity appreciably.

Curie's law.—Curie discovered experimentally that for many paramagnetic substances

$$\chi_m = \frac{A}{T}, \quad \text{or} \quad \chi_M = \frac{C}{T},$$

where A and C are constants, and C = AM*, if M* is the relative molecular mass of the substance. The latter equation implies that the molar susceptibility of a specimen varies inversely as the absolute temperature; this is the ***Curie-Langevin law.*** For oxygen the law holds exactly, but the molar paramagnetic susceptibilities of other substances tend to follow the law

$$\chi_M = \frac{C}{T - T_c}.$$

This is the ***Curie-Weiss law*** and the parameter T_c is known as the ***Curie point*** or, more precisely, the ***paramagnetic Curie point.*** To compute T_c, χ_M^{-1} is plotted against T and the graph is assumed to be linear.

The Curie-Weiss law is sometimes written

$$\chi_M = \frac{C}{\theta - \theta_c},$$

where θ is the temperature in degrees centigrade.

[N.B. The ***ferromagnetic Curie point,*** T_f, better known as the ***magnetic transformation point,*** is that temperature at which the substance loses its spontaneous magnetization, whereas T_c is a parameter occurring in the Curie-Weiss equation which represents the behaviour of a substance at temperatures well above T_f. For iron and steel there is a difference of about 15 degrees between the two temperatures.]

A simple oscillation experiment for the determination of T_f for nickel-steel.—This method is due to Bates, Gibbs and Reddi Pantalu (1936). The time of swing, t, of a ferromagnetic specimen in a field of strength H is given by

$$t = 2\pi\sqrt{\frac{I_0}{MH}},$$

where M is the magnetic moment of the specimen and I_0 its moment of inertia about the axis of suspension. If the specimen changes its magnetic condition so that M decreases there will be a corresponding increase in the period of oscillation. Thus the principle of this experiment is to determine the temperature at which the time of swing of a short nickel rod placed horizontally in a field of constant strength suddenly increases, i.e. M → 0.

The specimen, AB, Fig. 13·09, may be in the form of a cylinder 5 cm. long and 0·5 cm. in diameter. It is suspended horizontally in an electric furnace F by means of a thin phosphor-bronze wire W. A light concave mirror M is rigidly attached to a copper rod which connects the suspension W to the specimen. In this way the oscillations may be observed by a lamp and scale method. An electromagnet NS is placed centrally beneath the furnace.

AB is first given a small angular displacement and the current through the coils of the electromagnet NS adjusted so that the time of swing at room temperature is about one second. The furnace is then allowed to heat up to 225° C., the period of the magnet being determined at 50 deg. C. intervals. The heating current is then reduced so that the rate of temperature rise is about 0·5 deg. C. per minute.

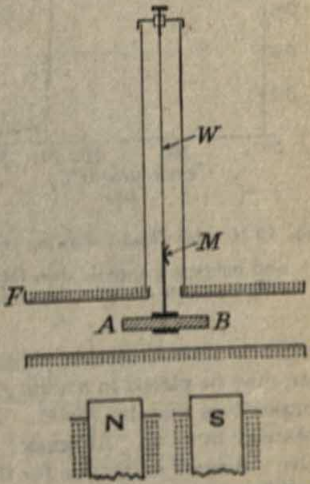


Fig. 13·09.—Bates' method for determining the magnetic transformation point for a nickel steel.

Observations of the period are then taken at intervals of five degrees until the temperature is about 235° C. when a slight increase in period is noticed. This temperature will vary with the type of nickel steel used. The heating current is then reduced so that the temperature remains constant at 235° C. and the period observed after 15 minutes. The temperature must then be allowed to rise very slowly (the rate must not exceed 0·2 deg. C. min.$^{-1}$) and the period determined at intervals of approximately 1 deg. C. Such a method must be adopted in order that the specimen may acquire the temperature of the furnace between each reading. At temperatures above 255° C. the rate of heating may be increased. Observations should also be made as the furnace is allowed to cool.

By plotting t^{-2} against temperature a graph similar to that shown in Fig. 13·10 (a) will be obtained. The magnetic transformation point is $(237 \pm 3)°$ C. Fig. 13·10 (b) shows how t^2 varies with the temperature above the magnetic transformation point and from this graph we deduce that χ^{-1} is a linear function of temperature.

[For pure nickel the paramagnetic Curie point is 376° C., provided the temperature of the specimen is within the range 480° C.–880° C.]

A similar experiment may be made with Monel metal. The magnetic transformation point is below 100° C. so that the specimen may be placed in a wide glass tube surrounded by water whose temperature may be varied. The necessary magnetic field may be obtained from an 'Alcomax' magnet fixed in the bottom of the tube. A small correction for the variation of the magnetic moment of this magnet with temperature must be made.

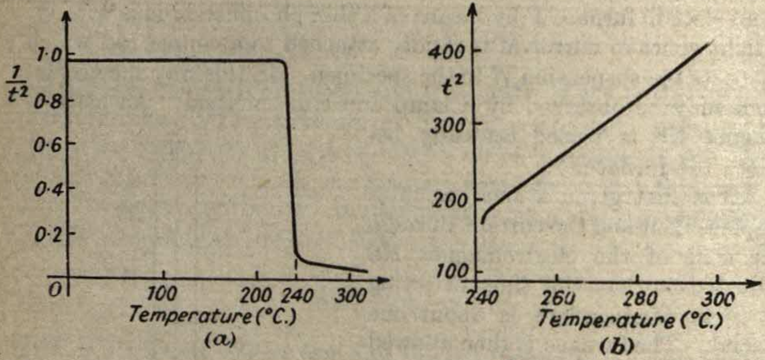


FIG. 13·10.—(a) Graph showing relation between $\frac{1}{t^2}$ and temperatures above and below a magnetic transformation point. (b) Graph showing relationship between t^2 and temperature above the same point.

Ferromagnetic gadolinium.—Metallic gadolinium was first isolated in 1935; it was prepared from its chloride, some 0·12 gm. with a silicon impurity of less than 1 per cent being obtained. Its ferromagnetic Curie point is 16° C. Between 90° and 360° C. it follows a Curie-Weiss law with $T_c = 302{\cdot}6°$ K., i.e. 29·5° C.

In the Curie-Weiss equation for paramagnetics

$$\chi_M = \frac{C}{T - T_c},$$

T_c may be positive or negative, and those paramagnetics which are ferromagnetic at ordinary temperatures are characterized by the fact that for them T_c is large and positive.

Gadolinium provides the only known example of an element ch has a magnetic transformation point in the region of room rature.

The correction for ferromagnetic impurities in a specimen whose magnetic susceptibility is being determined.—Since very large magnetic fields (7–25 kilo-oersted) are employed in a determination of the magnetic susceptibility of a paramagnetic or diamagnetic substance, HONDA and OWEN assume that any ferromagnetic impurity present is magnetized to saturation. In such an instance it seems legitimate to assume that the 'measured' or 'observed' intensity of magnetization, $J_{\text{obs.}}$, is given by

$$J_{\text{obs.}} = J + (J_{\text{Fe}})_{\text{sat.}},$$

where J is the intensity of magnetization for the pure substance and $(J_{\text{Fe}})_{\text{sat.}}$ is that of the ferromagnetic impurity. Now $(J_{\text{Fe}})_{\text{sat.}}$ is constant, while J is a linear function of the magnetizing field H; i.e. $\frac{J}{H} = \chi$, the volume susceptibility of the pure substance.

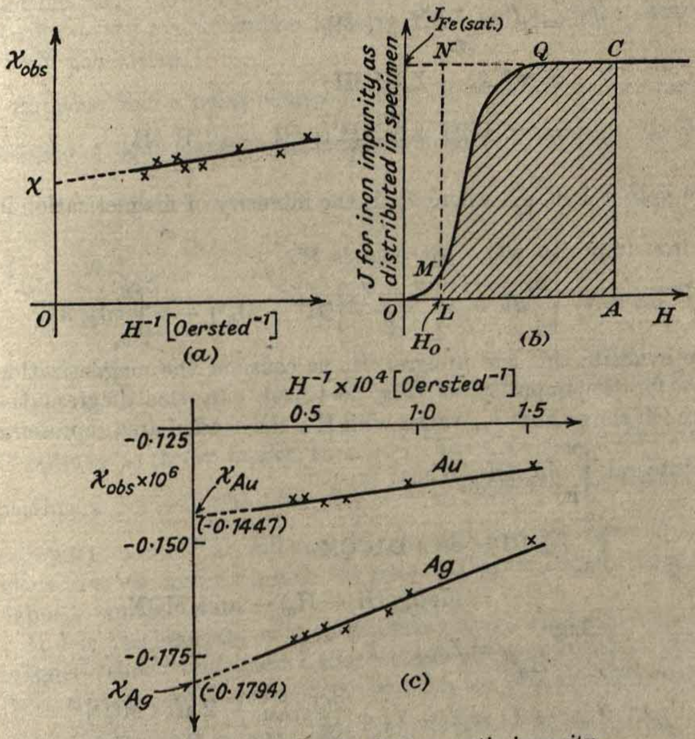


FIG. 13·11.—Correction for ferromagnetic impurity.

$$\therefore \chi_{\text{obs.}} = \chi + \frac{(J_{\text{Fe}})_{\text{sat.}}}{H}.$$

Hence, if H^{-1} is plotted against $\chi_{\text{obs.}}$, as in Fig. 13·11 (*a*), the intercept of the straight line on the $\chi_{\text{obs.}}$-axis gives χ, the quantity required.

Vogt has shown that this last expression does not apply to observations made in the Gouy method. The equation is only valid for those methods in which the test object lies wholly in a magnetic field of such strength that the saturation of the iron impurity is complete. In the Gouy method this state of affairs is not attained, for one end of the specimen lies in a magnetic field which is very small. The correction must then be made as follows :—If α is the cross-sectional area of the (uniform) rod, then, as on p. 422,

$$\begin{aligned}\delta F &= \alpha\frac{\mu_{\text{obs.}} - \mu_a}{8\pi}.2H\,\delta H\\ &= \alpha(\chi_{\text{obs.}} - \chi_a)H\,\delta H\\ &= \alpha\chi H\,\delta H + \alpha\frac{(J_{Fe})}{H}H\,\delta H - \alpha\chi_a H\,\delta H,\end{aligned}$$

since $\chi_{\text{obs.}} = \chi + \frac{J_{Fe}}{H}$, where J_{Fe} is the intensity of magnetization in the iron impurity when the field is H.

$$\therefore mg = -\int_H^{H_0} dF = \alpha\left[\frac{\chi - \chi_a}{2}\right](H^2 - H_0^2) + \alpha\int_{H_0}^{H} J_{Fe}\,dH.$$

To evaluate this last integral, let us consider the magnetization curve for the impurity, cf. Fig. 13·11 (*b*). In this diagram the curve OC shows how J_{Fe} varies with H ; the shaded area represents the integral $\int_{H_0}^{H} J_{Fe}\,dH$. Thus

$$\begin{aligned}\int_{H_0}^{H} J_{Fe}\,dH &= \text{area LACQM}\\ &= (J_{Fe})_{\text{sat.}}(H - H_0) - \text{area MQN}.\end{aligned}$$

But $$\frac{2mg}{\alpha(H^2 - H_0^2)} = \chi_{\text{obs.}} - \chi_a.$$

$$\therefore \chi_{\text{obs.}} - \chi_a = \chi - \chi_a + \frac{2(J_{Fe})_{\text{sat.}}}{H + H_0} - \frac{2\text{ area MQN}}{H^2 - H_0^2}.$$

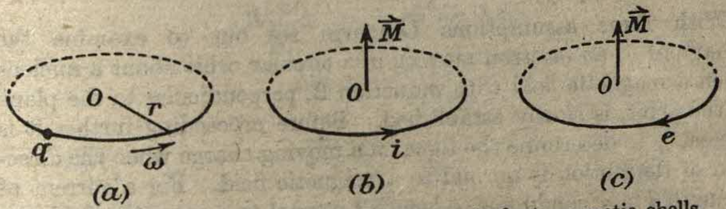
Since the last term in the expression is zero in all practical cases, we may write, if $H_0 \to 0$,

$$\chi_{\text{obs.}} = \chi + \frac{2(J_{Fe})_{\text{sat.}}}{H}.$$

If $\chi_{\text{obs.}}$ is plotted against H^{-1}, χ is readily obtained. Fig. 13·11 (*c*) shows the results obtained in this way by Vogt for gold and for silver.

Langevin's theory of diamagnetism.—The interpretation of the fact that all materials can acquire a certain degree of magnetization as an atomic phenomenon began when, in 1854, WEBER introduced his idea of molecular currents. In this theory he assumed that Ampèrian currents circulated within the molecules. It was not until 1905 that LANGEVIN, adopting the electron theory, as proposed by LORENTZ, gave a satisfactory interpretation of the processes at work. According to Lorentz all atoms consist of assemblages of electrons rotating in closed orbits about a centre of force and every rotating electron behaves like a uniform magnetic shell whose moment may be determined as follows. The theory now given attributes the magnetization to the orbital motion of the electrons within the atoms.

Suppose that a point charge q e.s.u., i.e. $\frac{q}{(c)}$ e.m.u., where c is the velocity of light, rotates with constant frequency f in a circular orbit



(a) (b) (c)

FIG. 13·12.—Rotating charges and their equivalent magnetic shells.

of radius r as shown in Fig. 13·12 (a). Then the charge per second passing a point in the orbit is $\frac{q}{(c)}f = i$, where i is the equivalent current in e.m.u. It will now be assumed that the rotation takes place in a vacuum for which the permeability is μ_0; its numerical value is unity.

If J is the intensity of magnetization in the equivalent uniform magnetic shell of thickness t, the strength of the shell is $\mathrm{J}t = \mu_0 i$. If $\mathrm{A} = \pi r^2$ is the area of the shell, since J is the pole strength per unit area, the magnetic moment of the shell is $\mathrm{AJ}t = \mathrm{A}\mu_0 i = \pi r^2 \mu_0 \frac{q}{(c)} \frac{\omega}{2\pi}$, where ω is the angular velocity of the charge. Thus

$$\mathrm{M} = \frac{1}{2}\mu_0 \cdot \frac{q}{(c)} \cdot r^2 \omega,$$

where M is the magnetic moment. Strictly speaking this is a vector, viz.

$$\vec{\mathrm{M}} = \hat{n}\frac{1}{2}\mu_0 \cdot \frac{q}{(c)} \cdot r^2 \omega,$$

where $\hat{n}$ is a unit vector whose direction bears the same relation to the current as does translation to rotation in a right-handed screw motion. Fig. 13·12 (*b*) shows the relation between $\overrightarrow{\mathrm{M}}$ and the direction of the current.

For an electron the direction of its motion is related to the magnetic moment of its equivalent shell as shown in Fig. 13·12 (*c*).

Now the motion of any particle of mass m within an orbit is determined by the equation

$$m(\ddot{r} - r\dot{\theta}^2) = \text{outward force},$$

and for motion in a circle, $\ddot{r} = 0$, so that the outward force on the particle is $-mr\dot{\theta}^2 = -mr\omega^2$. Hence for an electron moving in a circular orbit round a positively charged nucleus,

$$\begin{aligned} mr\dot{\theta}^2 &= mr\omega^2 \\ &= \text{force of attraction on the electron due to the nucleus} \\ &= \mathrm{F}\ \text{(say)}. \end{aligned}$$

With these assumptions Langevin set out to examine the behaviour of an electron moving in a circular orbit about a nucleus when a magnetic field with induction B, perpendicular to the plane of the orbit, is slowly established. Before proceeding further it is necessary to determine the force on a moving charge when the direction of its motion is normal to a magnetic field. For a current of strength i in a conductor of length l normal to a magnetic field, the force is Bil. If there are n charges per unit path, each moving with velocity u, and carrying a charge q e.s.u., the equivalent current in e.m.u. is given by

$$i = \frac{nqu}{(c)},$$

since the charge passing per unit time is nqu. Hence the force on all the charges in unit length of the path is

$$\mathrm{B}\left[\frac{nqu}{(c)}\right].$$

In this path there are n charges, so that the force on a single charge q is

$$\mathrm{B}\left[\frac{q}{(c)}\right]u.$$

Now suppose that when q is the rotating point charge and $\overrightarrow{\mathrm{M}}$ is directed as in Fig. 13·13 (*a*), a magnetic field H is applied where $\overrightarrow{\mathrm{H}}$ is parallel to $\overrightarrow{\mathrm{M}}$; let $\overrightarrow{\mathrm{B}}$ be the induction associated with this field. Then the force on the moving charge, due to its presence in the magnetic field, is $\mathrm{B}\frac{q}{(c)}u$ or $\mathrm{B}\frac{q}{(c)}.r\omega$, where r is the radius and

ω the angular velocity; this force is in the orbital plane and is directed away from the nucleus.

We have already seen that when the moving charge is an electron, e, it must rotate in an opposite sense, i.e. as in Fig. 13·13 (b), in order that $\vec{\mathrm{M}}$ may be directed as before. Suppose that the motion of the electron still takes place in a magnetic field $\vec{\mathrm{H}}$ parallel to $\vec{\mathrm{M}}$. The force on e due to $\vec{\mathrm{B}}$ will still be directed away from the nucleus since both the sign of the charge and the direction of its rotation have been reversed. To maintain equilibrium under these circumstances let the angular velocity of the electron become $\omega + \Delta\omega$. Thus the force on the electron, directed outwards along $\hat{r}$, is

$$-\mathrm{F} + \mathrm{B}\frac{|e|}{(c)}\omega r = [-mr(\omega + \Delta\omega)^2].$$

But $-\mathrm{F} = -mr\omega^2$, so that if $\Delta\omega$ is small, we have

$$\mathrm{B}\frac{|e|}{(c)}\omega r = -mr(2\omega\,\Delta\omega),$$

$$\therefore\ \Delta\omega = -\frac{1}{2}\frac{\mathrm{B}}{(c)}\left|\frac{e}{m}\right|.$$

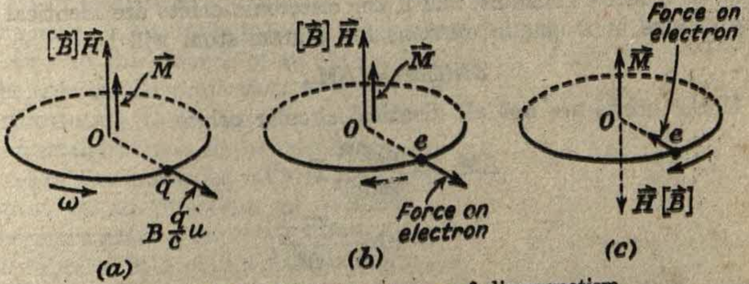


FIG. 13·13.—Langevin's theory of diamagnetism.

This causes a variation in the magnetic moment associated with the electron equal to

$$\Delta\mathrm{M} = \frac{1}{2}\mu_0\cdot\left[\frac{|e|}{(c)}\right]r^2\left\{-\frac{1}{2}\frac{\mathrm{B}}{(c)}\frac{|e|}{m}\right\},\ \text{since } \mathrm{M} = \frac{1}{2}\mu_0\cdot\frac{q}{(c)}\cdot r^2\omega$$

$$= -\frac{\mu_0 e^2\mathrm{B}}{4m(c)^2}\cdot r^2$$

Thus

$$\vec{\Delta\mathrm{M}} = -\hat{n}\frac{\mu_0 e^2\mathrm{B}}{4m(c)^2}\cdot r^2$$

$$= -\frac{\mu_0 e^2 r^2}{4m(c)^2}\cdot\vec{\mathrm{B}},$$

i.e. the change in $\vec{\mathrm{M}}$ is opposite to $\vec{\mathrm{H}}$.

If there is more than one electron in the atom, but all are in circular orbits normal to the applied magnetic field,

$$\Delta M = -\frac{e^2 B}{4m(c)^2}\Sigma r^2.$$

If the field $\vec{H}$ and $\vec{M}$ are antiparallel, as in Fig. 13·13 (*c*), and ω and $\omega + \Delta\omega$ are the angular velocities without and with the field respectively, then

$$-F - B\frac{|e|}{(c)}\omega r = -mr(\omega + \Delta\omega)^2,$$

since the direction of the force on the electron due to its motion in a magnetic field is now reversed. In this instance

$$\Delta M = \mu_0\frac{e^2 B}{4m(c)^2}.r^2,$$

so that the change in $\vec{M}$ is again opposite to $\vec{H}$.

In an atom with atomic number Z there are Z electrons, i.e. electrons in Z orbits will each contribute to the resultant magnetic moment of the atom. Now a gram-atom contains N atoms, where N is Avogadro's constant, and if the electronic orbits are identical the change in magnetic moment for a gram-atom will be

$$ZN(\Delta M) = \Delta M_{at}.$$

If the orbits are not all identical circular orbits,

$$\Delta M_{at} = \frac{\mu_0 e^2 B}{4m(c)^2}\Sigma\,(r^2)$$

$$= ZN\frac{\mu_0 e^2 B}{4m(c)^2}\overline{r^2},$$

where $\overline{r^2}$ is a suitable mean value.

For an orbit inclined at an angle θ to the applied field, the change in magnetic moment is

$$\Delta M = -\mu_0\frac{e^2 B}{4m(c)^2}r^2\cos\theta.$$

Resolving the vector $\overrightarrow{\Delta M}$ in the direction of $\vec{B}$ brings in an additional factor $\cos\theta$; hence

$$\Delta M_{at} = -\mu_0\frac{ZNe^2 B}{4m(c)^2}.\overline{r^2\cos^2\theta}.$$

To-day there is little value in determining $\overline{\cos^2\theta}$ since the classical theory assumes that all values of θ are possible, i.e. θ is continuously variable. From the point of view of the quantum theory this is no longer permissible, but the treatment given here is adequate to account for the susceptibility of a diamagnetic substance. In passing we note that

$$\chi_{at} = \frac{\Delta M_{at}}{H}.$$

$$\therefore \chi_m = \frac{\chi_{at}}{A^*} \quad \text{and} \quad \chi = \rho\frac{\chi_{at}}{A^*},$$

where χ_m is the mass susceptibility, A^* the relative atomic mass and ρ the density of the element considered.

Langevin's theory of a paramagnetic gas.—In 1905 Langevin applied the statistical kinetic theory to obtain an expression for the mass susceptibility of an ideal gas which is also paramagnetic. To develop this theory, which has been revised and finally replaced by quantum theory, it is necessary, with Langevin, to consider that each molecule is endowed with a permanent magnetic moment $\vec{M}$, and suppose that the state of magnetization in a substance is determined by two factors only, (*a*) the applied magnetic field which tends to produce alignment of the magnetic axis of each molecule in a given direction, and (*b*) the thermal agitation which endeavours to disorganize the orderly state so produced. On the whole a kind of statistical equilibrium is set up, in which the majority of the molecules contribute to the magnetization parallel to the applied field. In the absence of an external field the sum of the projections of the vectors representing the magnetic moments on any straight line is zero, since the magnetic axes of the molecules of a gas are distributed in all directions at random. If thermal agitation could be eliminated then any applied field would orientate the molecules so that their magnetic axes were at once parallel to the field, i.e. a state of magnetic saturation would be attained even in a weak field. If both field and thermal agitation operate simultaneously, the intensity of magnetization in the gas will increase with the field and decrease with temperature.

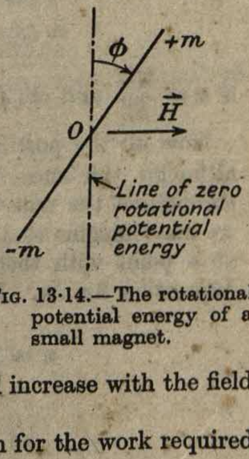


Fig. 13·14.—The rotational potential energy of a small magnet.

It is first necessary to obtain an expression for the work required to rotate an elementary magnet of moment $\vec{M}$ in a magnetic field $\vec{H}$; this will be called its rotational potential energy in order to avoid confusion with the potential energy of a magnet as derived on p. 388. Let the initial position of the magnet, defined by its magnetic moment vector, be at right angles to $\vec{H}$; its rotational potential energy is then taken to be zero. When the magnet is in a position such that its axis makes an angle ϕ with the zero position, cf. Fig. 13·14, the energy which must be supplied to return the magnet to its zero position is given by MH sin ϕ.

$\therefore$ [Rotational potential energy in position 'ϕ'] + MH sin $\phi = 0$,

i.e. the rotational potential energy of the magnet in position 'ϕ' is $-$ MH sin ϕ.

Now the energy in a given system is distributed, according to Maxwell, by the equation

$$\delta n = \mathrm{A} \exp\left(-\frac{\mathrm{E}}{k\mathrm{T}}\right).\delta\mathrm{E},$$

where δn is the number of 'particles' with energy between E and E + δE, A a constant, T the absolute temperature and k Boltzmann's constant, viz $\frac{\mathrm{R}}{\mathrm{N}}$, where R is the universal gas constant and N is Avogadro's constant.

If n is the number of elementary magnets in unit volume, and we take $\mathrm{E} = -\mathrm{MH}\sin\phi$, then

$$\delta n = \mathrm{A}\left\{\exp\left(\frac{\mathrm{MH}}{k\mathrm{T}}\sin\phi\right)\right\}\left\{-\mathrm{MH}\,\delta(\sin\phi)\right\}$$

$$= \mathrm{A}(-\mathrm{MH})\exp(\alpha x)\delta x,$$

if $\alpha = \frac{\mathrm{MH}}{k\mathrm{T}}$ and $\sin\phi = x$.

Now all the possible values of ϕ lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, although the axes of the elementary magnets are distributed throughout the unit volume considered. In order to determine A we may imagine that all the magnets have been collected together at a point with their axes in a plane. This justifies the above expression for δn. Integrating we get

$$n = \mathrm{A}(-\mathrm{MH})\int_{-1}^{+1}\exp(\alpha x)dx$$

$$= -\mathrm{AMH}\left[\frac{\exp(\alpha) - \exp(-\alpha)}{\alpha}\right].$$

Thus
$$\mathrm{A} = -\frac{\alpha n}{\mathrm{MH}}\left[\frac{1}{\exp(\alpha) - \exp(-\alpha)}\right].$$

To calculate J, the intensity of magnetization, we must note that an elementary magnet in the position 'ϕ' will contribute an amount

M sin ϕ to J. Thus

$$
\begin{aligned}
J &= \int_0^n M \sin\phi\, dn \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} M \sin\phi . \left(\frac{-\alpha n}{MH}\right)\left[\frac{1}{\exp(\alpha) - \exp(-\alpha)}\right] \exp(\alpha \sin\phi)(-MH)d(\sin\phi) \\
&= \frac{\alpha M n}{\exp(\alpha) - \exp(-\alpha)} \int_{-1}^{+1} x \exp(\alpha x) dx \\
&= \ldots \left[x . \frac{\exp(\alpha x)}{\alpha} - \frac{\exp(\alpha x)}{\alpha^2}\right]_{-1}^{+1} \\
&= \ldots \left[\frac{\exp(\alpha x)}{\alpha^2}(\alpha x - 1)\right]_{-1}^{+1} \\
&= \ldots \left[\frac{\exp(\alpha) + \exp(-\alpha)}{\alpha} - \frac{\exp(\alpha) - \exp(-\alpha)}{\alpha^2}\right] \\
&= Mn\left[\coth\alpha - \frac{1}{\alpha}\right].
\end{aligned}
$$

Now $Mn = J_0$, the saturation value of the intensity of magnetization, so that

$$\frac{J}{J_0} = \coth\alpha - \frac{1}{\alpha} = L(\alpha), \text{ say,}$$

where $L(\alpha)$ is known as the ***Langevin function.***

This function $L(\alpha)$ may be plotted against α as shown in Fig. 13·15.

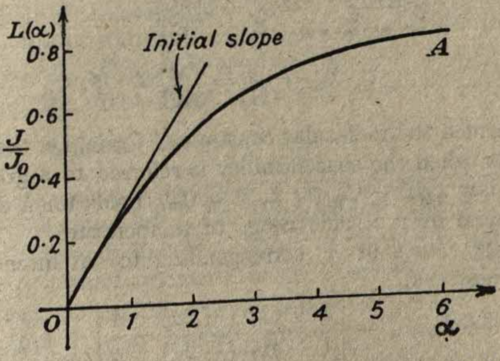


FIG. 13·15.—The Langevin function. $L(\alpha) = [\coth\alpha] - \frac{1}{\alpha}$.

For a fixed value of T it is equivalent to plotting $L(\alpha)$ against H, since α is proportional to H. Thus for a paramagnetic gas the curve gives the fractional saturation as a function of the applied low temperatures. Although gadolium sulphate $Gd_2(SO_4)_3.8H_2O$ behaves in this way, the results do not fit the Langevin theory; the quantum theory comes to the rescue.

The molecular field theory of Weiss.—By postulating a 'molecular field', WEISS has extended the ideas of Langevin to include paramagnetics generally and, in particular, ferromagnetics. In this he was guided by the method used by van der Waals to develop a kinetic theory of fluids by extending the ideas which Bernoulli had applied to an ideal gas. Just as in the case of a gas, to account for the transition to the liquid state, there must be added to the external pressure an internal pressure due to the mutual attractions between molecules, so with a ferromagnetic substance, as it is cooled in a magnetic field from a temperature above that which has rendered it paramagnetic, the transition to a ferromagnetic state is explained by assuming that, due to the overlapping of the fields of individual molecules, there comes into existence an internal or molecular field which, added to the external field, accounts for the very large values of J which are characteristic of this state.

Weiss assumed that the overlapping of the fields of the molecules, existing in a given region, is equivalent to a uniform field proportional to J and parallel to it. Thus if the molecular field $H_M = \eta J$, where η is a constant, the effective field is $H + \eta J$.

Here we cannot discuss Weiss' theory in detail; we shall have to be content to quote the result, viz.

$$\chi_M = \frac{C_M}{T - T_c},$$

where T is the absolute temperature and C_M and T_c are constants characteristic of the materials; T_c is the *paramagnetic Curie-point.*

In other words, the substance obeys a Curie-Weiss law at temperatures above T_c. [Thus, if temperatures are measured from the Curie-point, the Curie-Weiss law simulates the Curie law for paramagnetics. A linear variation of $\frac{1}{\chi_M}$ with T is observed for many paramagnetics over considerable ranges of temperature in accordance with the above law. The extrapolated value of T_c is sometimes negative (corresponding to a negative value for η, and so incompatible with a purely magnetic origin of the molecular field). When χ_M has been measured over a range of temperature with $T > T_c$, both C_M and T_c may be determined.]

The domain theory of magnetization in ferromagnetic materials.—In developing his theory of ferromagnetism WEISS was led to the conclusion that at any temperature below the magnetic transformation point any ferromagnetic material is saturated magnetically to a value appropriate to that temperature;

this is true even when the applied external field is zero. To account for the fact that a ferromagnetic material in the demagnetized state does not appear to be magnetized at all, Weiss found it necessary to postulate that, although the substance is everywhere magnetically self-saturated, it is divided into small regions in which the directions of the magnetization vectors are so orientated that the resultant intensity of magnetization is zero. Each such region is known as a ***domain,*** and in each domain the material is magnetized to saturation but the directions of magnetization for different domains are not necessarily parallel. When some of the magnetization vectors are caused to align themselves more or less parallel to an applied field, the specimen exhibits bulk magnetization.

In 1949 Kittel[*] wrote that the essential aspects of ferromagnetism are illustrated by the implications of the following experimental fact: ***It is possible to change the over-all magnetization of a suitably prepared ferromagnetic specimen from an initial value of zero (in the absence of an applied magnetic field) to a saturation value of the order of 1000 gauss, by the application of a field whose strength may be of the order of 0·01 oersted.***

Such a magnetization curve, derived from a B–H loop obtained by Williams and Shockley in 1949, is shown in Fig. 13·16. It indicates that it is possible in some instances to attain saturation magnetization by the application of a very weak magnetic field and also for the intensity of magnetization to be zero in zero (or nearly zero) applied field.

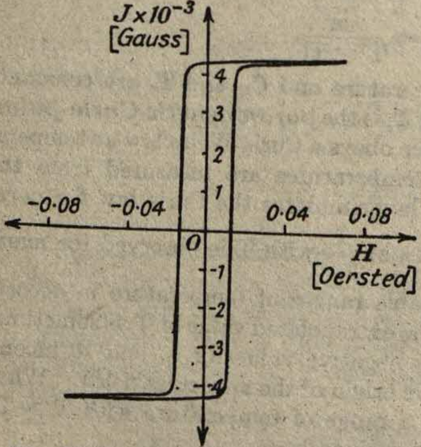


Fig. 13·16.—Magnetization curve for a single crystal of silicon iron.
[Derived from an oscillograph trace.]

The first of the above statements is remarkable for it is known from the study of paramagnetism that the application of a field of 0·01 oersted has an almost negligible effect on the magnetization of a system of free and independent elementary magnets. The small effect in the case of paramagnetics is known to be caused by thermal agitation which acts to oppose the ordering influence of the applied magnetic field. Now we have already learned that Weiss pointed out that this difficulty could be overcome if there is postulated in ferromagnetic materials the existence of a powerful internal ‘ molecular field ’. It is estimated that this field is more than twenty times as strong as any actual field produced in a laboratory. Weiss accounted for the second fact by introducing the idea of magnetic domains.

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[*] *Rev. Mod. Physics*, **21**, 541, 1949.

A digression on crystal structure.—Since ferromagnetics, like all solids, are crystalline in character it is necessary to refer briefly to some essential properties of simple crystals. Frequently the crystals, or grains, in a polycrystalline specimen may be observed directly with the unaided eye, and since 1920 many techniques have been developed for the production of large single crystals. One such is known as the ‘ strain-and-anneal ’ method, and when it is used to form a large iron crystal, an iron rod is elongated by stretching by about three per cent and then annealed at 880° C. for a period of several days. After cooling to room temperature single crystal specimens may then be cut from the large grains which develop in the rod under these conditions. In a second method, which is used to obtain large single crystals of nickel and of silicon-iron, the molten metal is allowed to cool very slowly through a temperature range which includes the melting point of the metal.

In connexion with some remarks which are to be made about single crystals it is convenient at this stage to recall that in crystallography a unit cell is defined as the smallest three-dimensional unit which, when repeated in three dimensions, gives the structure of the crystal concerned. Such a cell has shape and orientation but it does not necessarily have position except for convenience in discussing a certain problem ; for any one crystal many unit cells, all with the same volume, may be chosen. Fig. 13·17 shows such a unit cell,

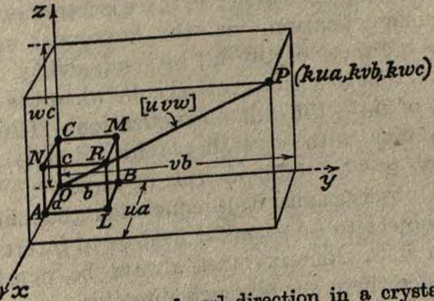


Fig. 13·17.—The [uvw] direction in a crystal.

which for simplicity, is taken to be a rectangular parallelepiped with sides a, b and c parallel respectively to the axes Ox, Oy and Oz. The position of any point P on the crystal lattice may always be expressed in terms of a, b and c, e.g. as kua, kvb and kwc, where u, v, w and k are integers. P is known as the point u, v, w, and the straight line OP is commonly called the $[uvw]$ direction, square brackets being used to avoid confusion with the Miller indices of planes, which are always written in parentheses.

Considerable advances in our fundamental conceptions about ferromagnetism were made as soon as large single crystals of iron, nickel and cobalt became available. Now although these crystals all belong to the cubic class and, generally speaking, such crystals are isotropic, i.e. any particular property does not vary with the direction to which it is referred or in which it is measured, yet, magnetically, their properties depend upon the direction along which each is measured ; we have a peculiarity known as ***magnetic anisotropy*** and to this reference will be made later, cf. p. 459.

Fig. 13·18 (a), based on the experimental work of HONDA and KAYA, shows the magnetization (or rather the intrinsic induction) curves for iron, which at temperatures below its magnetic transformation point exists, as shown in Fig. 13·18 (b), as a body-centred crystal, when the applied field is in turn parallel to the [100], [110], and [111] directions ; these are parallel to the edge of a cube, a face diagonal, or a body diagonal, respectively. The $4\pi J$-H curve rises most steeply in the first instance and since $\oint H\,dJ$ is the energy of magnetization per unit volume, this is least when $\vec{H}$ is parallel to a [100] direction. Such a direction is termed a direction of ***easy magnetizability*** or of ***preferred magnetization.***

Fig. 13·18 (c) shows similar curves for nickel, which, as shown in Fig. 13·18 (d), has a face-centred cubic structure. For this metal a body diagonal or [111] direction is that of preferred magnetization.

Finally, for cobalt, which at temperatures below 450° C. has a closely packed hexagonal structure, there is found only one direction of preferred magnetization, viz. the hexagonal axis ; directions normal to this are those of difficult magnetizability. Above 450° C. the structure is that of a face-centred cube when the above direction becomes that of most difficult magnetization. Thus the variation of magnetizability with crystallographic direction in hexagonal cobalt is much greater than in iron or nickel.

From such observations it is concluded that in a single iron crystal, not subject to an external field or to internal stresses, the magnetization in a domain must always be parallel to a [100] direction. Since there are six such directions in a crystal and no means for distinguishing one from the other exist, it follows that a demagnetized crystal of iron will generally have one-sixth of its domains orientated in each of these directions of easy magnetizability or preferred magnetization. In each domain the intensity of magnetization has a saturation value appropriate to the temperature of the specimen.

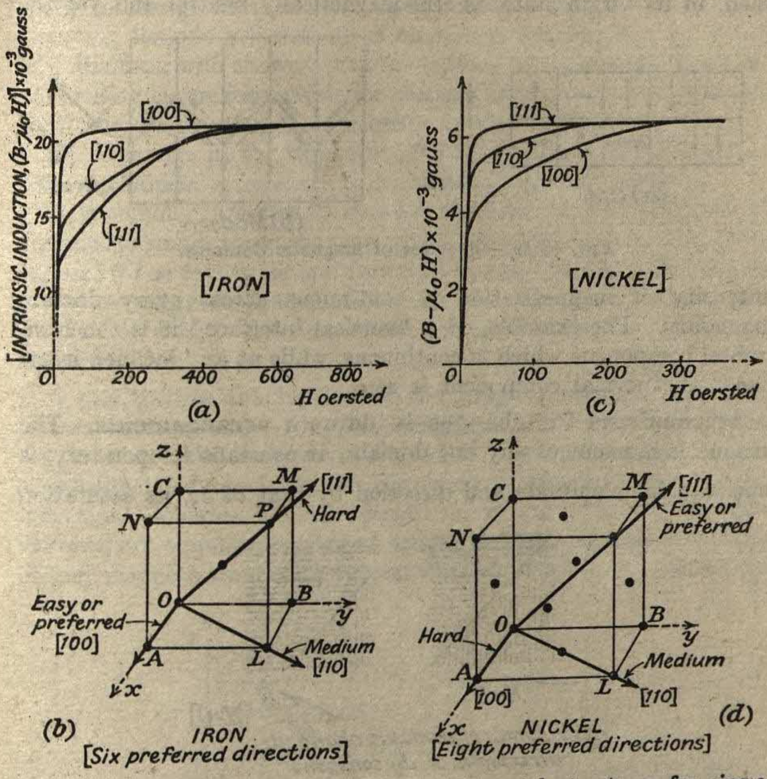
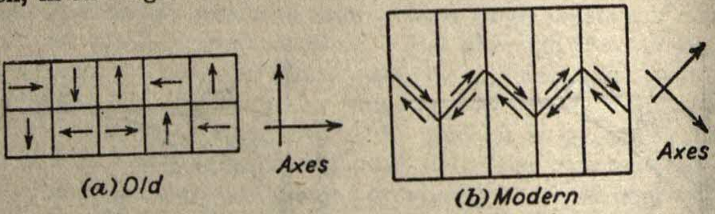


FIG. 13·18.—Intrinsic induction curves and crystal structures for single crystals of iron and nickel.

Since for nickel the [111] direction is that of preferred magnetization, one-eighth of its domains will, in general, be orientated in each of these eight directions.

Some aspects of the original and more modern domain theories.—The old familiar picture shown in Fig. 13·19 (*a*) represents a ferromagnetic crystal as being made up of small cubes in each of which the domain vector is parallel to one of the directions of preferred magnetization; for a single crystal of iron such a direction is always parallel to one of the edges of the unit cell when this is assumed to be a cube. This is now abandoned in favour of a picture of the type shown in Fig. 13·19 (*b*). The first picture is radically wrong for in it one of the fundamental theorems of magnetic theory is violated, viz. the normal component of the magnetic induction is not continuous; since domains are magnetized to saturation this essential condition is that the normal component of the intensity of magnetization must be continuous. In the new picture, the specimen, in its virgin state, is still magnetically neutral and yet the

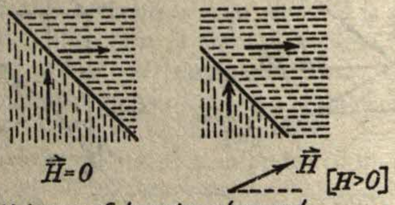


(a) Old (b) Modern

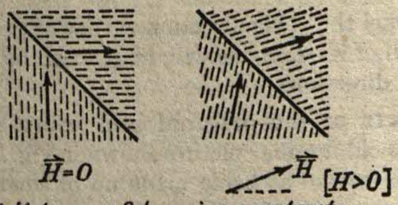
FIG. 13·19.—Theories of magnetic domains.

intensity of magnetization is continuous across every domain boundary. For example, at a 'vertical interface' it is the horizontal component which is continuous, while at an 'inclined interface' the normal component is zero.

Mechanisms for changes in domain arrangements.—The magnetic moment of any one domain, at constant temperature, is specified in magnitude and direction by that of $\vec{J}_s$, its saturation



(a) Volumes of domains changed; directions of J_S constant



(b) Volumes of domains constant; directions of J_S changed.

FIG. 13·20.—Two processes whereby a change in the magnetic state of a ferromagnetic material may be produced. (a) Volumes of domains changed; directions of $\vec{J}_s$ constant [the arrows give the direction of $\vec{J}_s$; their lengths are proportional to the magnetic moment of the domain]. (b) Volumes of domains constant; directions of $\vec{J}_s$ changed.

magnetization at that temperature, and its volume. Changes in the magnetic moment of a domain are usually considered to be brought about by one or other of the two following mechanisms :—

(*a*) A change in volume of the domain. This is the so-called 'moving boundary' method of change, a concept due to **Bloch** and **Becker**, who showed that the moving boundary process is the controlling factor responsible for changes in ferromagnetic materials until the magnetizing field exceeds several oersteds.

(*b*) A change in the direction of magnetization of the domain ; this mechanism is referred to as that of 'domain rotation'.

It is common practice to classify, according to the principles of thermodynamics, these changes, which are represented in Fig. 13·20 (*a*) and (*b*) as reversible and irreversible, respectively. It is found that the energy dissipated as heat in these changes is respectively a small or a large fraction of the potential energy stored in the domain.

A qualitative description of the magnetization process in terms of domains.—A typical magnetization curve for a ferromagnetic material is shown in Fig. 13·21. Let us consider that in a demagnetized specimen the directions of magnetization of the individual domains, i.e. the directions of the so-called ***domain vectors***, are equally distributed among various possible directions of easy magnetization. The initial state of affairs is represented by

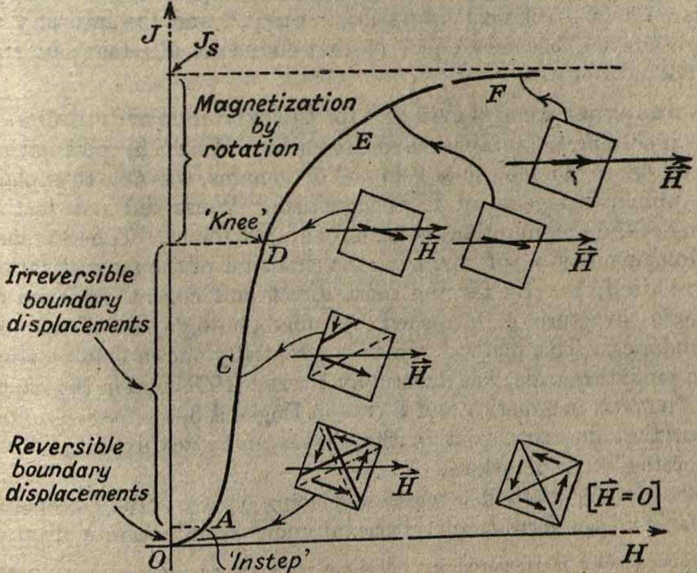

J
J_s
Magnetization by rotation
F
E
$\vec{H}$
'Knee'
D
$\vec{H}$
$\vec{H}$
Irreversible boundary displacements
C
$\vec{H}$
Reversible boundary displacements
$\vec{H}$
$[\vec{H}=0]$
A
O
'Instep'
H

Fig. 13·21.—A qualitative description of the magnetization process in a ferromagnetic material.

the point O on the diagram and four adjacent domains have their domain vectors so orientated that the resultant magnetization is zero. When a small external field is applied there is an increase in the resultant magnetic moment of the material. The change in magnetization which has occurred when the point A on the curve is reached must be attributed to boundary displacements which increase the sizes of those domains whose domain vectors are favourably orientated with respect to the applied field. The associated boundary displacements are mainly reversible. When the applied field is further increased irreversible boundary displacements predominate and the magnetization increases more rapidly ; this is shown at C. Although such movements are irreversible, at the same time the magnetization may also increase in virtue of the fact that some of the domains still grow as they did in the very early stages of the magnetization process. When the applied field is such that the *knee* of the curve is reached—this is a somewhat indefinite point D on the curve—the domain vectors will be so orientated that they are all parallel to that preferred direction associated with those domains whose volumes increased in the early stages. When the applied field is still further increased the magnetization increases much more slowly ; here no displacement of the boundaries can occur but the increase in magnetization is due, as shown at E, to the rotation of the domain vectors into a direction more nearly parallel to that of the applied field. Finally, in large fields, technical saturation is reached and the intensity of magnetization becomes equal to that found at all stages in any individual domain.

Some experimental evidence for the existence of domains.—The preceding paragraphs have shown how Weiss, by postulating the existence of a molecular field and of domains, was able to explain the principal aspects of ferromagnetism. Weiss did not justify either of these assumptions in terms of atomic forces. The existence of domains may be inferred from the character of the magnetization curve itself, but by far the most direct and cogent evidence of domain structure is furnished by microphotographs of domain boundaries. This method, which is essentially one of field-plotting on a miniature scale, was devised by BITTER (1931) and in the hands of WILLIAMS in America and BATES in England has provided ample proof that domains exist in the shapes and sizes expected from theoretical considerations.

The powder method consists in placing on a carefully prepared surface* of the ferromagnetic crystal under examination a drop of ethyl acetate containing in the colloidal state a finely divided ferromagnetic material such as haematite, Fe_2O_3. The ethyl acetate evaporates so that the surface becomes coated with a very finely divided layer of haematite. When the coated surface is examined by means of a microscope it is found that particles of iron oxide have become strongly concentrated about certain well-defined lines, which represent the boundaries between domains magnetized in different directions. The reason why the particles concentrate near these boundaries is that in their vicinity there exist very strong local magnetic fields which attract the magnetic particles.

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* The surface is prepared by polishing with emery papers down to the finest grade; the next and final stage of the process is done electrolytically. The specimen is washed and then placed on the base of an electrolytic cell;

The Barkhausen effect.—AB, Fig. 13·22 (*a*), is a rod of ferromagnetic material lying within a suitable solenoid, S, so that it may be magnetized to different degrees in the usual way. Upon this rod and close to one of its ends there is wound a search coil, S_1, consisting of about 10,000 turns ; this coil is joined in opposition to a coil S_2 forming the secondary of a mutual inductance M. The primary of M is in the magnetizing circuit. With the rod AB removed from the circuit the position of S_2 is adjusted so that the total linkages with the circuit in which it lies is zero for the largest current used in the solenoid. The secondary circuit is connected to a multi-stage amplifier and a telephone T.

The rod AB is replaced inside the solenoid. Then if the changes in the magnetization of the material of the rod were continuous and there were no disturbing effects, there would be no sounds heard in the phones when the magnetizing current changed slowly. In an actual experiment a series of 'clicks' or 'noises', corresponding to abrupt changes in different parts of the rod is heard. Sometimes these noises may be detected even without the use of an amplifier. In the light of the domain theory of ferromagnetism it has been customary to attribute the noises to a rapid succession of very small e.m.fs. induced in the coil S_1 as the orientation of the saturation magnetization within the domains is suddenly altered. This phenomenon was first investigated by BARKHAUSEN (1919) and it is known as the longitudinal ***Barkhausen effect.*** It has been described as the magnetic analogue of the 'cry of tin' and is intimately connected with magnetization processes.

It is clear from the above experiments that if we plotted on a very large scale the J–H curve for a ferromagnetic material, we should find it to be made up of a series of steps. VAN DER POHL (1920)

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it forms the anode while the cathode is a sheet of iron. The electrolyte is a perchloric acid ($HClO_4$)—acetic anhydride ($CH_3COOCOCH_3$)—water mixture. The underside of the specimen is coated with polystyrene glue to resist attack and with a current density of 0·1 A. cm.$^{-2}$ a satisfactory upper surface is obtained in about six minutes; after washing and drying the specimen is annealed in vacuo.

examined the magnitude of these steps in several portions of the complete hysteresis cycle and found that they were only pronounced when the ferromagnetic material was in a state corresponding to the steep portions of the curve, i.e. where $\overrightarrow{J}$ changes rapidly with $\overrightarrow{H}$, and were extremely small in the region of saturation. It is in this region that great amplification in the detector must be used.

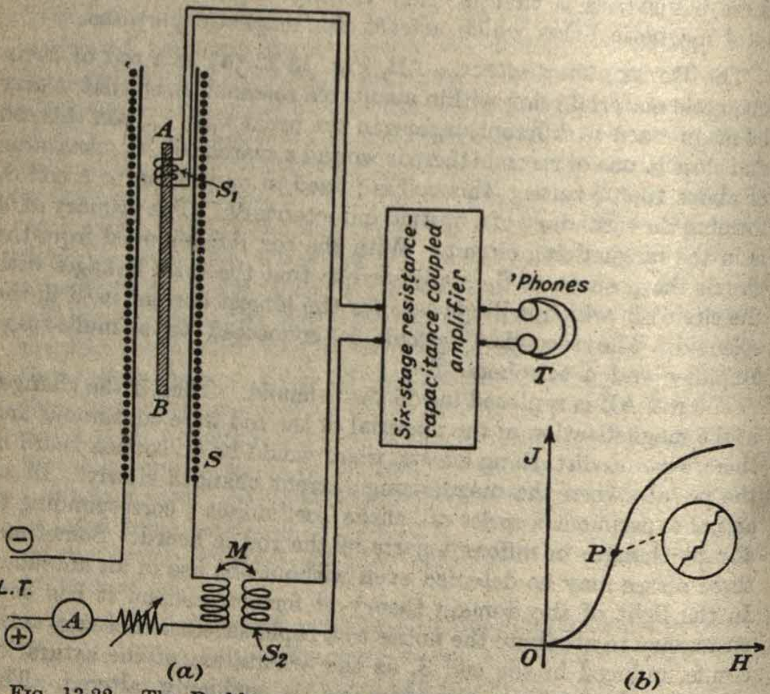

FIG. 13·22.—The Barkhausen effect. [(*b*) shows a portion of the curve of magnetization at P greatly enlarged.]

Fig. 13·22 (*b*) shows a portion of the magnetization curve at P greatly enlarged; the discrete steps are clearly shown.

Recent experiments by **Williams** and **Shockley** (1949) have established almost beyond doubt that there is no direct connexion between the Barkhausen discontinuities and the size of the domains. The Barkhausen discontinuities correspond to irregular fluctuations in the motion of a domain boundary under the influence of an applied field rather than to an abrupt change in the orientation of the magnetic vector associated with the domain. Experiments on single crystals have shown that the domain boundaries move as the applied magnetic field is changed and it is now thought that the Barkhausen discontinuities are due to irregular fluctuations in the motion of the domain boundaries. These irregularities may be due to small defects, such as impurities, holes, etc., in the interior of the specimen or to non-uniformity of the stresses within it.

The energy and origin of ferromagnetic domains.—Theoretical investigations by NÉEL, amply supported by the work of BOZORTH, WILLIAMS, BATES and others, begin with the assumption that any system of self-saturated domains will have energy associated with it. This energy comprises several independent components, and Néel has shown that when the sum of these components is a minimum we have the necessary condition for equilibrium.

On account of magnetic anisotropy, cf. p. 452, we have the first such component, viz. the ***energy of magnetic anisotropy***. Fig. 13·23 (a) shows a unit cell (a cube), which is part of a domain in an iron crystal. Normally the saturation intensity $\vec{J}_s$ will be parallel to an edge of the cube and to rotate $\vec{J}_s$ away from this direction energy must be expended; this energy is stored in the material and is known as the energy of magnetic anisotropy. The manner in which this varies with the direction of $\vec{J}_s$ in a (100) plane is shown in Fig. 13·23 (b); there is a distinct minimum whenever the magnetization vector is parallel to a crystallographic cube edge.

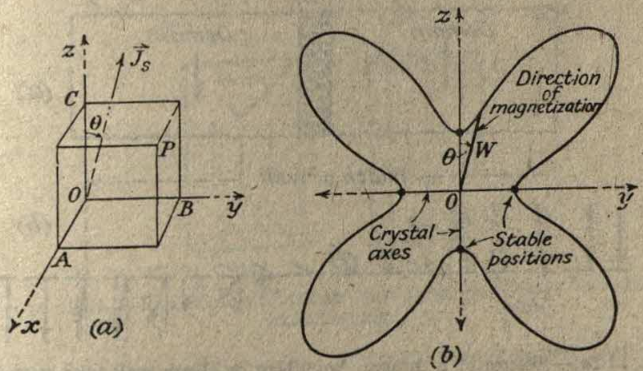

FIG. 13·23.—Energy of magnetic anistropy in the (100) plane of an iron crystal.

The second component is the so-called ***energy of demagnetization.*** To account for this it must be remembered that, unless free poles are absent, every magnetized ferromagnetic material is subject to a demagnetizing field of magnitude DJ, where D is the demagnetizing factor, cf. p. 365, and J is the intensity of magnetization. To increase J work must be done by an external agent since the increase takes place in the presence of free poles.

The third component is the energy associated with the boundary between adjacent domains, and for these the directions of the magnetization vectors must be antiparallel. In 1932 BLOCH showed that on an atomic scale the boundary between adjacent domains is not sharply defined but has a thickness which is spread over an area containing many atoms. Such a boundary is known as a ***Bloch wall*** and, as shown in Fig. 13·24, the resultant magnetic moment of the atoms within it gradually changes as we move across it from one domain to the next. Calculations by BLOCH and by NÉEL have shown that in iron, free from magnetostriction, the energy per unit area in a Bloch wall is of the order of 1 erg.cm.$^{-2}$; the walls themselves are about 500×10^{-8} cm. thick.

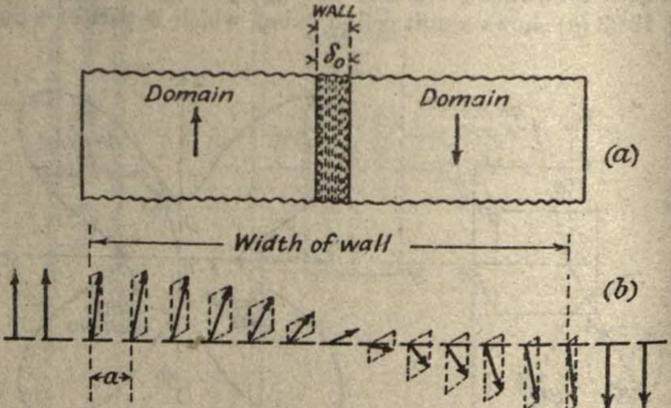

FIG. 13·24.—Nature of a domain boundary or Bloch wall; the variation of the resultant magnetic moment of atoms in the wall.

The fourth component of the energy arises from the presence of closure domains and so cannot be discussed until the formation of these triangular end domains has been described. With this object in view let us consider the structures shown in Fig. 13·25, where each represents a longitudinal cross section of a single ferromagnetic crystal. In (*a*) we have a saturated configuration consisting of a single domain in which $\vec{J}_s$ is parallel to a vertical edge; consequently the energy of magnetic anisotropy will be a minimum. However, free poles will appear at the ends of the domain and these will tend to demagnetize the specimen. The field outside it will have energy $\frac{\mu}{8\pi}\int H^2\,dv$, where the integration covers the whole field; it is called ***energy of demagnetization.*** Suppose, however, that this single domain subdivides itself into two equal domains as shown in Fig. 13·25 (*b*). The energy of magnetic anisotropy and the intensity of magnetization, which is still $\vec{J}_s$, are not altered, but the former single pole at each end is replaced by two poles, equal in magnitude but opposite in sign. The magnetostatic energy has therefore been reduced to about one-half its previous value, but at the same time the energy of the Bloch wall between the two domains

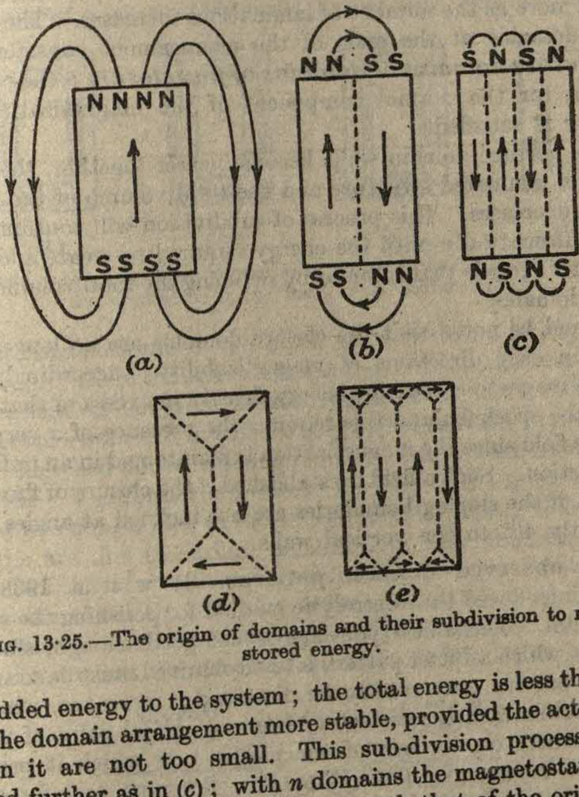

FIG. 13·25.—The origin of domains and their subdivision to reduce stored energy.

has added energy to the system; the total energy is less than before and the domain arrangement more stable, provided the actual grains within it are not too small. This sub-division process may be carried further as in (*c*); with *n* domains the magnetostatic energy will be reduced approximately to one-*n*th that of the original configuration; this is because of the reduced spatial extension of the field.

It is possible to devise an arrangement of domains for which the magnetic field energy is zero. Such an arrangement is shown in Fig. 13·25 (*d*) but this state of affairs brings with it another seat of potential energy, for as Becker and Döring showed in 1935, even if there is no change in volume on magnetization, a single crystal elongates in the associated [100] direction and contracts laterally. Thus, when the domains are assembled as in (*d*) mechanical constraints arise in those domains for which $\vec{J}_s$ is horizontal. Consequently such domains must be strained if they are to fit the spaces allotted to them, and the energy thus stored is known as ***magnetostrictive-strain energy.*** If the total volume of the strain-producing horizontal domains can be reduced, there follows at once a diminution in the above strain energy. Such reductions occur when the domain arrangement changes from that of (*d*) to that of (*e*), and still more as the number of laminations increases. The wedge-shaped domains at the ends of the arrangement considered are called ***closure domains*** or ***domains of flux closure*** ; the essential condition for the normal component of the magnetization at a boundary is satisfied.

As the vertical domain walls become nearer together, the entity acquires a laminated structure and the total volume of the closure domains decreases. This process of subdivision will continue more or less automatically until the energy required to create additional boundaries exceeds that released by reducing the total volume of the closure domains.

It should be noted that the closure domains are no longer magnetized in easy directions of magnetizability ; accordingly they possess a magneto-crystalline energy density in excess of that in the bulk of the specimen and this requires the presence of a very large magnetic field since the magnetization is maintained in an unfavourable direction. Such a field is established if the closure of flux is not ideal, i.e. if the sloping boundaries are not inclined at angles which are exactly 45° to the vertical walls.

Some observed domain patterns.—It was in 1938 that Elmore introduced the electrolytic method for polishing the surface of a crystal. Mechanical polishing is not good enough, since the surface on which a Bitter pattern is to be obtained must be so smooth that the pattern may readily be discerned ; moreover, while the mechanical method is satisfactory from an optical point of view it is useless here on account of the plastic flow which it produces in surface layers. In 1949 Williams, Bozorth and Shockley obtained patterns which showed the structure of those domains which form the bulk of a ferromagnetic material, i.e. closure domains did not appear.

In one of their experiments they cut out from a thin single crystal of three per cent silicon iron, a specimen resembling a picture frame with dimensions about 2 cm. × 1 cm. × 0·07 cm. ; its sides were parallel to the [100] and [010] directions. The temperature of the specimen was raised above its magnetic transformation point and after it had been allowed to cool slowly it was found to consist of four domains as shown in Fig. 13·26 (*a*). In such an instance the energy of magnetic anistropy and the total area of the boundaries will be a minimum, and at the same time the demagnetization energy will be zero and the magnetostrictive energy very small ; thus the essential conditions for stability are inherent. When a magnetizing coil was wound as uniformly as possible on the frame, fields with directions opposite to that of the magnetization in the several parts of the framework could be applied. The change in direction of magnetization from that of (*a*) to that of (*d*) is brought about by the movement of a single 180°-wall* across each side of the frame; intermediate positions of the Bloch wall are shown in Fig. 13·26 (*b*) and (*c*), and these indicate an inward or outward movement of the walls so that one set grows at the expense of the other. The powder photographs obtained in this work revealed that the main structure was as sketched above, although ' fir tree ' and other small patterns appeared. The former are due to the fact that the surface of the crystal was not parallel to the selected crystal plane, while the small patterns are due to surface inclusions or imperfections.

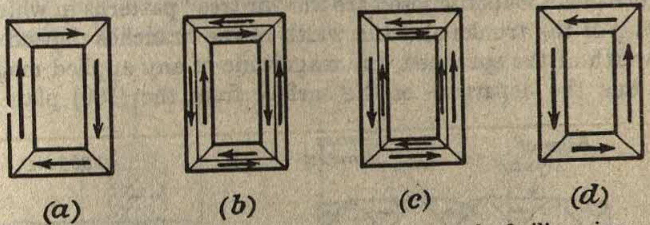

FIG. 13·26.—Domain arrangement in a single crystal of silicon iron, when the crystal is in the form of a 'picture frame'.

Much of the information derived from powder patterns is obtained by making use of the following technique. If a scratch is made on the surface of a ferromagnetic material then, in some instances, the lines of induction are disturbed as shown in Fig. 13·27; we have a flux leakage so that any haematite powder scattered on the surface will be attracted to the scratch. In the course of work on Bitter patterns it was found that if the scratch on the exposed surface of a domain is parallel to $\vec{J}_s$, then the flux leakage is small, but it assumes considerable proportions when the scratch is normal to the above direction.

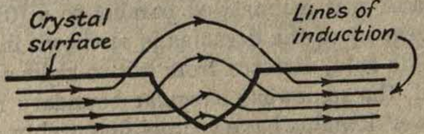

FIG. 13·27.—Effect of a surface scratch on the distribution of the lines of magnetic induction.

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* Two adjacent domains, whose vectors are parallel to the Bloch wall separating them but oppositely directed, are known as 180 degree domains; if the vectors are normal to each other we have 90 degree domains.

Proceeding in this way it was found that for a single crystal of the above mentioned silicon iron, the directions of magnetization are as shown in Fig. 13·28 (*a*). The primary structure consists of domains of width W magnetized alternately in the [001] and $[00\bar{1}]$ directions; in addition, the domains of closure, predicted by Néel, are revealed. Superimposed are the 'fir tree' patterns in which the spacing of the trunks and the width of the branches depend upon the width of the specimen, the magnitude of any applied magnetic field and the departure of the surface from the (100) plane.

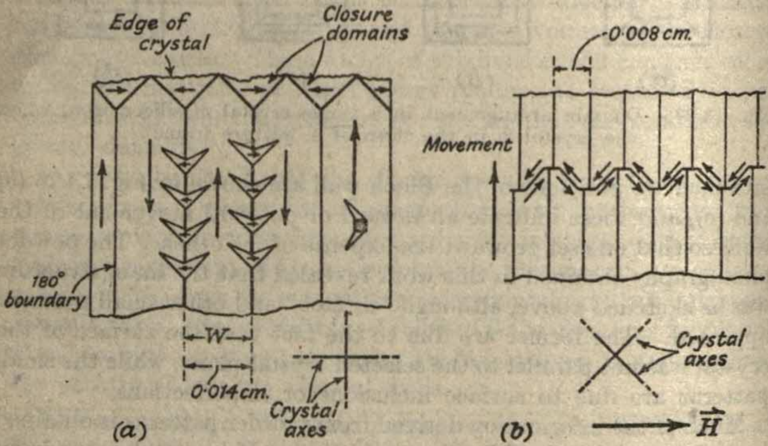

FIG. 13·28.—Some observed domain patterns.

When the surface of the crystal under investigation is parallel to a (100) plane and it is magnetized parallel to a [011] direction, a possible arrangement of the domains is as shown in Fig. 13·28 (*b*). When a magnetic field, directed from left to right across the specimen, is applied, the somewhat irregular boundary moves upwards because the lower domains have vectors favourably orientated with respect to the field. This movement takes place in the opposite direction when the applied field is reversed.

Some effects of small surface 'inclusions' in ferromagnetic crystals.—For the sake of simplicity it will be assumed that the inclusion is a very small piece of non-ferromagnetic material in the form of a cube. To reach a stable state small closure domains, similar to those found at the external boundaries of specimens, must be introduced. When a Bloch wall nearly bisects the inclusion, so that the flux of induction intercepted by the latter is about zero, an arrangement of closure domains, similar to that shown in Fig. 13·29 (*a*), prevents the appearance of free poles.

On the other hand, if an inclusion is not near to a wall, the intercepted flux is not zero and the appearance of free poles is a necessary consequence ; cf. the upper part of Fig. 13·29 (*b*). Néel has pointed out, however, that the energy of the demagnetizing field is much reduced if closure domains, such as those shown in the lower half of the diagram, are introduced so that the effective free poles are spread over a wider area. Powder patterns show the presence of these tapered domains.

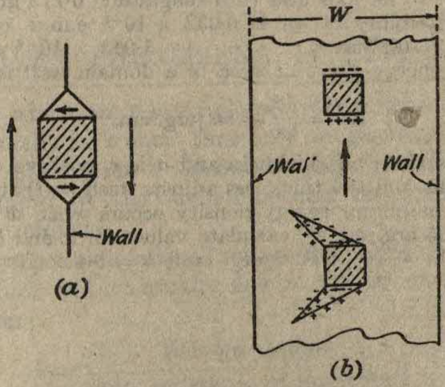

FIG. 13·29.—Domain arrangements near to a cubical non-ferromagnetic inclusion in a crystal of silicon iron.

Theoretical calculations on the presence of inclusions are intricate, but in some simple cases calculations have shown that as the size of the inclusions in a ferromagnetic material is reduced, so should the coercivity of the material increase. When the inclusions have linear dimensions comparable with those of the wall separation, the arrangements shown in the last diagram cannot exist; at the present time the theory of surface inclusions is very incomplete.

EXAMPLES XIII

13·01. Distinguish between *ferromagnetic*, *paramagnetic* and *diamagnetic* substances.

Describe and give the theory of a method for measuring the magnetic susceptibility of (*a*) sulphur, (*b*) an aqueous solution of manganese sulphate.

13·02. The magnetic susceptibility of copper is -0.76×10^{-6} e.m.u. cm.$^{-3}$.

Discuss fully the information contained in this statement and indicate how its accuracy may be tested in the laboratory.

13·03. Describe how you would determine accurately a value for the magnetic induction associated with a magnetic field of approximately 6000 oersted.

Calculate a value for the apparent change in the weight of a cylinder of sulphur of cross-section 2·00 cm.^{2} when it is suspended vertically with its lower end in a magnetic field of strength 10,000 oersteds, and its upper end in a field of strength 100 oersteds. [The volume susceptibilities of sulphur and of air may be taken as $-1{\cdot}00 \times 10^{-6}$ and $+0{\cdot}030 \times 10^{-6}$ e.m.u. respectively.] [103 dyne. (decrease).]

13·04. A rod of sulphur, 1·50 cm.^{2} in cross-sectional area, is placed in air in a non-uniform magnetic field, the direction of the field at each end of the specimen being at right angles to its axis. The magnetic field strength at one end P is 12,000 oersteds, and at the other end Q 2000 oersteds. The force on the rod due to the field acts from P to Q parallel to its axis and is of magnitude 0·112 gm.-wt. If the magnetic susceptibility of air is $0{\cdot}032 \times 10^{-6}$ e.m.u. cm.$^{-3}$ find the susceptibility of sulphur. [$-1{\cdot}013 \times 10^{-6}$ e.m.u. cm.$^{-3}$.]

13·05. The energy per unit area of a domain wall is given by

$$W = \left(\frac{a}{w} + bw\right) \text{erg. cm.}^{-2},$$

where w cm. is the wall thickness and a is a positive constant; b is another constant and the thickness adjusts itself so that W is a minimum. If the minimum energy density occurs when $w = 950$ Å and has a value 1·05 erg. cm.$^{-2}$, calculate values for a and b.

[$a = 4{\cdot}8 \times 10^{-6}$ erg. cm.; $b = 5{\cdot}4 \times 10^{-4}$ erg. cm.$^{-3}$.]

CHAPTER XIV

TRANSIENT PHENOMENA IN ELECTRICITY

Phenomena which are not simple periodic functions of time and which exist usually for a short time only are known as 'transients' or 'transient phenomena'. Here we shall discuss the transient currents produced by the sudden opening or closing of various simple electric circuits.

When a steady unidirectional voltage is applied to a circuit the current, I, which will eventually flow in the circuit is determined by the relation

$$I = \frac{\text{Voltage applied}}{\text{Total resistance of the circuit}}.$$

At the instant of closing a circuit the current is essentially zero; we have to examine how the current changes with time until its final steady value is reached.

The growth of current in a coil of inductance L and resistance R.—Such a coil may be represented by a pure resistance R in series with a pure inductance L. Let such a combination be connected to a source of steady e.m.f. V, as indicated in Fig. 14·01 (*a*). The internal resistance of the source is neglected.

As soon as the charging key is closed current will begin to flow and a corresponding flux will be linked with the coil. By Faraday's law, as given by Neumann, the increasing flux will cause a back e.m.f. $L\frac{di}{dt}$ to be introduced into the circuit, where i is the instantaneous current. Hence the voltage available for the production of current will be $V - L\frac{di}{dt}$. If we imagine a source of this voltage to act on R, as in Fig. 14·01 (*b*), then

$$V - L\frac{di}{dt} - Ri = 0,$$

i.e.
$$V = Ri + L\frac{di}{dt}.$$

To integrate this equation we may write *

$$\frac{dt}{\mathrm{L}} = \frac{di}{\mathrm{V} - \mathrm{R}i}$$

whence
$$\frac{t}{\mathrm{L}} = -\frac{1}{\mathrm{R}}\ln(\mathrm{V} - \mathrm{R}i) + \mathrm{A},$$

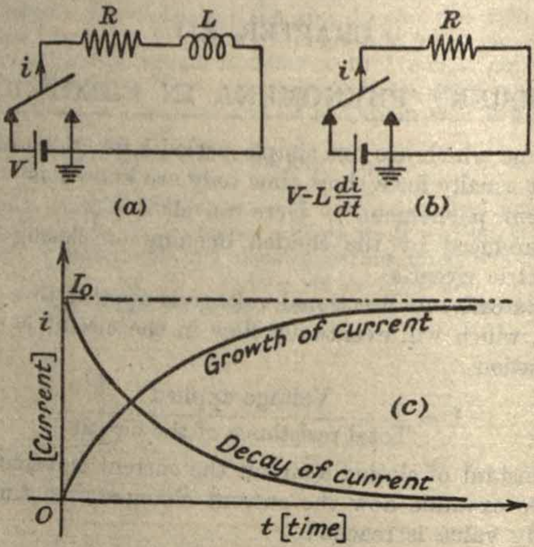

FIG. 14·01.—Growth and decay of a current in an inductive circuit.

where A is an integration constant. The initial conditions are $t = 0$, $i = 0$. $\therefore \mathrm{A} = \frac{1}{\mathrm{R}}\ln \mathrm{V}$.

$$\therefore \frac{t}{\mathrm{L}} = \frac{1}{\mathrm{R}}\ln\frac{\mathrm{V}}{\mathrm{V} - \mathrm{R}i}.$$

$$\therefore \exp\left(\frac{\mathrm{R}}{\mathrm{L}}t\right) = \frac{\mathrm{V}}{\mathrm{V} - \mathrm{R}i}, \quad \text{or} \quad 1 - \frac{\mathrm{R}}{\mathrm{V}}i = \exp\left(-\frac{\mathrm{R}}{\mathrm{L}}t\right).$$

$$\therefore i = \frac{\mathrm{V}}{\mathrm{R}}\left[1 - \exp\left(-\frac{\mathrm{R}}{\mathrm{L}}t\right)\right] = \mathrm{I}_0\left[1 - \exp\left(-\frac{\mathrm{R}}{\mathrm{L}}t\right)\right],$$

where I_0 is the steady value of the current $\left(\frac{\mathrm{V}}{\mathrm{R}}\right)$ reached when $t \rightarrow \infty$. Thus the current in the circuit rises according to an exponential law, the current-time graph being as indicated in

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* Alternatively, we may use the integrating factor $\exp\left(\frac{\mathrm{R}}{\mathrm{L}}t\right)$ [cf. p. 472].

Fig. 14·01 (*c*). It has therefore been established that when a circuit possesses self inductance, the current only reaches its steady value, as deduced from Ohm's law, after infinite time; it may, however, rise to within a very small percentage of this current in a quite brief interval after closing the switch.

Example.—Calculate the value of the current after 1 second when a p.d. of 10 volts is applied to a circuit of resistance 20 ohm. and self-inductance 0·1 H. We have

$$i = \frac{10}{20}\left[1 - \exp\left(-\frac{20}{0\cdot1} \times 1\right)\right] = 0\cdot5[1 - \exp(-200)].$$

If $x = \exp(-200)$,

$$\log x = -[200 \times 0\cdot4343] = -86\cdot86 = \overline{87}\cdot14.$$
$$\therefore x = 1\cdot38 \times 10^{-87}.$$
$$\therefore i = 0\cdot5[1 - (1\cdot38 \times 10^{-87})] = 0\cdot5 \text{ amp.}$$

Hence the current rapidly attains a value which, for all practical purposes, differs by a negligible amount from its final steady value. If, however, the resistance had been 0·1 ohm., we should have

$$i = \frac{10}{0\cdot1}[1 - \exp(-1)] = 100[1 - 0\cdot3679]$$
$$= 63\cdot2 \text{ amp.},$$

and this differs appreciably from the steady value of 100 amp.

Rate of growth of current in an inductive coil (R and L).— The rate of increase of the current in an inductive circuit is $\frac{di}{dt}$ and since

$$i = I_0\left[1 - \exp\left(-\frac{R}{L}t\right)\right],$$

we have

$$\frac{di}{dt} = I_0\frac{R}{L}\exp\left(-\frac{R}{L}t\right) = \frac{R}{L}(I_0 - i).$$

Hence the rate at which the current grows depends only on $\frac{R}{L}$ and not upon the separate values of L and R. The quantity $\frac{L}{R}$ is called the ***time constant*** of the circuit and is denoted by λ. The equation for the current may therefore be written

$$i = I_0\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right].$$

Unless the coil has a very large number of turns or ferromagnetic materials are present in it (when L is no longer a constant), L is usually very small compared with R, so that the current approximates to its steady value in a very small fraction of a second.

If we put $t = \lambda$, we have

$$[i]_{t=\lambda} = I_0[1 - \exp(-1)] = 0{\cdot}632\, I_0.$$

Thus the time constant is the time in which the current differs from its final value by exp (– 1) times the final value.

The four curves shown in Fig. 14·02 are drawn for values of $\lambda = 0{\cdot}5 \times 10^{-2}$, 1×10^{-2}, 2×10^{-2}, 4×10^{-2} sec. and it will be observed that the time taken for each current to reach a value $ON = I_0 \exp(-1)$, is equal to the time constant. It will be seen that the smaller λ is, the more rapidly does the current reach its final value.

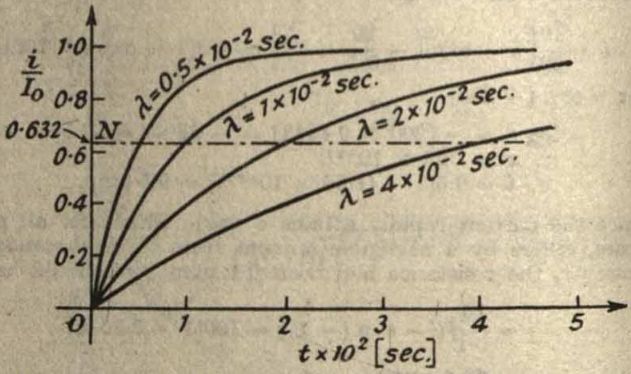


FIG. 14·02.—Curves showing the effect of the time constant on the growth of a current in an inductive circuit.

Decay of current through an inductive resistance.—When the switch is thrown over as in Fig. 14·03 (*a*) so that the steady voltage applied to the circuit becomes zero, the differential equation becomes

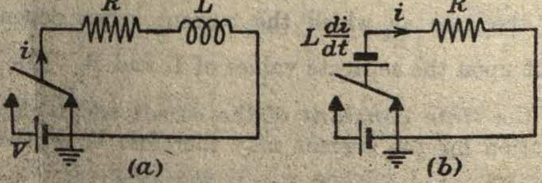


FIG. 14·03.—Decay of a current in an inductive circuit.

$$Ri + L\frac{di}{dt} = 0.$$

This equation may be written down at once if Fig. 14·03 (*b*) is considered. It may be re-written in the form

$$\frac{dt}{\mathrm{L}} = -\frac{di}{\mathrm{R}i}.$$

Integrating, we have

$$\frac{t}{\mathrm{L}} = \mathrm{B} - \frac{1}{\mathrm{R}}\ln i,$$

where B is an integration constant.

Since when $t = 0$, $i = \dfrac{\mathrm{V}}{\mathrm{R}}$ or I_0, we have $\mathrm{B} = \dfrac{1}{\mathrm{R}}\ln\dfrac{\mathrm{V}}{\mathrm{R}}$.

$$\therefore \frac{t}{\mathrm{L}} = \frac{1}{\mathrm{R}}\ln\frac{\mathrm{V}}{\mathrm{R}i}, \quad \text{or} \quad i = \frac{\mathrm{V}}{\mathrm{R}}\exp\left(-\frac{\mathrm{R}}{\mathrm{L}}t\right) = \mathrm{I}_0\exp\left(-\frac{\mathrm{R}}{\mathrm{L}}t\right).$$

The current-time curve is indicated in Fig. 14·01 (*c*); it will be noted that the growing and decaying currents are complementary.

Again, since $\lambda = \dfrac{\mathrm{L}}{\mathrm{R}}$, we have

$$i = \mathrm{I}_0\exp\left(-\frac{t}{\lambda}\right),$$

so that the smaller the time constant the more rapidly does the current become zero.

The physical meaning of the time constant λ.—(*a*) ***Growth of current in an inductive circuit.*** Since

$$i = \mathrm{I}_0\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right], \qquad \text{where } \lambda = \mathrm{LR}^{-1},$$

$$\frac{di}{dt} = \mathrm{I}_0.\frac{1}{\lambda}.\exp\left(-\frac{t}{\lambda}\right).$$

$$\therefore \left(\frac{di}{dt}\right)_{t=0} = \frac{\mathrm{I}_0}{\lambda} = \frac{\mathrm{V}}{\mathrm{L}}.$$

Let τ be the time in which the current would reach its final value if the initial rate of increase of the current were maintained. Then $\tau\left(\dfrac{di}{dt}\right)_{t=0} = \mathrm{I}_0 = \dfrac{\mathrm{V}}{\mathrm{R}}$.

$$\therefore \tau = \mathrm{I}_0 \div \frac{\mathrm{V}}{\mathrm{L}} = \frac{\mathrm{V}}{\mathrm{R}} \div \frac{\mathrm{V}}{\mathrm{L}} = \frac{\mathrm{L}}{\mathrm{R}}.$$

Also, the time constant is a measure of the rate at which a current grows or decays in an inductive circuit.

(1) In a circuit of given resistance the rate of growth or decay is proportional to λ^{-1}.

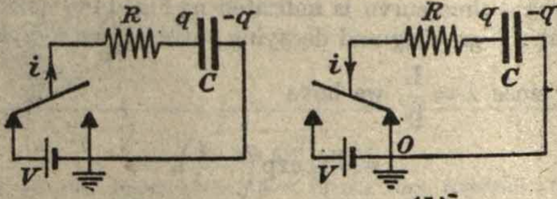
(2) In a circuit with given inductance the rate of growth is increased by increasing the resistance of the circuit.

(*b*) ***Decay of current in an inductive circuit.*** In this instance we have $i = I_0 \exp\left(-\frac{t}{\lambda}\right)$, so that

$$\left(\frac{di}{dt}\right)_{t=0} = -\frac{I_0}{\lambda}.$$

The physical meaning of λ is obvious.

Charge and discharge of a capacitor.—(*a*) ***Charging.***—Let a steady potential difference V be applied to the circuit shown in Fig. 14·04 (*a*). One pole of the cell and one plate of the condenser are earthed and therefore at zero potential at all times. If i is the instantaneous value of the current through R, the potential of the positive plate of the condenser will be $V - Ri$ at the instant considered. If q is the instantaneous charge on this plate we have



(*a*) (*b*)

FIG. 14·04.—(*a*) The charging and (*b*) the discharging of a condenser.

$$q = C(V - Ri),$$

where C is the capacitance of the condenser and since $i = \dot{q}$ this equation may be written

$$\frac{dq}{dt} + \frac{q}{CR} = \frac{V}{R}.$$

The integrating factor is $\exp\left(\int \frac{dt}{CR}\right) = \exp\left(\frac{t}{CR}\right)$. Hence

$$\exp\left(\frac{t}{CR}\right) \times \frac{dq}{dt} + \exp\left(\frac{t}{CR}\right) \times \frac{q}{CR} = \frac{V}{R}\exp\left(\frac{t}{CR}\right).$$

$$\therefore \frac{d}{dt}\left(q \exp\frac{t}{CR}\right) = \frac{V}{R}\exp\left(\frac{t}{CR}\right).$$

$$\therefore q \exp\left(\frac{t}{CR}\right) = VC \exp\left(\frac{t}{CR}\right) + A,$$

where A is an integration constant.

$$\therefore\ q = \mathrm{VC} + \mathrm{A} \exp\left(-\frac{t}{\mathrm{CR}}\right).$$

Since when $t = 0$, $q = 0$,

$$\mathrm{A} = -\,\mathrm{VC} = -\,\mathrm{Q}_0.$$

$$\therefore\ q = \mathrm{Q}_0\left[1 - \exp\left(-\frac{t}{\mathrm{CR}}\right)\right].$$

$$\therefore\ i = \dot{q} = \frac{\mathrm{Q}_0}{\mathrm{CR}} \exp\left(-\frac{t}{\mathrm{CR}}\right) = \frac{\mathrm{V}}{\mathrm{R}} \exp\left(-\frac{t}{\mathrm{CR}}\right).$$

Since $q = \mathrm{Q}_0\left[1 - \exp\left(-\frac{t}{\mathrm{CR}}\right)\right]$, it follows that v, the instantaneous value of the potential of the positive plate of the condenser, is given by

$$v = \frac{q}{\mathrm{C}} = \mathrm{V}\left[1 - \exp\left(-\frac{t}{\mathrm{CR}}\right)\right].$$

Hence T, the time required for v to grow from v_1 to v_2, is given by

$$\mathrm{T} = \mathrm{CR} \ln \frac{\mathrm{V} - v_1}{\mathrm{V} - v_2}.$$

(*b*) ***Discharging.*** When the cell is removed from the circuit, as indicated in Fig. 14·04 (*b*), the condenser begins to discharge. Let q be the charge on the condenser at time t after the cell has been removed by means of the key indicated. Then v, the instantaneous value of the potential of the positive plate is given by $v = \frac{q}{\mathrm{C}}$. If i is the instantaneous current through R and in the direction indicated, then

$$\frac{q}{\mathrm{C}} - \mathrm{R}i = \text{potential at O} = 0.$$

But

$$i = -\frac{dq}{dt}.$$

$$\therefore\ \frac{q}{\mathrm{C}} + \mathrm{R}\frac{dq}{dt} = 0,$$

a solution of which is

$$q = \mathrm{Q}_0 \exp\left(-\frac{t}{\mathrm{CR}}\right),$$

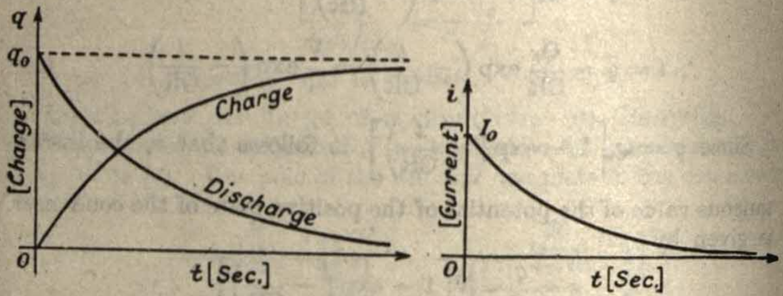
Hence

$$i = -\frac{dq}{dt} = \frac{\mathrm{Q}_0}{\mathrm{CR}} \exp\left(-\frac{t}{\mathrm{CR}}\right),$$

and since $[i]_{t=0} = \mathrm{I}_0 = \frac{\mathrm{Q}_0}{\mathrm{CR}}$, we have

$$i = \mathrm{I}_0 \exp\left(-\frac{t}{\mathrm{CR}}\right).$$

It must be noted that both in the charge and the discharge of a condenser the current begins with its greatest value and decays exponentially to zero.



(a) (b)

FIG. 14·05.—The charging and discharging of a condenser.

Curves showing how the charge on the positive plate of the condenser varies with time during the processes of charging and discharging are shown in Fig. 14·05 (*a*). The variation of the current with time is shown in Fig. 14·05 (*b*).

The time constant for a C–R circuit.—The time constant λ is now defined by $\lambda = \mathrm{CR}$. Since $q = \mathrm{Q}_0\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right]$,

$$\frac{dq}{dt} = i = \frac{\mathrm{Q}_0}{\lambda}\exp\left(-\frac{t}{\lambda}\right).$$

$$\therefore \left(\frac{dq}{dt}\right)_{t=0} = \frac{\mathrm{Q}_0}{\lambda},$$

so that, as in the case of an inductive circuit, λ is the time in which the final conditions on the condenser would be attained if the initial rate of growth of q were maintained. For then

$$\frac{dq}{dt} = \frac{\mathrm{Q}_0}{\lambda}.$$

$$\therefore \Big[q\Big]_0^{\mathrm{Q}_0} = \frac{\mathrm{Q}_0}{\lambda}\tau, \quad \text{i.e. } \tau = \lambda,$$

where τ is the time of charging under the conditions stipulated.

The energy dissipated during the charging of a condenser. It is interesting to inquire into the energy dissipated in the resistance R during the charging of a condenser. We have

$$q = Q_0\left[1 - \exp\left(-\frac{t}{CR}\right)\right].$$

$$\therefore\ i = \dot{q} = \frac{Q_0}{CR}\exp\left(-\frac{t}{CR}\right).$$

Now δW, the energy dissipated in time δt, is given by

$$\delta W = i^2 R\,\delta t = \frac{Q_0{}^2 R}{C^2R^2}\exp\left(-\frac{2t}{CR}\right)\delta t.$$

$$\therefore\ W = \frac{Q_0{}^2}{C^2R}\int_0^\infty \exp\left(-\frac{2t}{CR}\right)dt = \frac{1}{2}\frac{Q_0{}^2}{C} = \tfrac{1}{2}Q_0V.$$

[N.B. This is independent of R.]

The energy supplied by the battery is Q_0V; that gained by the condenser $\frac{1}{2}Q_0V$. Thus the total energy supplied by the battery is accounted for.

Example.—Two condensers of capacitance C_1 and C_2 are charged to potentials V_1 and V_2 respectively. They are then insulated from the supply and connected in parallel through a resistance R. Find an expression for the current which flows and hence show that the energy dissipated is independent of the value of R. [Assume $V_1 > V_2$.]

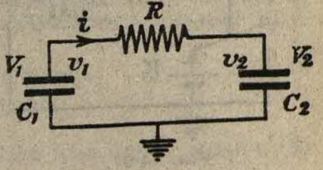


FIG. 14·06.

Let v_1 and v_2 be the instantaneous potentials on the positive plates of the condensers—cf. Fig. 14·06. Then, if i is the instantaneous current,

$$v_1 - Ri = v_2.$$

$$\therefore\ \frac{q_1}{C_1} - Ri = \frac{q_2}{C_2},$$

where q_1 and q_2 are the instantaneous charges on the positive plates of the condensers.

$$\therefore\ \frac{1}{C_1}\frac{dq_1}{dt} - R\frac{di}{dt} = \frac{1}{C_2}\frac{dq_2}{dt}.$$

But

$$-\frac{dq_1}{dt} = i = +\frac{dq_2}{dt},$$

$$\therefore\ \left(\frac{1}{C_1} + \frac{1}{C_2}\right)i = -R\frac{di}{dt}.$$

If $\dfrac{1}{C} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$, the above equation may be written

$$\frac{di}{i} = -\frac{1}{RC}dt,$$

which on integration becomes

$$\ln i = -\frac{1}{RC}t + \beta,$$

where β is an integration constant.

At $t = 0$, $i = \frac{V_1 - V_2}{R}$; hence $\beta = \ln\left(\frac{V_1 - V_2}{R}\right)$.

$$\therefore\ i = \frac{V_1 - V_2}{R}\exp\left(-\frac{t}{CR}\right).$$

The energy dissipated in the process is

$$\int_0^\infty i^2 R\,dt = \int_0^\infty \frac{(V_1 - V_2)^2}{R}\exp\left(-\frac{2t}{CR}\right)dt$$

$$= \frac{(V_1 - V_2)^2}{2}C = \tfrac{1}{2}C_1C_2\frac{(V_1 - V_2)^2}{(C_1 + C_2)},$$

which is the well-known formula.

Example.—A condenser of capacitance C is charged from a cell, e.m.f. V, through a resistance R; when the condenser is fully charged it is connected to a cell, e.m.f. $V_0 < V$. Determine the voltage across the condenser t sec. later.

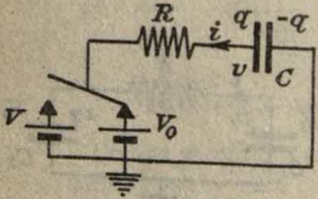


Fig. 14·07.

The system is shown in Fig. 14·07. The charge on the condenser before being connected to V_0 is given by $Q = CV$. At time t after making the above connexion let the charge on C be q, its potential being v.

Let i be the instantaneous current in the direction indicated so that $i = -\dot{q}$.

Then

$$v - Ri = V_0,$$

or

$$v + R\frac{dq}{dt} = V_0,$$

or

$$v + RC\frac{dv}{dt} = V_0.$$

This may be written

$$\frac{dv}{dt} + \frac{1}{RC}v = \frac{V_0}{RC}.$$

The integrating factor is $\exp\left(\int\frac{1}{RC}dt\right) = \exp\left(\frac{t}{RC}\right)$, and we have

$$v\exp\left(\frac{t}{RC}\right) = \frac{V_0}{RC}\int\exp\left(\frac{t}{RC}\right)dt$$

$$= V_0\exp\left(\frac{t}{RC}\right) + A,$$

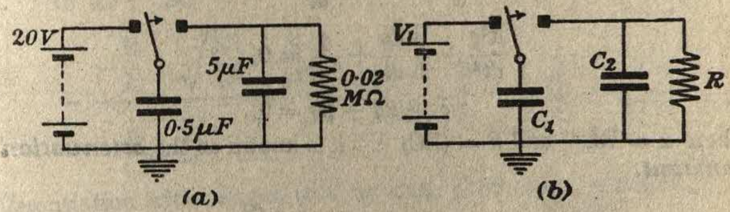
where A is an integration constant.

When $t = 0$, $v = V$. $\therefore\ V - V_0 = A$.

$$\therefore\ v = V_0 + (V - V_0)\exp\left(-\frac{t}{RC}\right).$$

Example.—By means of a rotary switch operating at 6000 rev.min.$^{-1}$, the 0·5 μF. condenser shown in Fig. 14·08 (*a*) is connected in turn to the battery of e.m.f. 20 V. and to the 5 μF. condenser. Across this condenser there is fixed a 0·02 MΩ resistor. What is the final voltage attained by the 5 μF. condenser and what is the voltage fluctuation on this condenser ?

Fig. 14·08 (*b*) shows the arrangement in general. The charge on the condenser C_1 after being connected momentarily to V_1 is $Q_1 = C_1V_1$. Let V_2 be the final steady potential on the combination of C_2 and R. Then the final steady charge on C_2 is $Q_2 = C_2V_2$.



(*a*) (*b*)

FIG. 14·08.

At the instant when C_1 and C_2 are connected together the charges Q_1 and Q_2 are shared instantaneously and the common potential V_3 is given by

$$V_3 = \frac{Q_1 + Q_2}{C_1 + C_2} = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}.$$

Now $V_2 = V_3 \exp\left(-\frac{t}{C_2R}\right)$, where t is the time during which C_2 is allowed to discharge before being recharged from C_1, i.e. $t = 0·01$ sec.

But
$$\frac{t}{C_2R} = \frac{0·01}{5 \times 10^{-6} \times (0·02 \times 10^6)} = 0·1.$$

$$\therefore\ V_2 = V_3 \exp(-0·1) = 0·905V_3.$$

$$\therefore\ V_3 = \frac{V_2}{0·905} = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}.$$

$$\therefore\ V_2 = \frac{C_1V_1(0·905)}{C_1 + C_2 - 0·905C_2}$$

$$= \frac{0·5 \times 20 \times 0·905}{0·5 + 5(0·095)} = \frac{9·05}{0·975}$$

$$= 9·28 \text{ volt.}$$

$$\therefore\ V_3 = \frac{9·28}{0·905} = 10·27 \text{ volt.}$$

$\therefore$ Fluctuation = 0·99 volt.

The discharge of a capacitor through an inductive coil.—The system is shown in Fig. 14·09 (*a*) ; it is assumed that the capacitor has been charged and we have to examine how its charge varies with time. If q is the instantaneous charge on the positive plate of the condenser, and if i is the instantaneous current,

$$i = -\frac{dq}{dt},$$

and the circuit may be replaced by that shown in Fig. 14·09 (*b*). From this we have

$$\frac{q}{C} - Ri - L\frac{di}{dt} = 0,$$

or

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0,$$

or

$$\ddot{q} + \alpha\dot{q} + \beta q = 0,$$

where $\alpha = RL^{-1}$ and $\beta = (LC)^{-1}$. α is known as the ***attenuation constant.***

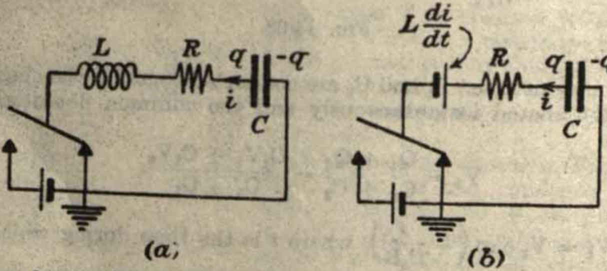


FIG. 14·09.—Discharge of a condenser through a resistance and an inductance in series.

A solution of this equation is

$$q = A_1 \exp(\lambda_1 t) + A_2 \exp(\lambda_2 t),$$

where λ_1 and λ_2 are roots of the auxiliary equation $(\lambda^2 + \alpha\lambda + \beta) = 0$,

i.e. $\lambda_1 = \dfrac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}$ and $\lambda_2 = \dfrac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$.

The final form of the expression for the instantaneous charge depends upon whether the term under the square-root sign is positive, zero, or negative, i.e. whether $R^2 >=< 4\frac{L}{C}$.

(i) ***Suppose*** $R^2 > \frac{4L}{C}$. Under these conditions both λ_1 and λ_2 are real and the initial conditions $t = 0, q = Q_0, i = 0$ determine the constants A_1 and A_2. We have

$$A_1 + A_2 = Q_0,$$

and

$$A_1\lambda_1 + A_2\lambda_2 = 0.$$

From these two equations we find

$$A_1 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2\sqrt{\alpha^2 - 4\beta}} Q_0 ; \quad A_2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2\sqrt{\alpha^2 - 4\beta}} Q_0 .$$

$$\therefore q = \frac{\frac{R}{L} + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}} Q_0 \exp \frac{1}{2}\left(-\frac{R}{L} + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right)t$$

$$+ \frac{-\frac{R}{L} + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}} Q_0 \exp \frac{1}{2}\left(-\frac{R}{L} - \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right)t.$$

By differentiation with respect to t, we find, after some reduction,

$$i = -\dot{q} = \frac{V}{\sqrt{R^2 - \frac{4L}{C}}} \exp\left(-\frac{1}{2}\frac{R}{L}t\right)\left[\exp\left(\frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}.t\right) - \exp\left(-\frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}.t\right)\right].$$

(ii) *Suppose* $R^2 = \frac{4L}{C}$. In this instance the roots are equal; we may however suppose that the roots are not equal and write for them λ_1 and $\lambda_1 + h$ where $h \rightarrow 0$. Then it may be shown that

$$q = (A + Bt) \exp (\lambda_1 t)$$

is a solution, the constants A and B being determined by the initial conditions, cf. Vol. I, p 34.

Now $$q = (A + Bt) \exp\left(-\frac{1}{2}\frac{R}{L}t\right),$$

and $$i = -\dot{q} = -B \exp\left(-\frac{1}{2}\frac{R}{L}t\right) + (A + Bt)\frac{1}{2}\frac{R}{L}\exp\left(-\frac{1}{2}\frac{R}{L}t\right).$$

The initial conditions give $A = Q_0$, and $B = \frac{1}{2}\frac{R}{L}Q_0$.

$$\therefore q = Q_0\left[1 + \frac{1}{2}\frac{R}{L}t\right]\exp\left(-\frac{1}{2}\frac{R}{L}t\right).$$

Hence $$i = -\dot{q} = Q_0\frac{R^2 t}{4L^2}\exp\left(-\frac{1}{2}\frac{R}{L}t\right);$$

(iii) *Suppose* $R^2 < \frac{4L}{C}$. In this instance the resistance in the circuit is low and we have, if $j = \sqrt{-1}$,

$$\lambda_1 = \frac{-\alpha + j\sqrt{4\beta - \alpha^2}}{2}, \quad \lambda_2 = \frac{-\alpha - j\sqrt{4\beta - \alpha^2}}{2}.$$

$$\therefore\ q = \exp(-\tfrac{1}{2}\alpha t)[A_1 \exp(j\tfrac{1}{2}\sqrt{4\beta - \alpha^2}\,t) \\ + A_2 \exp(-j\tfrac{1}{2}\sqrt{4\beta - \alpha^2}\,t)],$$

where A_1 and A_2 are constants. This expression is easily rearranged to give

$$q = A \exp(-\tfrac{1}{2}\alpha t)\cos(\tfrac{1}{2}\sqrt{4\beta - \alpha^2}\,t - \phi),$$

where A and ϕ are constants. Thus

$$q = A \exp(-\tfrac{1}{2}\alpha t)\cos(\omega' t - \phi),$$

where $\alpha = \frac{R}{L}$, and [illegible] $= \tfrac{1}{2}\sqrt{4\beta - \alpha^2} = \frac{1}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}$.

It remains to determine the constants A and ϕ from the initial conditions.

Now when $t = 0$, $q = Q_0$, i.e. $A\cos(-\phi) = Q_0$, or $\cos\phi = \frac{Q_0}{A}$.

Also $\quad i = -\dot{q} = -Ar\exp(-\tfrac{1}{2}\alpha t)\cos(\omega' t - \phi + \psi)$,

where $\quad r = +\sqrt{\frac{\alpha^2}{4} + \omega'^2} = \frac{1}{\sqrt{LC}}$, $r\cos\psi = -\tfrac{1}{2}\alpha$ and $r\sin\psi = \omega'$. Also since $i = 0$ when $t = 0$, $-\phi + \psi = \frac{\pi}{2}$.

Hence
$$\sin\psi = \cos\phi = \frac{Q_0}{A},$$

from which we find $A = Q_0 \dfrac{\frac{2}{\sqrt{LC}}}{\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}}$.

$$\therefore\ q = Q_0 \frac{\frac{1}{\sqrt{LC}}}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \exp\left(-\frac{1}{2}\frac{R}{L}t\right)\cos\left(\frac{1}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}\,.t - \phi\right),$$

where
$$\cos\phi = \frac{Q_0}{A} = \sqrt{1 - \frac{R^2C}{4L}}.$$

Hence

$$i = -\dot{q} = \frac{\dfrac{V}{L}}{\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}} \exp\left(-\frac{1}{2}\frac{R}{L}t\right) \sin\frac{1}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}\,.t.$$

The natural frequency, f', of the circuit is given by

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

If R is small compared with $2\sqrt{\dfrac{L}{C}}$, $f' = \dfrac{1}{2\pi\sqrt{LC}} = f_0$, where f_0 is the frequency of the free oscillations.

If q_1 and q_2 are two successive maxima and T the period,

$$\frac{q_1}{q_2} = \frac{\exp(-\frac{1}{2}\alpha t)}{\exp[-\frac{1}{2}\alpha(t+T)]} = \exp(\tfrac{1}{2}\alpha T).$$

The quantity δ defined by

$$\delta = \ln\frac{q_1}{q_2} = \tfrac{1}{2}\alpha T = \frac{2\pi\left(\dfrac{R}{2L}\right)}{\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}},$$

is known as the ***logarithmic decrement per cycle*** of the circuit.

When the resistance of the circuit is very small

$$\delta = \frac{\pi R}{L}\sqrt{LC} = \pi R\sqrt{\frac{C}{L}} = \frac{\pi R}{L\omega_0} = \pi\frac{R}{\left(\dfrac{1}{\omega_0 C}\right)},$$

so that δ is π times the ratio of the resistance to the inductive or capacitive reactance in the circuit at resonance. The curves shown in Fig. 14·10 illustrate the nature of the discharge occurring for three different values of the circuit resistance when a 1 μF. condenser, with a potential difference of a few volts across it, is discharged through an inductance of 1 H.

The charging of a capacitor through an inductive coil.—The circuit is shown in Fig. 14·11. If V is the potential difference across the terminals of the cell, and it is assumed to be steady, the charge q on the condenser at any instant is determined by the equation

$$Ri + L\frac{di}{dt} + \frac{q}{C} = V,$$

and since $i = \dot{q}$, this becomes

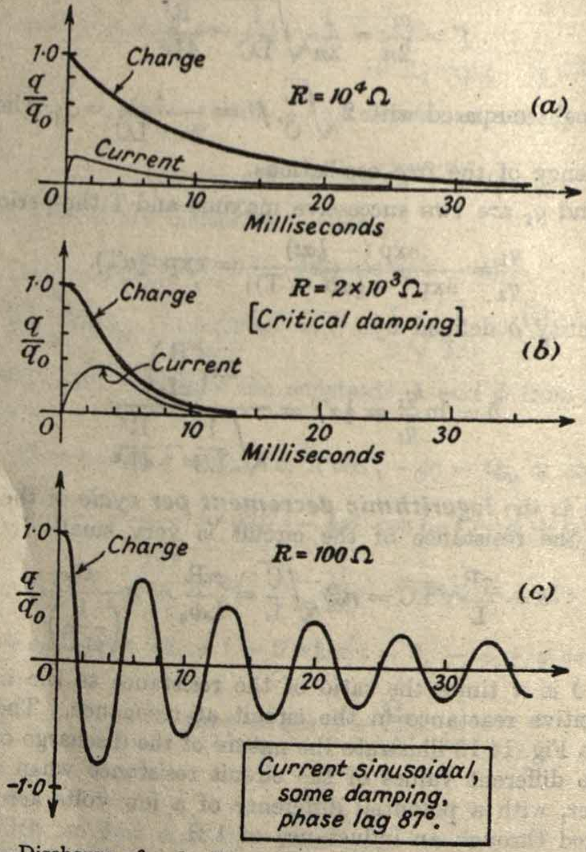
$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = \frac{V}{L}.$$

The solution consists of two parts; the particular integral given by

$$q_1 = VC = Q_0,$$

together with the complementary function

$$q_2 = A \exp\left(-\tfrac{1}{2}\alpha t\right) \cos(\omega' t - \phi),$$



14·10.—Discharge of a 1 μF. condenser through a self inductance of 1 H. for different values of the circuit resistance.

(From Harnwell—'Principles of Electricity and Magnetism', McGraw Hill Book Co. Inc.)

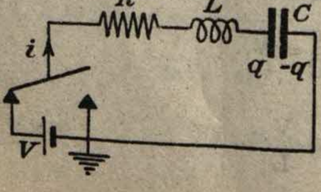


FIG. 14·11.—The charging of a condenser through an inductive coil.

if the resistance R is so small that the circuit is oscillatory ; A and ϕ are constants.

$$\therefore\ q = q_1 + q_2 = Q_0 + A \exp\left(-\tfrac{1}{2}\alpha t\right) \cos(\omega' t - \phi).$$

Now at $t = 0$, $q = 0$,

i.e. $$\cos\phi = -\frac{Q_0}{A},$$

and $$i = \dot{q} = Ar \exp\left(-\tfrac{1}{2}\alpha t\right)\cos(\omega' t - \phi - \psi),$$

where $r = +\sqrt{\dfrac{\alpha^2}{4} + \omega'^2}$, $r\cos\psi = \tfrac{1}{2}\alpha$, and $r \sin\psi = \omega'$.

Since $i = 0$, when $t = 0$,

$$-\phi - \psi = \frac{\pi}{2};$$

$$\therefore\ -\frac{Q_0}{A} = \cos\phi = \sin\psi = \frac{2\omega'}{\sqrt{4\omega'^2 + \alpha^2}}.$$

$$\therefore\ A = -Q_0 \frac{\dfrac{2}{\sqrt{LC}}}{\sqrt{\dfrac{4}{LC} - \dfrac{R^2}{L^2}}}.$$

$$\therefore\ q = Q_0\left[1 - \frac{\dfrac{1}{\sqrt{LC}}}{\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}} \exp\left(-\frac{1}{2}\frac{R}{L}t\right)\cos\left\{\frac{1}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}.t - \phi\right\}\right].$$

The expression for the current becomes, after some reduction,

$$i = \dot{q} = \frac{V}{\sqrt{\dfrac{L}{C} - \dfrac{R^2}{4}}} \exp\left(-\frac{1}{2}\frac{R}{L}t\right) \sin\frac{1}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}.t.$$

Otherwise we can differentiate the fundamental differential equation when we get

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0,$$

a solution of which is

$$i = A_1 \exp\left(-\frac{1}{2}\frac{R}{L}t\right)\cos(\omega' t - \phi_1),$$

where A_1 and ϕ_1 are constants. Since $i = 0$ when $t = 0$, $\phi_1 = \tfrac{1}{2}\pi$,

i.e. $$i = A_1 \exp\left(-\frac{1}{2}\frac{R}{L}t\right)\sin\ \omega' t,$$

and since $q = 0$ also when $t = 0$, the original differential equation shows that $\left(\frac{di}{dt}\right)_{t=0} = \frac{V}{L}$. But

$$\left(\frac{di}{dt}\right)_{t=0} = A_1\omega' = A_1\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Hence

$$A_1 = \frac{V}{\sqrt{\frac{L}{C} - \frac{R^2}{4}}},$$

which leads at once to the expression for the current already obtained.

[N.B. When $R^2 = \frac{4L}{C}$,

$$i = \frac{V}{L}t\exp\left(-\frac{1}{2}\frac{R}{L}t\right),$$

and when $R^2 > \frac{4L}{C}$,

$$i = \frac{V}{\sqrt{\frac{R^2}{4} - \frac{L}{C}}}\exp\left(-\frac{1}{2}\frac{R}{L}t\right)\sinh\frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}.t.\Big]$$

Fig. 14·12 shows how the charge on the condenser during the process of charging varies with time, the constants of the circuit being such that the charging process is oscillatory.

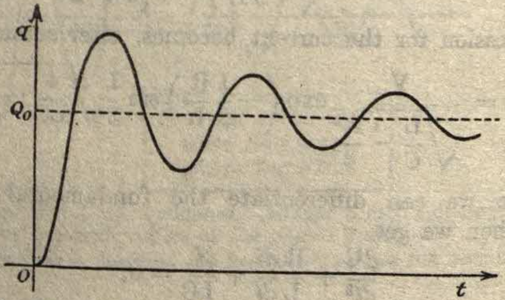


FIG. 14·12.—The variation of the charge on a condenser with time when the charging process is oscillatory.

It will be noted that the maximum charge, i.e. the first 'surge', is much greater than the final charge on the condenser. Hence if a condenser is charged from a high potential source through a small resistance and large inductance the insulation of the condenser may be punctured although its insulation could have withstood the final steady potential drop across its terminals. To avoid this the condenser must be charged through a resistance with a sufficiently high value.

The logarithmic decrement of energy during the discharge of a capacitor through an inductive coil.—If V is the initial potential difference across a capacitor of capacitance C, which is allowed to discharge through an inductive coil, R–L, then provided conditions are such that the discharge is oscillatory, the charge on the positive plate of the capacitor at time t is given, cf. p. 480, by

$$q = Q_0 \frac{\frac{1}{\sqrt{LC}}}{(\omega')^2} \exp\left(-\frac{1}{2}\frac{R}{L}t\right) \cos(\omega' t - \phi),$$

where $Q_0 = CV$, $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ and $\cos\phi = \sqrt{1 - \frac{R^2C}{4L}}$.

Hence

$$i = -\dot{q} = Q_0 \frac{\frac{1}{LC}}{\omega'} \exp\left(-\frac{1}{2}\frac{R}{L}t\right) \sin\omega' t$$

$$= \frac{V}{\omega' L} \exp\left(-\frac{1}{2}\frac{R}{L}t\right) \sin\omega' t.$$

The total energy associated with the condenser and the magnetic field at time t is

$$W = \tfrac{1}{2}\frac{q^2}{C} + \tfrac{1}{2}Li^2$$

$$= \tfrac{1}{2}Q_0{}^2 \cdot \frac{1}{LC^2\omega'^2} \cdot \exp\left(-\frac{R}{L}t\right) \cos^2(\omega' t - \phi)$$

$$+ \tfrac{1}{2}LQ_0{}^2 \cdot \frac{1}{L^2C^2\omega'^2} \exp\left(-\frac{R}{L}t\right) \sin^2\omega' t$$

$$= \tfrac{1}{2}\frac{V^2}{L\omega^2} \exp\left(-\frac{R}{L}t\right)[\cos^2(\omega' t - \phi) + \sin^2\omega' t].$$

If T is the period of the oscillatory discharge, we find, if we express the above energy at times t and $t + T$ by the symbols W_t and W_{t+T} respectively, then

$$\frac{W_t}{W_{t+T}} = \frac{\exp\left(-\frac{R}{L}t\right)}{\exp\left\{-\frac{R}{L}(t+T)\right\}} = \exp\left(\frac{R}{L}T\right)$$

$$= \exp\Delta, \text{ (say)},$$

where Δ is the decrement per cycle of the 'stored' energy. Thus

$$\Delta = \frac{R}{L}T = 2\delta,$$

where δ is the decrement of the charge (or current) per cycle.

The energy expended during the oscillatory discharge of a condenser through an inductive coil.—Let W_R be the energy dissipated in the resistance R in time t from the commencement of the discharge. Then if R is constant

$$W_R = R\int_0^t i^2\,dt = R\left(\frac{V}{\omega L}\right)^2\int_0^t \exp\left(-\frac{R}{L}t\right)\sin^2\omega t\,dt,$$

where, for convenience, ω is written instead of ω'. Now

$$\int_0^t \exp\left(-\frac{R}{L}t\right)\sin^2\omega t\,dt = \int_0^t \exp\left(-\frac{R}{L}t\right)\left\{\frac{1-\cos 2\omega t}{2}\right\}dt$$

$$\left[-\frac{1}{2}\cdot\frac{L}{R}\exp\left(-\frac{R}{L}t\right) - \frac{1}{2}\frac{\exp\left(-\frac{R}{L}t\right)}{\sqrt{\frac{R^2}{L^2}+4\omega^2}}\cos\left\{2\omega t - \tan^{-1}\frac{2\omega}{\left(-\frac{R}{L}\right)}\right\}\right]_0^t$$

Since $\cos(x+y) = \dfrac{1-\sin^2 x - \sin^2 y}{\cos(x-y)}$, then if we write

$x = \omega t - \tan^{-1}\left(\dfrac{2\omega}{-\frac{R}{L}}\right)$, $y = \omega t$, we have

$$W_R = R\left(\frac{V}{\omega L}\right)^2\left[-\frac{1}{2}\cdot\frac{L}{R}\exp\left(-\frac{R}{L}t\right)\right.$$

$$-\frac{\sqrt{LC}}{4}\exp\left(-\frac{R}{L}t\right)\left\{1 - \sin^2\overline{\omega t - \tan^{-1}\left(\frac{2\omega}{-\frac{R}{L}}\right)} - \sin^2\omega t\right\}\frac{2}{R}\sqrt{\frac{L}{C}}$$

$$\left.+\frac{1}{2}\frac{L}{R} - \frac{L}{2R}\{1-(\omega\sqrt{LC})^2\}\right]$$

$$=\frac{1}{2}\frac{V^2}{\omega^2}\left[\frac{1}{L}-\frac{1}{L}+\omega^2C\right]$$

$$-\frac{1}{2}\cdot\frac{V^2}{L\omega^2}\exp\left(-\frac{Rt}{L}\right)\left[\sin^2\left\{\omega t+\tan^{-1}\left(\frac{2\omega}{\frac{R}{L}}\right)\right\}+\sin^2\omega t\right]$$

$$=\tfrac{1}{2}V^2C-\frac{1}{2}\frac{V^2}{L\omega^2}\exp\left(-\frac{R}{L}t\right)\left[\sin^2\left\{\omega t+\tan^{-1}\left(\frac{2\omega}{\frac{R}{L}}\right)\right\}+\sin^2\omega t\right].$$

It should be noted that the energy dissipated in the resistance plus the energy left in the circuit is $\frac{1}{2}V^2C$, which is the initial store of energy in the capacitor. Also, if $t \to \infty$,

$$[W_R]_{t\to\infty}=\tfrac{1}{2}V^2C,$$

i.e. all the energy originally stored in the capacitor is finally dissipated in the resistance.

Measurement of high resistance by leakage.—When a condenser of capacitance C and initial charge Q is allowed to discharge through a resistance R for a time t, the charge remaining on the condenser is given by the formula [cf. p. 473]

$$q=Q\exp\left(-\frac{t}{CR}\right),\quad\text{or}\quad R=\frac{t}{C\ln\frac{Q}{q}}.$$

It follows that if the ratio $\frac{Q}{q}$ can be determined experimentally, then R can be found. To do this, a ballistic galvanometer G, Fig. 14·13, the resistance R, a condenser of known capacitance C and an accumulator A, are arranged as shown; one terminal of each component is at a common potential while the other terminals are joined to four highly insulated mercury cups, a, b, c and d.

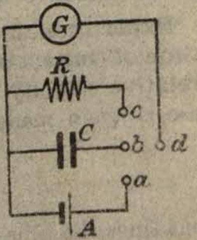


Fig. 14·13.—Measurement of high resistance by the method of leakage.

The condenser C is first charged by joining a and b. The insulated copper connecting piece enabling this to be done is then removed and at once placed across bd. The condenser discharges itself through G and the throw σ_0 is proportional to Q. The condenser is then recharged; by placing the connector across bc the charge is allowed to leak through R for a known time t. The charge left on the condenser is then discharged through G; the ballistic throw σ is proportional to q. Then

$$R = \frac{t}{C \ln \frac{\sigma_0}{\sigma}} = \frac{t}{2{\cdot}303 C \log \frac{\sigma_0}{\sigma}}.$$

A series of corresponding values of t and σ should be obtained and from the linear graph obtained by plotting t against $\log \frac{\sigma_0}{\sigma}$ a value for R may be found.

In practice it often happens that the condenser C is slightly defective. To test this the condenser is charged and immediately discharged through the ballistic galvanometer. It is then recharged by momentarily placing the connector across *ab* and after the lapse of say one to ten minutes, the condenser is discharged through the galvanometer. If the kick of the galvanometer is different from that obtained when the interval between charging and discharging is practically zero, then the condenser has an insulation resistance which is finite. If σ_1 is the throw after the condenser has been left for t_1 seconds, then its insulation resistance, R_1, is given by

$$R_1 = \frac{t_1}{C \ln \frac{\sigma_0}{\sigma_1}}.$$

Under such circumstances the value of R determined in the first part of the experiment is really the effective resistance of a resistance R_1 in parallel with R_2, the resistance of the resistor actually in parallel with C. Since $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$, a value for R_2 is readily obtained. [N.B. Unless R_1 exceeds R_2 many times the condenser is too defective for use.]

Experimental determination of the relation between the time of contact of a metal sphere with a metal plate and their relative velocity at the moment of impact.—In this experiment use is again made of the formula

$$t = CR \ln \frac{\sigma_0}{\sigma},$$

but since the time of contact is small it follows that if σ_0 and σ are to differ by an accurately measurable amount, then CR must be comparable with t. If $C = 1\ \mu F.$ and $R = 1000\ \Omega$, $CR = 10^{-3}$ second, so that such a condenser and resistance are each of suitable size. The condenser, accumulator, ballistic galvanometer and mercury cups are arranged as shown in Fig. 14·14 (*a*). The resistance R is then arranged between the mercury cup *d* and the well-insulated metal plate P. Any clamp used to hold the plate in position must be insulated from it. S is a metal sphere hanging from a fine metal wire about a metre in length; the head H from which the wire is suspended is connected metallically to G. When at rest S is just in contact with P.

On joining the mercury cups *a* and *b* the condenser C is charged; by connecting *b* to *c* the condenser may be discharged completely. Let σ_0 be the ballistic throw. Now let the condenser be charged and then connected to R, the sphere S being withdrawn a measured horizontal distance from P. It is convenient to hold S in position by means of a piece of cotton; when this is burnt S collides with P and during the time the metals are in contact the condenser discharges through R. The time of contact is short but, given a suitable value of R, the change in the charge on C is appreciable and readily measured. Let σ be the ballistic throw when the condenser is discharged through the galvanometer. The time of contact is then calculated from the formula already given.

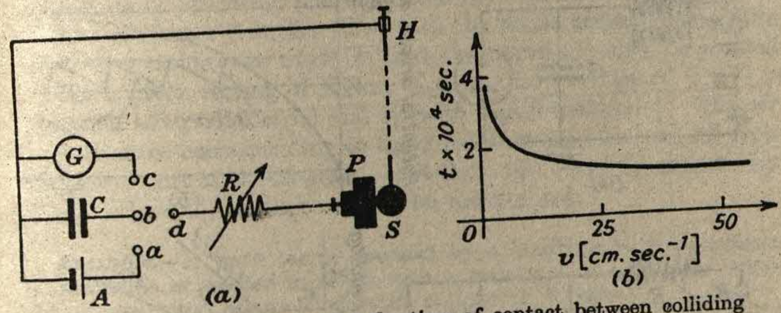


FIG. 14·14.—Determination of the time of contact between colliding metal objects.

If l is the length of the simple pendulum formed by S and the supporting wire, and if S is drawn aside a horizontal distance x, the velocity of S on impact is given by

$$v \doteqdot x\sqrt{\frac{g}{l}\left[1+\frac{x^2}{4l^2}\right]},$$

where g is the intensity of gravity.

Fig. 14·14 (*b*) shows how the time of contact varies in a particular instance with the velocity of the sphere at the moment of impact.

Experimental determination of the permittivity of an insulating liquid.—When a tube containing neon at a suitable low pressure has a small potential difference applied between two electrodes fitted to the tube no measurable current through the tube can be detected. If the applied potential difference is increased it is found that when a certain critical value V_1 is reached the neon becomes ionized, the resistance falls to a low value and a current flows. V_1 is known as the ***striking potential.*** If the potential difference across the tube is then decreased current continues to flow until the potential difference reaches a value V_2; this is the ***extinction potential.*** [For neon, $V_1 = 180$ volt., $V_2 = 140$ volt.]

In order to show this remarkable property of a neon lamp, an air-condenser C, Fig. 14·15 (*a*), (in parallel with the lamp L), is arranged in series with a resistance R (about 1 MΩ) and a high tension battery B (200 V.). Let V be the potential drop across the condenser. As soon as this reaches a value V_1, the lamp glows and the condenser begins to discharge rapidly. This continues until $V = V_2$, the extinction potential. While this discharge occurs the energy required to maintain the glow comes almost entirely from the condenser since the battery current is very small. When the lamp ceases to glow its resistance assumes a very high value and the condenser begins to charge again. This charging and discharging process occurs regularly, once the initial charging process is complete, so that the potential difference across the lamp varies as in Fig. 14·15 (*b*). [If the value of C is about 10 μF. several seconds elapse between successive flashes of the lamp.]

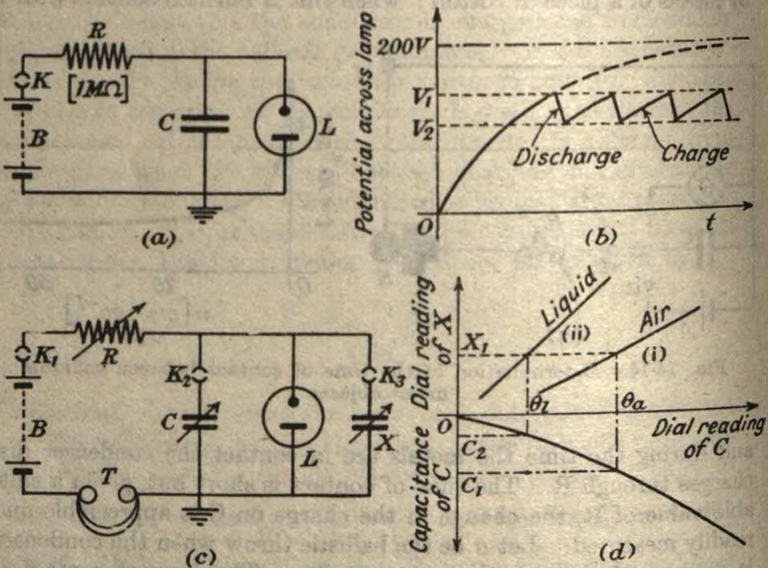


FIG. 14·15.—Determination of the permittivity of an insulating liquid.

To use this property of a neon lamp in a determination of the permittivity, κ, of an insulating liquid, two air-condensers, a resistance R, a high tension battery and a pair of phones T are arranged as in Fig. 14·15 (*c*). C is a variable standard air-condenser while X is an air-condenser with a scale but whose capacitance need not be known. Keys K_2 and K_3 enable either condenser to be placed in turn in parallel with the lamp L.

With C in parallel with L and at a setting near the upper end of its range, the value of R should be adjusted so that an audible note is heard in the phones. C is then replaced by X and this is adjusted until a note of the same frequency is heard. Several corresponding observations are made and the curve (i), Fig. 14·15 (*d*), is plotted.

The condenser C is then filled with the liquid under investigation and observations are made so that the curve (ii) may be plotted.

For a given setting of X, say X_1, the dial readings of C will be θ_a (with air as dielectric) and θ_l (with liquid dielectric). The air-capacitances corresponding to these readings may be read off from the lower part of the diagram. Let them be C_1 and C_2 respectively. Then $\kappa C_2 = C_1$, so that κ may be determined.

Example.—A neon lamp, shunted by a condenser of capacitance C farads, is connected in series with a resistance R ohms, the whole being connected across a constant supply of V volts.

Assuming that the lamp glows when the p.d. across it is V_1 [$V_1 < V$], and is extinguished when the p.d. falls to V_2, find the periodic time of the flicker produced by the above arrangement. Assume that when the lamp glows its resistance is r ohms.

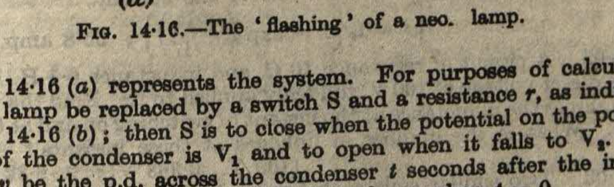


FIG. 14·16.—The 'flashing' of a neon lamp.

Fig. 14·16 (*a*) represents the system. For purposes of calculation let the lamp be replaced by a switch S and a resistance r, as indicated in Fig. 14·16 (*b*); then S is to close when the potential on the positive plate of the condenser is V_1 and to open when it falls to V_2.

Let v be the p.d. across the condenser t seconds after the instant when the lamp first glows, i.e. S first closes when $t = 0$.

Let $Q = CV_1$ be the charge on the positive plate of C at time $t = 0$. Then, with the notation indicated on the diagram,

$$(i - i_c)r = v.$$

Now $$i_c = C\frac{dv}{dt}, \quad \text{and} \quad i = \frac{V - v}{R}.$$

$$\therefore \frac{V - v}{R} = i = \frac{v + i_C r}{r} = \frac{v + Cr\dfrac{dv}{dt}}{r}.$$

$$\therefore \frac{dv}{dt} = \frac{V}{RC} - \frac{v}{C}\left(\frac{1}{R} + \frac{1}{r}\right)$$

$$= \frac{V\xi - vR}{CR\xi}, \quad \text{if } \xi = \frac{rR}{r + R}.$$

$$\therefore dt = \left(\frac{CR\xi}{V\xi - vR}\right)dv = -\frac{CR\xi}{R} \cdot \frac{d(V\xi - vR)}{V\xi - vR}.$$

Hence the time t_1 required for the potential drop across C to fall from V_1 to V_2 is given by

$$t_1 = C\xi \ln \frac{V\xi - V_1R}{V\xi - V_2R}.$$

Before the lamp can begin to glow again the potential drop across C must rise from V_2 to V_1. The time required for this to take place is given by

$$t_2 = CR \ln \left(\frac{V - V_2}{V - V_1}\right). \quad \text{[cf. p. 473.]}$$

The period required is $t_1 + t_2$ and this is

$$CR \ln \frac{V - V_2}{V - V_1} + C\xi \ln \left(\frac{V\xi - V_2R}{V\xi - V_1R}\right).$$

If $\xi \longrightarrow 0$, as is usual, the frequency of flashing becomes

$$\frac{1}{CR \ln \left(\dfrac{V - V_2}{V - V_1}\right)}.$$

EXAMPLES XIV

14·01. Derive an expression for the growth of current in an inductive circuit across which a steady voltage is applied. If the final current in such a circuit is 5 A., and the time taken to reach 1 A. is 0·06 sec., find the initial rate of rise of the current.

$\left[i = I_0\left\{1 - \exp\left(-\frac{R}{L}t\right)\right\}; \text{ 55·8 amp.sec.}^{-1}.\right]$

14·02. A circuit of resistance 10 Ω. and inductance 0·1 H. in series has a direct voltage of 100 V. suddenly applied to it. Find the voltage-drop across the inductance at the instant of switching on and at 0·01 sec. later. What is the number of linkages at these instants?

[100 V., 0·368 V.; 0, 6·32 × 10^8.]

14·03. Two similar coils each having 1000 turns of closely wound wire are arranged so that 60 per cent of the effective flux associated with one coil links with the other coil. A current of 10 A. in either coil produces a flux of 10^4 maxwells in that coil. Obtain values for the mutual inductance of the system and the self inductance of each coil. If the current in one coil changes uniformly from 2·5 A. to − 2·5 A. in 0·005 sec., find the e.m.f. induced in the other.

[6 mH., 10 mH., 6 V.]

14·04. A telegraph circuit consists of a battery with an electromotive force of 10 V. and internal resistance 1 Ω., a non-inductive line of 40 Ω. resistance, and a relay of 19 Ω. resistance and 1·2 H. inductance. If the armature of the relay is not attracted until the current rises to 0·05 A., estimate the interval of time that elapses between the closing of the circuit and the operation of the relay. [0·0071 sec.]

14·05. When a coil of resistance R ohms and inductance L henries is connected to a d.c. supply the current reaches 0·632 of its final steady value, I_0, in 1 sec. After the current has reached its final steady value the circuit is suddenly short-circuited. What will be the value of the current, in terms of I_0, 3 sec. later ? [0·050 I_0.]

14·06. The resistance and inductance of a series circuit are 4 Ω. and 20 H. respectively. At the instant of closing the supply switch, the current increases at the rate of 5 amp.sec.$^{-1}$. Calculate (*a*) the applied voltage, (*b*) the rate of growth of current when 10 amp. flows in the circuit, (*c*) the stored energy at this instant.
[100 V., 3 amp.sec.$^{-1}$, 1000 joule.]

14·07. Calculate the voltage at the terminals of a coil of resistance 10 Ω. and inductance 20 H. at the instant when the current is 16 A. and (*a*) increasing, (*b*) decreasing, at the rate of 8 amp.sec.$^{-1}$. Find also the stored energy under both conditions.
[320 V., 160 V., 2560 joule.]

14·08. Derive an expression for the electromagnetic energy associated with a steady current flowing in a coil. A steady voltage is applied suddenly across an inductive coil. Obtain an expression for the current in the coil at any subsequent instant. If the final current through such a coil is 10 amp., and the time taken for the current to reach 1 amp. is 0·02 sec., find the initial rate of rise of the current. Use your result to give a physical meaning to the term 'time-constant'.
[52·7 amp.sec.$^{-1}$.]

14·09. An e.m.f. of 10 volts is applied to a coil of wire having a resistance of 2 ohms and a self inductance of 1 henry. Derive an expression for the variation of current with time and calculate the time taken for the current to rise to half its steady value of 5 amperes. How would the time of rise be affected if the resistance of the coil were increased by 18 ohms and the applied e.m.f. increased so that the final steady value of the current remained 5 amperes ? (G)
[0·346 sec., 0·0346 sec.]

14·10. Define self inductance and the practical unit of inductance. Obtain, from energy considerations, the relation between the e.m.u. of inductance and the henry. A reactor of self-inductance 15 H. and resistance 3 Ω. is supplied at 30 V. Find (*a*) the time-constant of the reactor, (*b*) the maximum linkages, (*c*) the maximum value of stored energy. [5 sec., 1·5 × 10^{10} maxwell-turns, 750 joule.]

14·11. An inductive coil (R, L) is joined to a cell whose e.m.f. is V. Prove that after a time *t* seconds the current will be $VR^{-1}(1 - \alpha)$, where $L \ln \alpha + Rt = 0$. Also show that if the contact with the cell is then broken, the current *t* seconds later will be $VR^{-1}\alpha(1 - \alpha)$.

14·12. Describe and give the theory of a method for measuring a high resistance by allowing a charged condenser to leak through it for a known time. A charged 1 μF. condenser loses 10 per cent of its charge in 10 min. When a high resistance is connected between the terminals it loses 50 per cent of its charge in 2 min. Calculate the value of the high resistance. (A)
[1·78 × 10^8 ohm.]

14·13. A resistance R is connected across the terminals of a charged condenser of capacitance C. Find an expression for the time required for the potential difference across the condenser to fall from V_1 to V_2. Describe an experiment in which this result is used to measure a resistance of the order of 100 megohms, paying attention to the arrangements necessary for accuracy.

14·14. A condenser, of capacitance 2 μF., has a high resistance, R, connected across its terminals. In one minute the charge on the condenser is reduced by one-quarter. If the natural leak of the condenser is such that the charge on it falls by 2 per cent in one minute, obtain a value for R. [$1{\cdot}10 \times 10^8$ ohm.]

14·15. Two condensers of capacitance C_1 and C_2 are charged to potentials V_1 and V_2 respectively. They are then insulated from the supply and connected in parallel through a resistance R. Find an expression for the current which flows and hence show that the energy lost is independent of the value of R.

$$\left[i = \frac{V_1 - V_2}{R} \exp\left(-\frac{t}{CR}\right), \text{ where } \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}.\right]$$

14·16. Describe experiments you would perform to measure (*a*) a high resistance of the order of 10 megohms; (*b*) a low resistance of about 0·10 ohm.

The voltage across a condenser falls from 100 units to 90 units in 5 minutes due to defective insulation. When the terminals are wired by a resistance of 10 megohms the voltage drops from 100 units to 20 units in 2 minutes. Calculate the insulation resistance of the condenser in megohms. (G) [395 MΩ.]

14·17. A condenser of capacitance 9×10^{-4} μF. is connected in series with a coil of self inductance 0·4 mH., and of negligible resistance. Calculate the frequency of the electrical oscillations which would take place as a result of charging the condenser. Establish the formula you use. [$2{\cdot}65 \times 10^5$ cycle.sec.$^{-1}$.]

14·18. A condenser of capacitance C charged to a potential difference V has its terminals joined to the ends of a resistance R and self inductance L to form an oscillatory circuit. Obtain, from first principles, a value for (*a*) the period, (*b*) the decrement per cycle.

If L = 5 mH., C = 2 μF. and V is 10 V. and R is adjusted to the value for critical damping, find (*a*) the time at which the maximum current occurs, (*b*) the maximum value of the discharge current.

[10^{-4} sec., 0·0736 A.]

14·19. A condenser of capacitance C μF. is charged from a battery through a resistance of R MΩ. The electromotive force of the battery is V volt. When the condenser is fully charged the electromotive force is reduced to V_0 volt. Obtain a value for the voltage, v, across the condenser t sec. thereafter.

$$\left[v - V_0 = (V - V_0) \exp.\left(-\frac{1}{CR}t\right).\right]$$

14·20. A 10 μF. condenser is connected through a 2 MΩ. resistance to a direct-current supply. After being on charge for 25 sec. the condenser is disconnected and discharged through a spark-gap. Determine what percentage of the energy input from the supply is liberated in the spark. [35·7.]

14·21. A condenser C, Fig. Ex. 14·21, has a resistance R in series and this combination is shunted by a resistance R_0. Find how long it takes after the key, K, is closed before the total current from the supply reaches 45 mA., when V = 20 V., R_0 = 500 Ω., R = 1000 Ω., and C = 200 μF. [0·277 sec.]

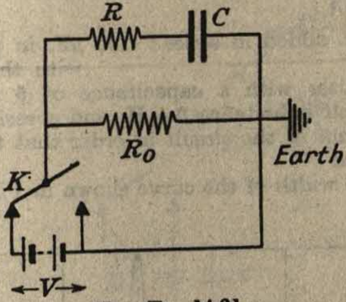


FIG. Ex. 14·21.

14·22. A resistance is shunted across the terminals of a 2 μF. condenser which has been previously charged to a potential difference of 50 V. If in 2·5 min. the potential difference falls to 40 V., calculate a value for the resistance in megohms. [336 MΩ.]

14·23. The insulated plate of a condenser C is given a charge Q_0. This plate is then connected through a resistance R to a battery of e.m.f. V. Show that after a time t the charge on the condenser is given by $q = VC + (Q_0 - VC)\exp\left(-\frac{t}{CR}\right)$.

What is the value of this charge after a long time?

A neon lamp is connected across the terminals of a condenser. This combination is in series with a 0·1 MΩ. resistance and a 100 V. battery. When the potential drop across the condenser reaches a value of 80 V. the lamp glows momentarily, the current through it ceasing when the potential difference across it drops to 70 V. If the condenser has a capacitance of 10 μF. determine the interval between successive flashes of the lamp. [VC, 0·405 sec.]

14·24. A neon lamp, shunted by a condenser of capacitance C farad., is connected in series with a resistance R ohm., the whole being connected across a constant potential difference of V volt. Assuming that the lamp glows when the potential drop across it is $V_1 (V_1 < V)$ and is extinguished when this potential falls to V_2, find the periodic time of the flicker produced by the above arrangement. The lamp when glowing has a resistance (a) zero, and (b) a small value r ohm.

$$\left[(a)\ CR\ln\frac{V - V_2}{V - V_1};\quad CR\ln\frac{V - V_2}{V - V_1} + C\xi\ln\frac{V\xi - V_1 r}{V\xi - V_2 r},\quad \text{where } \xi = \frac{rR}{r + R}.\right]$$

14·25. A condenser with a capacitance of 10 μF. is charged to a potential of 100 V. and is then discharged through a resistor of resistance 10^4 Ω. Calculate values for (a) the current at the commencement of the discharge, (b) the time required for the charge on the condenser to be halved and (c) the time required for the energy of the condenser to be halved. [(a) 0·01 amp., (b) 0·0697 sec., (c) 0·0349 sec.]

14·26. A condenser with a capacitance 10 μF. is charged to a potential of 100 V. and is then discharged through a coil having a resistance of 100 Ω. and a self inductance 0·4 H.; discuss the nature of the discharge. Find values for (*a*) the resistance, and (*b*) the capacitance of the condenser which must be added in order that the discharge may be critically damped.

[300 Ω. added in series; 150 μF. in parallel with the first condenser.]

14·27. A condenser with a capacitance of 5 μF. is discharged through a coil of self inductance 0·4 H. and a resistor. What must be the total resistance of the circuit in order that the discharge shall be critically damped? [566 ohms.]

14·28. From the width of the curve shown in Fig. Ex. 14·28 when $1 = 0{\cdot}2\ I_0$, obtain a value for the decrement of the oscillating circuit to which the curve refers. Construct resonance curves similar to that shown for a series circuit whose decrement is (*a*) 0·02, (*b*) 0·005. [0·0100.]

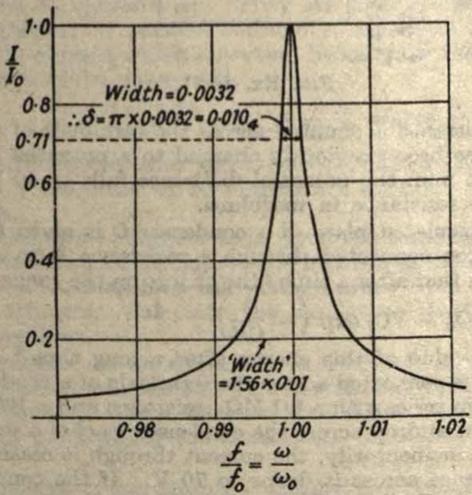


FIG. Ex. 14·28.

14·29. A condenser of capacitance 0·2 μF. is charged to a potential of 2000 V. and then discharged through a coil of resistance 5 Ω. and self-inductance 0·0125 H.

Calculate (*a*) the circular frequency, (*b*) the phase angle and decrement per cycle, (*c*) the maximum value of the current, and (*d*) the energy which is dissipated as heat in the first half-cycle.

[(*a*) 2×10^4 sec.$^{-1}$, (*b*) 5° 6′, 0·0628, (*c*) 7·87 A., (*d*) 0·024 joule.]

14·30. A condenser of capacitance 0·4 μF. is charged to a potential of 1000 V. and then discharged through a resistance of 10 Ω. and an inductance of 0·025 H.

Calculate (*a*) the initial energy of the condenser, (*b*) the time and magnitude of the current when it reaches its highest value, (*c*) the energy in the magnetic field at this instant, (*d*) the potential drop across the condenser at the same moment, and (*e*) the energy of the condenser at the same instant.

[(*a*) 0·20 joule., (*b*) $1{\cdot}57 \times 10^{-4}$ sec., 3·88 A., (*c*) 0·188 joule., (*d*) 122 V., 0·003 joule.]

14·31. The electrical system shown in Fig. Ex. 14·31 is charged from a 10 V. battery. Find the charge on the 3 μF. condenser three seconds after introducing the battery into the circuit, the key K being closed. If at time $t = 3$ sec. the key K is opened, what is the charge on the 3 μF. condenser 2 seconds later ?

[0·26 × 10^{-4} coulomb., 1·87 × 10^{-5} coulomb.]

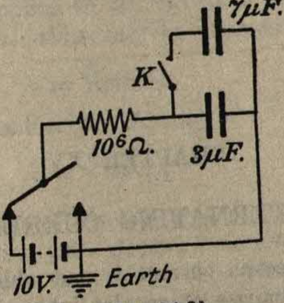


FIG. Ex. 14·31.

CHAPTER XV

ALTERNATING CURRENTS

An alternating electric current is one which passes through a complete cycle of changes at regular intervals. Each cycle consists of two half-cycles during one of which the current is chiefly or entirely positive whereas in the following half-cycle the current is chiefly or entirely negative.

Figs. 15·01 (*a*) and 15·01 (*b*) represent the wave-forms of two alternating currents, the times being plotted as abscissae, the zero of time in Fig. 15·01 (*a*) being chosen as the moment when the current passes through one of its zero values, whereas in Fig. 15·01 (*b*) the zero of time is chosen at random.

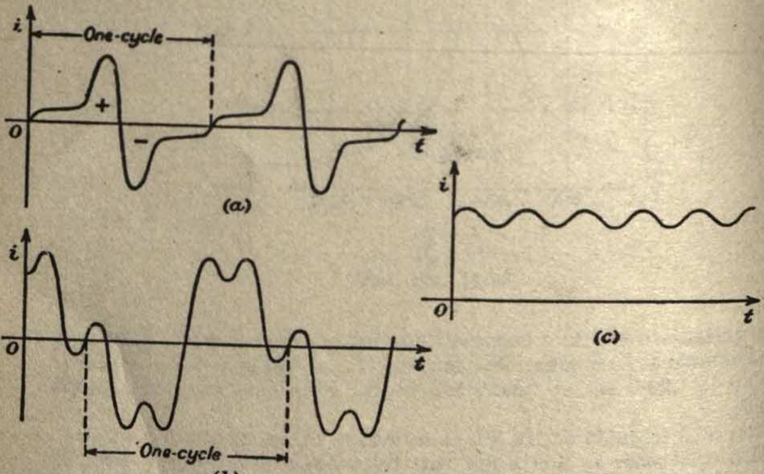


FIG. 15·01.—(*a*) and (*b*) Alternating currents, (*c*) a unidirectional current with 'ripple'.

If the current varies but does not change sign, i.e. it varies but is always positive (or negative), i.e. unidirectional, the current is said to ***pulsate.*** If the pulsations are small in magnitude there is said to be a ***ripple*** on the current—cf. Fig. 15·01 (*c*).

Alternating current of sinusoidal wave-form.—Calculations dealing with alternating currents are usually made on the assumption that the voltage applied to the circuit is sinusoidal, i.e. the variations of voltage with time may be expressed by

$$v = \hat{V} \sin \omega t,$$

or, if we chose a different origin of time,

$$v = \hat{V} \cos \omega t,$$

where $\hat{V}$ is the peak value of the voltage and $\omega = 2\pi f$, where f is the frequency. In dealing with electrical circuits we shall *always* assume that the wave-form is given by the 'cosine' form, as this eventually leads to a considerable simplification of the subject. It is somewhat difficult to obtain an applied voltage which has a pure sine-wave form. In general, the magnetic field in which a coil rotates to produce alternating current is not uniform; the departure from non-uniformity gives rise to small alternating voltages which are added to the fundamental; their frequencies are usually simple multiples of the fundamental frequency. The departure of the wave-form from a purely sinusoidal one is due entirely to these 'overtones', which, together with the fundamental, constitute the 'harmonics'. The first overtone is the second harmonic, etc.

More complex wave-forms.—Fig. 15·02 shows a voltage-time curve with a complex (or distorted) wave-form; the two components, each of sine-wave form, which go to make up the complex

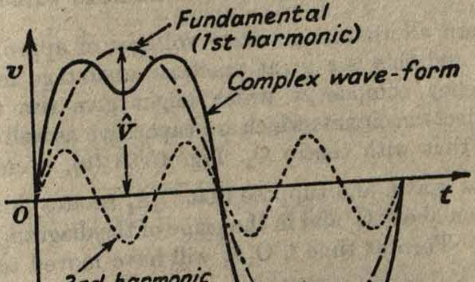


Fig. 15·02.—A complex wave-form and its harmonic components.

wave-form are shown by the dotted curves. More complex wave-forms have a large number of such components. The sine-curve whose frequency is equal to that of the 'complex' wave is known as the fundamental. The superposed curve, which together with the fundamental produces the wave-form shown in Fig. 15·02, has a circular frequency 3ω and amplitude $\frac{1}{3}\hat{V}$. Hence

$$v = \hat{V}\sin\omega t + \tfrac{1}{3}\hat{V}\sin 3\omega t$$
$$= \hat{V}\sin 2\pi f t + \tfrac{1}{3}\hat{V}\sin 6\pi f t. \qquad [\omega = 2\pi f.]$$

In a similar wave we may have

$$v = \hat{V}\cos\omega t + \tfrac{1}{3}\hat{V}\cos 3\omega t.$$

Purely sinusoidal wave-forms.—If the voltage and current are purely sinusoidal they are expressed by the equations

$$v = \hat{V}\cos\omega t, \quad \text{and} \quad i = \hat{I}\cos\omega t,$$

and these are the simplest wave-forms to consider. It can be shown, cf. Vol. I, p. 14, that all continuous periodic functions may be analysed into a series of sine and cosine functions of different frequencies.

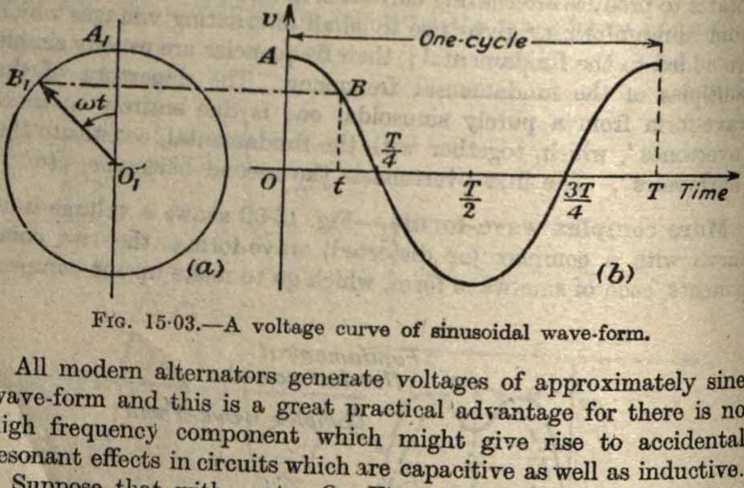


FIG. 15·03.—A voltage curve of sinusoidal wave-form.

All modern alternators generate voltages of approximately sine wave-form and this is a great practical advantage for there is no high frequency component which might give rise to accidental resonant effects in circuits which are capacitive as well as inductive.

Suppose that with centre O_1, Fig. 15·03 (*a*), a circle of radius $O_1A_1 = \hat{V}$ is drawn and suppose that O_1A_1 rotates in an anticlockwise direction about O_1 and in the plane of the diagram with angular velocity ω. Then at time t, O_1A_1 will have moved to O_1B_1 where $A_1\widehat{O_1}B_1 = \omega t$. Hence the projection of B_1 on O_1A_1 gives v, the instantaneous value of the applied voltage. Thus in Fig. 15·03 (*b*), if we mark off $Ot = t$ and $tB = v$, we get the point B, (t, v), on the voltage curve and if the process is continued the whole of the wave-form of the applied voltage may be constructed. It is seen at once that

$$f = \frac{\omega}{2\pi}, \quad \text{and} \quad T = \frac{1}{f} = \frac{2\pi}{\omega}.$$

The applied voltage necessary to send a current of sinusoidal wave-form through an R–L–C series combination.— Suppose that $i = \hat{I} \cos \omega t$ is the current in the circuit, shown in Fig. 15·04 (*a*). We have to determine the value of the supply voltage v necessary to maintain this current. Since the presence of the self inductance introduces an e.m.f. $-L\frac{di}{dt}$ into the circuit, this may be replaced by the equivalent circuit shown in Fig. 15·04 (*b*). Considering the rise in potential as we pass from E through C and R to the generator, we have

$$\frac{q}{C} + Ri = v - L\frac{di}{dt},$$

where q is the instantaneous value of the charge on the positive plate of the capacitor. The above equation may be written

$$\begin{aligned} v &= Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \\ &= R\hat{I} \cos \omega t - \omega L\hat{I} \sin \omega t + \frac{1}{\omega C}\hat{I} \sin \omega t + K, \end{aligned}$$

where K is an integration constant; its value is zero since when $i = 0$, v_c (across C) $= 0$. Hence

$$v = \hat{I}\left[R \cos \omega t - \left\{\omega L - \frac{1}{\omega C}\right\} \sin \omega t\right].$$

Now let

$$[\ldots] = A \cos \omega t \cos \phi - A \sin \omega t \sin \phi.$$

Then

$$A = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}, \quad \text{and} \quad \tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

$$\therefore v = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\,\hat{I} \cos (\omega t + \phi).$$

If ϕ is positive, i.e. $\omega L > \frac{1}{\omega C}$, the current *lags* behind the applied voltage.

If ϕ is negative, i.e. $\omega L < \frac{1}{\omega C}$, the current *leads* the applied voltage.

If ϕ is zero, i.e. $\omega L = \frac{1}{\omega C}$, the current and applied voltage are *in phase* and the circuit behaves as if it consisted of a pure resistance only.

If $v = \hat{V} \cos \omega t$, the value of the instantaneous current is given by

$$i = \frac{\hat{V}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos (\omega t - \phi).$$

The quantity $\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = Z = \sqrt{R^2 + X^2}$ is known as the ***impedance*** of the circuit; X is the ***reactance***. Both Z and X are measured in ohms.

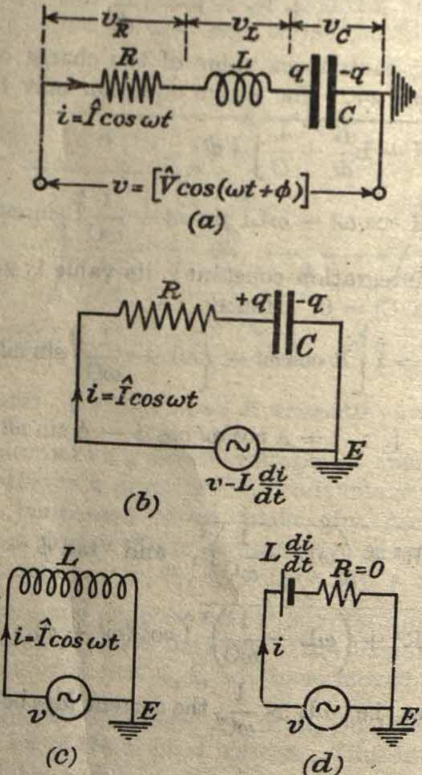

v_R
v_L
v_C
R
L
q
-q
C
$i = \hat{I} \cos \omega t$
$v = [\hat{V} \cos(\omega t + \phi)]$
(a)
R
+q
-q
C
$i = \hat{I} \cos \omega t$
E
$v - L\frac{di}{dt}$
(b)
L
$i - \hat{I} \cos \omega t$
v
E
(c)
$L\frac{di}{dt}$
R=0
i
v
E
(d)

FIG. 15·04.—An R–L–C series circuit fed from an a.c. supply.

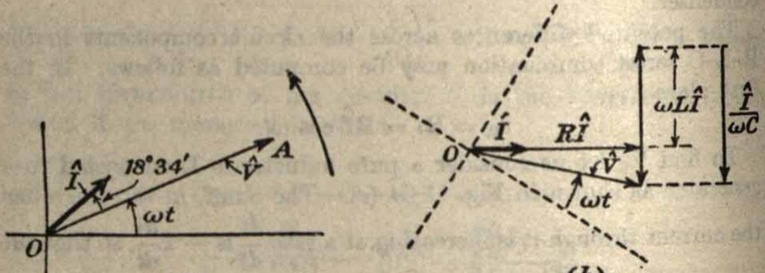
If $C = \infty$, we have the case of an inductive coil fed from an a.c. supply. The reactance is then purely inductive, i.e. $X = \omega L$, and the current lags the applied voltage by $\tan^{-1} \frac{\omega L}{R}$.

If $L = 0$, the system becomes that of a capacitor in series with

Also
$$\tan\phi = \frac{\omega L - \dfrac{1}{\omega C}}{R} = -\frac{3{\cdot}36}{10} = -0{\cdot}336.$$

$$\therefore\ \phi = -18^\circ\ 34'.$$

Fig. 15·05 (*a*) shows the phase relationship between the current and applied voltage, while Fig. 15·05 (*b*) gives the magnitudes and phase relations of the voltages across each of the three components.



(*a*) (*b*)

FIG. 15·05.—Phase differences in an R–L–C series circuit.

Important note. The methods adopted to find the current in a reactive circuit, i.e. in a circuit possessing reactance, only give expressions for the current at times when steady conditions have been reached. The complete analysis shows that initially transient currents are present but these generally die out very rapidly. In practice we have only to consider the state of affairs in a circuit when steady conditions have been reached.

Resonance.—If the inductive and capacitive reactances of a circuit are equal we have seen that the applied voltage and current are in phase. Now we have already seen that if a current is started in an R–L–C series circuit of small resistance, oscillations occur. If the applied voltage has the same frequency, viz. $\dfrac{1}{2\pi\sqrt{LC}}$, as the free frequency of the circuit, we have the phenomenon of ***current resonance;*** it corresponds, cf. p. 574, to velocity resonance in a mechanical system. In such an instance $\omega = \omega_0$, where ω_0 is the free circular frequency, and we have

$$i = \frac{\hat{V}}{R}\cos\omega_0 t,$$

so that if $R \longrightarrow 0$ the amplitude of the current may attain very high values even if the amplitude of the applied voltage is only moderately large. The current is in phase with the applied voltage and the acceptor circuit is of much value in radio receiving circuits. For example, if $R = 3\ \Omega$, $L = 400\ \mu H.$, $C = 0{\cdot}001\ \mu F.$

$$\omega_0 = \frac{1}{\sqrt{4 \times 10^{-4} \times 1 \times 10^{-9}}} \simeq 1{\cdot}6 \times 10^{6}\ \text{sec.}^{-1},$$

so that the voltage magnification is

$$\frac{\dfrac{\hat{I}}{\omega_0 C}}{R\hat{I}} = \frac{1}{\omega_0 CR} = 210.$$

Thus a signal of amplitude 5 mV. may result in a potential difference of more than 1 V. between the grid and filament of a valve arranged as indicated in Fig. 15·07 (*b*) if the damping effect of the

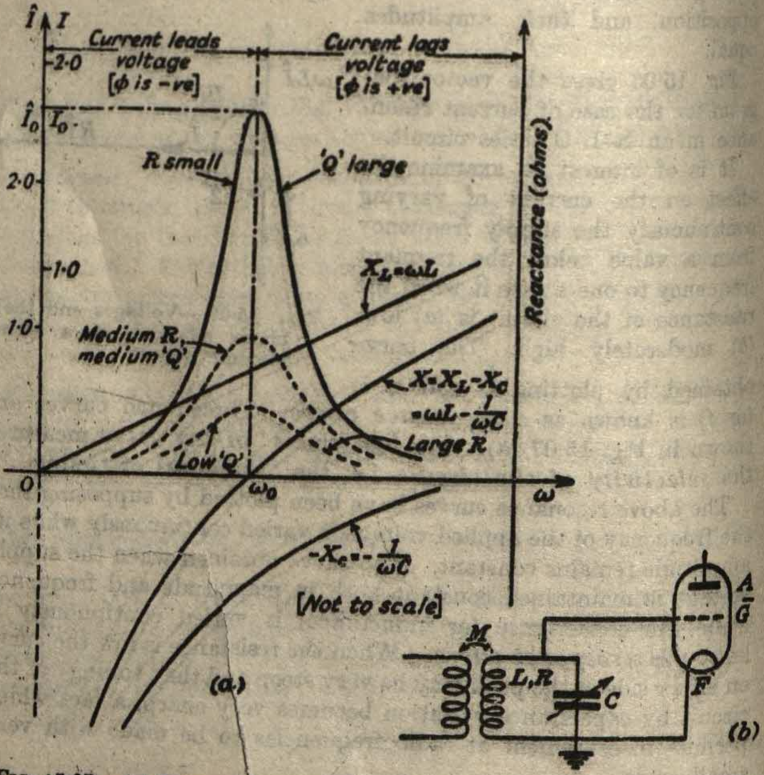


FIG. 15·07.—(*a*) Current resonance in an R–L–C series circuit; variation with ω of the inductive reactance (ωL) and the capacitive reactance $\frac{1}{\omega C}$, (*b*) voltage magnification by means of a tuned acceptor circuit.

internal circuit of the valve upon the acceptor current is not considered.

The lines X_L, $-X_C$, and X, which are superimposed on Fig. 15·07 (*a*), indicate how the inductive reactance, the capacitive reactance and the total reactance vary with the circular frequency ω. [X_C is essentially positive but $-X_C$ has been drawn to avoid confusion on the diagram.]

Root-mean-square values of applied voltage and current.—If a steady current of I amperes flows through a pure resistance R the rate of expenditure of energy (i.e. the power) is I^2R watt. or I^2R joule.sec.$^{-1}$. This power is expended in heating the resistance.

When the current passing through the resistance R is an alternating one the power varies from one moment to the next and the practical question which arises at once is to assign a value to that steady current which, when squared and multiplied by the resistance, will give the mean power dissipated in the resistance. If the instantaneous value of the current is

$$i = \hat{I}\cos\omega t,$$

and v is the instantaneous value of the potential difference across R, then at any instant the power expended is

$$p = vi = i^2R = R(\hat{I})^2\cos^2\omega t.$$

During one cycle (of period T), or any number of cycles, the mean power is

$$P = \frac{\int_0^T p\,dt}{\int_0^T dt} = \frac{\int_0^T R(\hat{I})^2\cos^2\omega t\,dt}{\int_0^T dt} = \frac{R}{T}(\hat{I})^2\int_0^T \cos^2\omega t\,dt$$

$$= \frac{R(\hat{I})^2}{T}\int_0^T\left[\frac{1+\cos 2\omega t}{2}\right]dt = \tfrac{1}{2}R(\hat{I})^2.$$

The equivalent or effective value of the current which, when squared and multiplied by R, gives the mean power is therefore $\frac{1}{\sqrt{2}}\hat{I}$. This is also known as the ***virtual*** or ***root-mean-square (r.m.s)*** value of the current and is denoted by I.

Similarly, the r.m.s. value of a sinusoidal voltage of amplitude $\hat{V}$ is $\frac{\hat{V}}{\sqrt{2}}$ and this we call V.

If the current is truly sinusoidal it may be represented by $i = \hat{I}\cos\omega t$, cf. Fig. 15·08, which indicates the wave-form of such a current. The graph of i^2 against t is also shown in Fig. 15·08 and the area below this curve is directly proportional to the energy dissipated

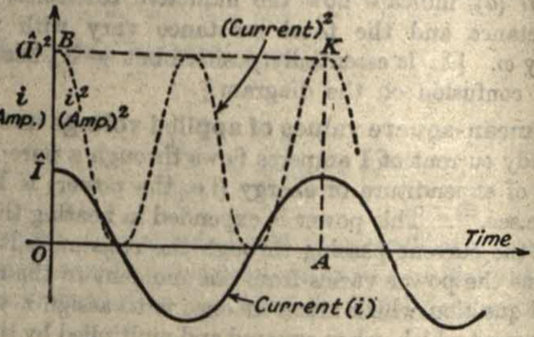


Fig. 15·08.—The r.m.s. current.

in a given time. Now if we consider a time interval equal to any integral multiple of the period $\left(\frac{2\pi}{\omega}\right)$, the rectangular area which encloses the i^2 curve, e.g. OAKB, is exactly twice the area below the i^2 curve, for $\int_0^{\mathrm{T}} i^2\,dt = \frac{(\hat{\mathrm{I}})^2}{2}\mathrm{T}$, where $\mathrm{T} = \frac{2\pi}{\omega}$. Hence the energy dissipated by the alternating current in a given time is one half that dissipated by a z.f. current* of magnitude $\hat{\mathrm{I}}$ in the same time; this agrees with the result obtained earlier.

Root-mean-square values of current and voltage may be measured by hot-wire instruments, cf. p. 522, or by electrodynamometers, cf. p. 324.

If the alternating current is not sinusoidal it may be represented by a Fourier series such as

$$i = \mathrm{I}_0 + \hat{\mathrm{I}}_1 \cos(\omega t - \phi_1) + \hat{\mathrm{I}}_2 \cos(2\omega t - \phi_2) + \hat{\mathrm{I}}_3 \cos(3\omega t - \phi_3) + \cdots$$

where I_0 indicates that, in effect, there is a direct current superimposed upon the alternating currents. The r.m.s. value of this current, which must be evaluated over an integral number of cycles of the fundamental current, is given by I, where

$$\mathrm{I}^2 = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} [\mathrm{I}_0 + \hat{\mathrm{I}}_1 \cos(\omega t - \phi_1) + \hat{\mathrm{I}}_2 \cos(2\omega t - \phi_2) + \ldots]^2 dt$$

$$= \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} [\mathrm{I}_0{}^2 + \ldots + (\hat{\mathrm{I}}_n)^2 \cos^2(n\omega t - \phi_n) + \ldots$$

---

* A direct current is now called a zero frequency (z.f.) current.

$$+ \mathrm{I}_0\hat{\mathrm{I}}_n \cos(n\omega t - \phi_n) + \cdots$$

$$+ \hat{\mathrm{I}}_n\hat{\mathrm{I}}_m \cos(n\omega t - \phi_n)\cos(m\omega t - \phi_m) + \cdots]dt$$

$$= \mathrm{I}_0{}^2 + \frac{(\hat{\mathrm{I}}_1)^2}{2} + \frac{(\hat{\mathrm{I}}_2)^2}{2} + \frac{(\hat{\mathrm{I}}_3)^2}{2} + \cdots$$

because the mean values of all terms which are cosines or the product of cosines, such as $\cos(n\omega t - \phi_n)\cos(m\omega t - \phi_m)$, are zero.

$$\therefore \mathrm{I} = \sqrt{\mathrm{I}_0{}^2 + \mathrm{I}_1{}^2 + \mathrm{I}_2{}^2 + \mathrm{I}_3{}^2 + \cdots},$$

where I_1, I_2, I_3 . . . are the effective values of the harmonic components.

Power in a.c. circuits ; power factor.—The instantaneous value of the power expended in an a.c. circuit, or part of a circuit, is the product of the instantaneous current and the instantaneous voltage across the circuit (or the part of the circuit considered). Suppose that $v = \hat{\mathrm{V}} \cos \omega t$ is the voltage applied to R, L and C in series.* Then it has been shown that

$$i = \frac{\hat{\mathrm{V}}}{\sqrt{\mathrm{R}^2 + \left(\omega\mathrm{L} - \dfrac{1}{\omega\mathrm{C}}\right)^2}} \cos(\omega t - \phi),$$

$$\left[\text{where } \tan\phi = \frac{\omega\mathrm{L} - \dfrac{1}{\omega\mathrm{C}}}{\mathrm{R}}\right].$$

$$= \hat{\mathrm{I}} \cos(\omega t - \phi), \text{ say.}$$

Hence the energy dissipated in time δt is $vi\,\delta t$, i.e.

$$\hat{\mathrm{V}}\hat{\mathrm{I}} \cos \omega t . \cos(\omega t - \phi)\delta t = \delta\mathrm{W} \text{ (say).}$$

The instantaneous rate at which energy is being spent is

$$\lim \frac{\delta\mathrm{W}}{\delta t} = \frac{d\mathrm{W}}{dt} = \hat{\mathrm{V}}\hat{\mathrm{I}} \cos \omega t \cos(\omega t - \phi) = p \text{ (say).}$$

Hence, P, the mean power is given by

$$\mathrm{P} = \frac{\int_0^{\mathrm{T}} p\,dt}{\int_0^{\mathrm{T}} dt} = \frac{\int_0^{\mathrm{T}} \hat{\mathrm{V}}\hat{\mathrm{I}} \cos \omega t \cos(\omega t - \phi)dt}{\int_0^{\mathrm{T}} dt}$$

$$= \frac{1}{\mathrm{T}} \hat{\mathrm{V}}\hat{\mathrm{I}} \int_0^{\mathrm{T}} \cos \omega t \cos(\omega t - \phi)dt$$

---

* The theorem is not limited to any particular circuit for if $v = \hat{\mathrm{V}} \cos \omega t$, $i = \hat{\mathrm{I}} \cos(\omega t - \phi)$ whatever form the load may take.

$$= \frac{\hat{V}\hat{I}}{T}\cdot\frac{1}{2}\int_0^T [\cos(2\omega t - \phi) + \cos\phi]dt$$

$$= \frac{1}{2}\frac{\hat{V}\hat{I}}{T}[0 + T\cos\phi)] = \tfrac{1}{2}\hat{V}\hat{I}\cos\phi$$

$$= VI\cos\phi,$$

where V and I are the r.m.s. values of the applied voltage and the current. Hence $P = \frac{1}{2}\frac{(\hat{V})^2}{Z}\cos\phi$, where Z is the impedance of the circuit.

The factor $\cos\phi$ is known as the *power factor* of the circuit and since $\tan\phi = \frac{X}{R} = \frac{\omega L - \frac{1}{\omega C}}{R}$, it is given by

$$\cos\phi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{R}{Z}.$$

Since $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, $\cos\phi$ is always positive, i.e. energy is always expended by the generator.

Moreover, since $\cos\phi = \frac{R}{Z}$ and $ZI = V$, we have

$$P = VI\cos\phi = ZI^2.\frac{R}{Z} = I^2R.$$

This equation shows that it is only on account of the presence of resistance that power is expended in an a.c. circuit.

Let us examine the power, p, at any instant in the above circuit and its variation with time. We have

$$p = \tfrac{1}{2}\hat{V}\hat{I}[\cos(2\omega t - \phi) + \cos\phi] = VI[\cos(2\omega t - \phi) + \cos\phi],$$

so that p consists of a constant quantity $VI\cos\phi$ and a quantity $VI\cos(2\omega t - \phi)$ which varies with time with double the frequency of the applied voltage or current.

Example.—$v = 20\cos\omega t$, $i = 10\cos(\omega t - 0{\cdot}628)$, $f = 50$ sec.$^{-1}$.

Then $$p = vi = \tfrac{1}{2}.200[\cos 0{\cdot}628 + \cos(628t - 0{\cdot}628)]$$
$$= 80{\cdot}9 + 100\cos(628t - 0{\cdot}628).$$

The power factor is cos 36° = 0·809 [0·628 rad. = 36°].

The wave-forms of the applied voltage and current are shown in

Then

$$\frac{dy}{dC} = \frac{1}{Z^2}\left[0 - R.\frac{1}{2\left\{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right\}^{\frac{1}{2}}}.2\left(\omega L - \frac{1}{\omega C}\right) + \left(\frac{1}{\omega C^2}\right)\right]$$

$$= \frac{-R\left(\omega L - \frac{1}{\omega C}\right)\left(\frac{1}{\omega C^2}\right)}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1\cdot 5}}.$$

This is zero when $\left(\omega L - \frac{1}{\omega C}\right) = 0$, i.e. the combination is in resonance with the supply. When this condition is satisfied $\phi = 0$ and the power factor is unity; at the same time the capacitor will have a capacitance C_0 given by

$$C_0 = \frac{1}{\omega^2 L}.$$

When $C = 0$, the impedance is infinite, i.e. the power factor is zero, while when $C \longrightarrow \infty$, i.e. no condenser is present, the power factor is

$$\left(\frac{R}{Z}\right)_{C\to\infty} = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}.$$

The variation of power factor with capacitance is therefore as shown in Fig. 15·10; it is a maximum when $C = C_0$.

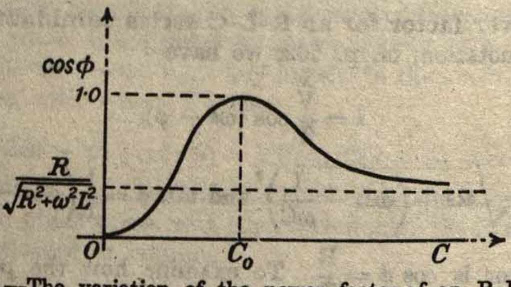


FIG. 15·10.—The variation of the power factor of an R–L–C series combination with C.

Now the expression for y may be written

$$y = \frac{\omega CR}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}},$$

so that when $C \rightarrow 0$, but is not actually zero,

$$[y]_{C \rightarrow 0} = \omega CR,$$

and hence at the origin the slope of the curve is ωR.

Power factor improvement.—If V is the potential difference across a load, I the current through it and ϕ the phase difference between the potential and current vectors, the mean power, cf. p. 510, is VI cos ϕ, where $0 < \phi < \frac{1}{2}\pi$. If ϕ is large, cos ϕ is small, so that to obtain the necessary power in the load a larger current must be used than when ϕ is small. For example, consider a single phase generator at 500 V. With a power factor of 0·7, the current I required to supply 1000 kW. is given by

$$500 \times I \times 0{\cdot}7 = 1000 \times 1000,$$

i.e.

$$I = 286 \text{ A}.$$

If the power factor could be raised to 0·9 the current would be 222 A. Thus, with a low power factor, more generating plant is required to supply the extra current. When the power factor is raised the size of the plant can be reduced and a cable of smaller cross-sectional area used. In engineering practice the simplest method for improving the power factor is to connect a condenser of suitable capacitance in parallel with the load. The following example shows how such a condenser raises the power factor and reduces the total current from the generator.

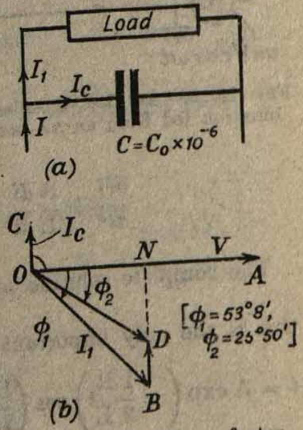


FIG. 15·11.—Power factor improvement.

Example.—A single-phase motor running off a 440 V., 50-cycle.sec.$^{-1}$ supply, takes 40 A. at a power factor 0·60 lagging. Obtain a value for the capacitance of the shunting condenser necessary to raise the power factor to 0·90 lagging, the load remaining unchanged.

Let ϕ_1 be the phase difference when no condenser is present. Since $\cos\phi_1 = 0{\cdot}60$, $\phi_1 = 53°\ 8'$. If ϕ_2 is the phase difference when a condenser is placed in parallel with the load, as shown in Fig. 15·11 (*a*), $\cos\phi_2 = 0{\cdot}90$, i.e. $\phi_2 = 25°\ 50'$. In Fig. 15·11 (*b*), let OA represent the voltage vector (r.m.s. value is used since this will only affect the scale of the diagram). Then $OB = I_1 = 40$ amp. is the current vector and this lags at 53° 8′. If C is the capacitance of the condenser, its reactance is $X_C = \frac{1}{\omega C}$, where $\omega = 2\pi f = 314$. Hence the current in the condenser branch is given by

$$I_C = 440 \times \omega C_0 \times 10^{-6},$$

where C_0 is the capacitance of the condenser in microfarads. Thus

$$I_C = 0{\cdot}138\ C_0,$$

and on the diagram this is represented by OC, a vector of the correct magnitude and at right angles to OA.

Let I be the total current from the supply. Since the total power is unaltered (there is no power in the condenser arm)

$$440 \times 40 \times 0{\cdot}60 = 440 \times I \times 0{\cdot}9,$$

i.e.

$$I = 26{\cdot}7 \text{ amp.}$$

In the diagram this is represented by OD, where $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{OC}$, if $\overrightarrow{BD} = \overrightarrow{OC}$. Since BD = BN − DN, we have

$$I_C = I_1 \sin \phi_1 - I \sin \phi_2.$$

$$\therefore\ 0{\cdot}138\, C_0 = 40 \sin 53^\circ\, 8' - 26{\cdot}7 \sin 25^\circ\, 50'$$

$$= 20{\cdot}4.$$

$$\therefore\ C_0 = 148, \qquad \text{or } C = 148\mu F.$$

THE MORE COMPLETE THEORY OF ALTERNATING CURRENT CIRCUITS

Current in a circuit containing resistance, inductance and capacitance, in series connected to an a.c. supply.—[Forced oscillations in a R–L–C series circuit.] The circuit is shown in Fig. 15·12. With the usual notation the differential equation giving the instantaneous value of the current is

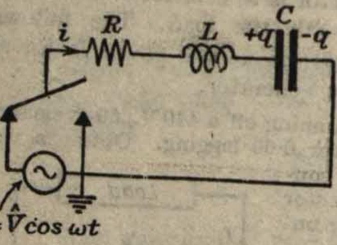


FIG. 15·12.—An R–L–C series combination fed from an a.c. supply.

$$\frac{q}{C} + Ri = v - L\frac{di}{dt},$$

where $v = \hat{V} \cos \omega t$. Thus

$$L\frac{di}{dt} + Ri + \frac{q}{C} = \hat{V} \cos \omega t.$$

Since $i = \dot{q}$, we have

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = -\frac{\omega \hat{V}}{L} \sin \omega t.$$

The complete solution of this equation is, if $\frac{R}{L} < 2\sqrt{\frac{1}{LC}}$, and this is the only important instance in practice,

$$i = A \exp\left(-\frac{1}{2}\frac{R}{L}t\right) \cos\left(\frac{1}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}.t - \phi'\right)$$

$$+ \frac{\left(-\frac{\omega \hat{V}}{L}\right)}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2\omega^2}{L^2}}} \sin(\omega t - \psi),$$

where A and ϕ' are integration constants and

$$\tan\psi = -\frac{R}{\left(\omega L - \frac{1}{\omega C}\right)}.$$

After the lapse of a finite time, which is not small, the term with the exponential vanishes and we have

$$i = -\frac{\frac{\omega\hat{V}}{L}}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2\omega^2}{L^2}}}\sin(\omega t - \psi)$$

$$= \frac{\hat{V}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos(\omega t - \phi).$$

where $-\cot\psi = \tan\phi$, so that

$$\tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

Hence
$$i = \hat{I}\cos(\omega t - \phi),$$

where, if Z is the impedance, $\hat{I} = \hat{V}Z^{-1}$.

Thus, at first, there exists a natural oscillation whose frequency is $\frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$, together with a forced oscillation of frequency $\frac{\omega}{2\pi}$ and amplitude $\frac{\hat{V}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$. [In wireless circuits where $R \rightarrow 0$, the frequency of the natural oscillation becomes $\frac{1}{2\pi\sqrt{LC}}$.]

The natural oscillation has an attenuation constant $\frac{R}{2L}$. The period is

$$\frac{4\pi}{\sqrt{4\beta - \alpha^2}} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}},$$

and the logarithmic decrement per cycle is given by

$$\delta = \tfrac{1}{2}\alpha T = \frac{4\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}\cdot\frac{R}{4L}.$$

[In wireless circuits this becomes

$$\pi R\sqrt{\frac{C}{L}} = \pi R\frac{1}{\omega L} = \pi R\frac{1}{\left(\frac{1}{\omega C}\right)}.$$

In such circuits the logarithmic decrement per cycle is equal to π times the ratio of the resistance to the inductive reactance or to the capacitive reactance.]

After the natural oscillation has died away, there remains the forced vibration of amplitude $\dfrac{\hat{V}}{\sqrt{R^2+\left(\omega L-\dfrac{1}{\omega C}\right)^2}}$. This is a maximum when $\omega^2 = LC$. The current is then $\dfrac{\hat{V}}{R}\cos \omega t$, i.e. $\phi = 0$ and the circuit behaves as if it were a pure resistance; its power factor is unity.

A series-resonance curve.—Since the inductive and capacitive reactances are usually very large compared with R, slight inequalities in ωL and $\dfrac{1}{\omega C}$, i.e. slight departures from the condition for resonance, make great changes in the amplitude of the current. The curve obtained by plotting i against ω (or f) is known in this instance as a ***series-resonance curve.*** Three such curves are shown in Fig. 15·07 (*a*), cf. p. 506. The 'peakiness' of any curve measures the 'sharpness of tuning'. The curves are symmetrical only if we are dealing with a feebly damped circuit.

The impedance of the circuit is given by

$$Z = \sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}.$$

If we call $\dfrac{\omega}{\omega_0} = \xi$, where $\omega_0 = \dfrac{1}{\sqrt{LC}}$, the expression for Z becomes

$$Z = \left[R^2 + \left(L\xi.\frac{1}{\sqrt{LC}} - \frac{\sqrt{LC}}{C\xi}\right)^2\right]^{0\cdot5} = \sqrt{R^2+\frac{L}{C}\left(\xi-\frac{1}{\xi}\right)^2},$$

and
$$\tan\phi = \frac{\sqrt{\frac{L}{C}}\left(\xi-\frac{1}{\xi}\right)}{R}.$$

Hence for 'frequencies' less than ω_0, ϕ is negative, so that the current leads the impressed voltage. At resonance, $\phi = 0$, i.e. the current is in phase with the applied voltage. At higher 'frequencies', ϕ is positive and the current lags behind the applied voltage.

Further remarks on the decrement per cycle of an oscillatory circuit. When $\ddot{\xi} + \alpha\dot{\xi} + \beta\xi = 0$ is the equation representing an oscillatory motion the appropriate solution is

$$\xi = A \exp(-\tfrac{1}{2}\alpha t) \cos(\tfrac{1}{2}\sqrt{4\beta - \alpha^2}.t - \phi),$$

where A and ϕ are arbitrary constants. The quantity $\frac{1}{2}\alpha$ is known as the ***attenuation constant*** and the ***decrement per cycle*** is $\frac{1}{2}\alpha T$, where T is the period of the natural oscillations; it is given by

$$T = \frac{4\pi}{\sqrt{4\beta - \alpha^2}}.$$

The natural circular frequency is $\frac{1}{2}\sqrt{4\beta - \alpha^2}$, so that the natural frequency is $\frac{1}{4\pi}\sqrt{4\beta - \alpha^2}$; this will be called f'.

For an oscillatory circuit consisting of R, L and C in series we have

$$\alpha = \frac{R}{L}, \quad \text{and} \quad \beta = \frac{1}{LC}.$$

When an alternating voltage given by $v = \hat{V} \cos \omega t$ is applied to the above circuit the instantaneous current, after the initial transient effects have died away, is given by

$$i = \frac{\hat{V}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \phi),$$

where $\tan \phi = \dfrac{\omega L - \frac{1}{\omega C}}{R}$. For such a circuit the decrement is

$$\delta = \tfrac{1}{2}\alpha T = \frac{1}{2}\frac{\alpha}{f'} = \frac{R}{2f'L}.$$

[Mechanical oscillators, e.g. the pendulums of astronomical clocks, can be made with extremely low decrements; such low values for the decrement could not be approached in electrical circuits unless some material of vastly better conductivity than copper were available—one might use lead at liquid air temperature.]

Now ω_0, the resonant circular frequency, is given by $\omega_0\sqrt{LC} = 1$, so that the resonant frequency is given by

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}},$$

while the natural frequency, f', is given by

$$f' = \frac{1}{4\pi}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}.$$

$$\therefore\ (f')^2 = \frac{1}{16\pi^2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right) = f_0^2 - \frac{1}{16\pi^2}\frac{R^2}{L^2} = f_0^2 - \frac{4(f')^2}{16\pi^2}\delta^2.$$

$$\therefore\quad \frac{f_0}{f'} = \sqrt{1 + \frac{\delta^2}{4\pi^2}},$$

so that even when δ is as large as 0·1, $\frac{f_0}{f'}$ does not differ from unity by more than one part in ten thousand. Hence, in practice, it is never necessary to differentiate between

$$\delta = \frac{R}{2f'L}, \text{ and } \delta = \frac{R}{2f_0L} = \pi\sqrt{\frac{C}{L}}R.$$

Practical methods for the determination of the decrement for an oscillatory circuit are given later, cf. pp. 578 *et seq.*

On the effect of introducing a pure inductance into an R–C circuit fed from an a.c. supply, $v = \hat{V}\cos\omega t$.—With the condenser and resistance alone in the circuit, the impedance is $\sqrt{R^2 + \frac{1}{\omega^2C^2}}$, so that the r.m.s. current is

$$I = \frac{V}{\sqrt{R^2 + \frac{1}{\omega^2C^2}}} = I_{L\to 0} \text{ (say)}.$$

When the inductance is added the impedance becomes

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

and hence the r.m.s. current will exceed its initial value $I_{L\to 0}$ if

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 < R^2 + \frac{1}{\omega^2C^2},$$

i.e. $\quad \omega L - \frac{1}{\omega C} < \frac{1}{\omega C}$, or $\omega L - \frac{1}{\omega C} < -\frac{1}{\omega C}$

i.e. $\quad L < \frac{2}{\omega^2C}$, or $L < 0$.

The second condition is impossible.

The r.m.s. current is a maximum when $\omega L - \frac{1}{\omega C} = 0$, a fact which can be formally established by using

$$\frac{dI}{dL} = \frac{V(-\frac{1}{2})}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1\cdot 5}} . 2\left(\omega L - \frac{1}{\omega C}\right) . \omega.$$

For a stationary value of I, the above expression is zero, i.e.

$$\omega = 0, \quad \text{or} \quad \omega L - \frac{1}{\omega C} = 0.$$

The condition $\omega = 0$ corresponds to a d.c. circuit and hence is

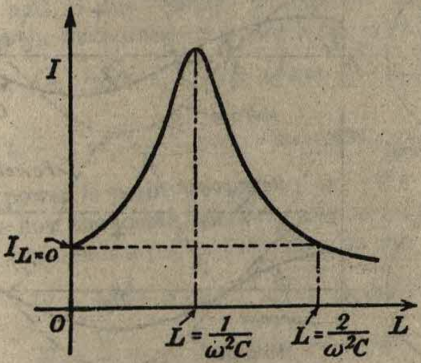


FIG. 15·13.—Variation with L of the current in an R–L–C series circuit.

excluded. Hence the condition $\omega L - \frac{1}{\omega C} = 0$ gives a maximum r.m.s. current. The variation of the r.m.s. current with L is shown in Fig. 15·13.

Power and its variation with phase difference in an a.c. circuit.—It has been shown that when a sinusoidal voltage given by $v = \hat{V} \cos \omega t$ is applied to a R–C–L series circuit the current at any instant, after the transient current has died away, is given by $i = \frac{\hat{V}}{Z} \cos (\omega t - \phi) = \hat{I} \cos (\omega t - \phi)$, where ϕ and Z are constants for the load. The amount of energy supplied during a time δt is given by

$$\delta W = vi\, \delta t.$$

The sign of δW is therefore positive as long as the two cosine functions have the same sign. Under such circumstances the energy is then supplied by the generator to the circuit where it is partly transformed into mechanical work, partly into Joule heat, and partly stored as energy in the magnetic and electric fields. As the angles ωt and $\omega t - \phi$ may each be anywhere within the range 0 to 2π, the functions will at certain times be negative when δW may then be negative. At such times the generator is receiving back energy from the circuit and it is the store of electromagnetic energy in the field which supplies this energy.

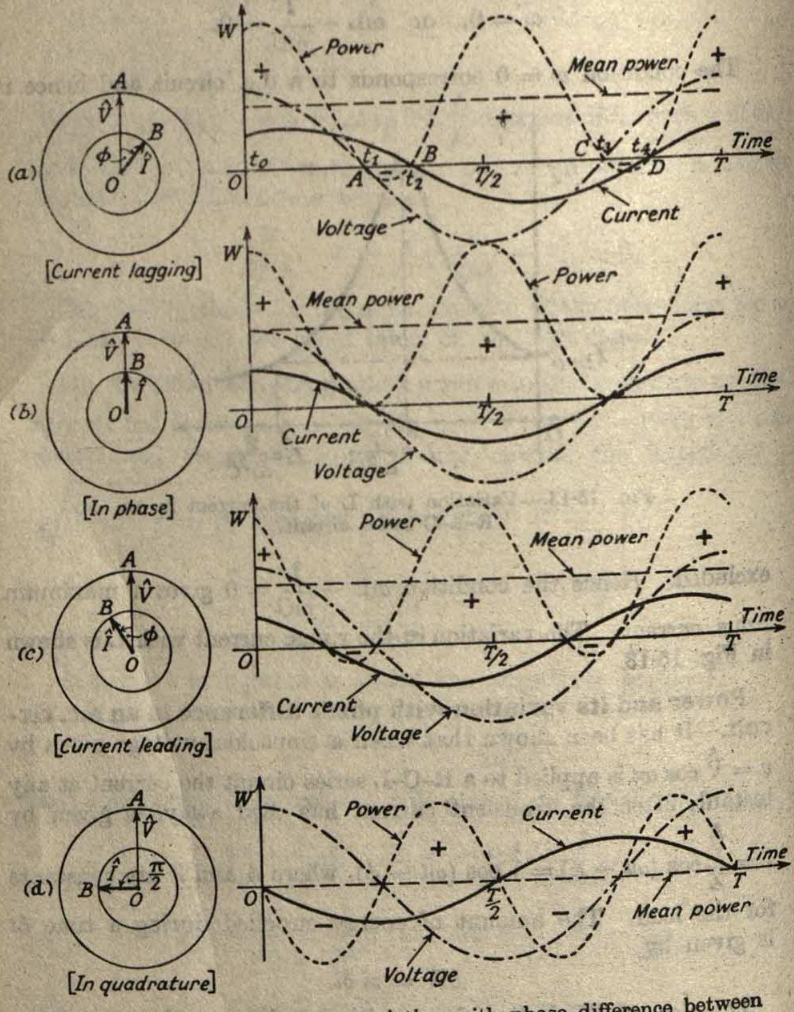


FIG. 15·14.—Power and its variation with phase difference between applied voltage and current.

Figs. 15·14 (*a*)–(*d*) indicate how the power developed in the circuit varies as the initial phase difference ϕ changes. Let us consider the first diagram in greater detail. The voltage-time and the current-time curves begin at an instant $t = t_0$ when both v and i have positive values; at A the voltage is zero but the current is not zero until B is reached. The time interval AB corresponds to the phase lag ϕ. Now initially

$$v_{t=t_0} = \hat{V} \cos \omega t_0, \quad \text{and} \quad i_{t=t_0} = \hat{I} \cos (\omega t_0 - \phi).$$

If the voltage is zero at time t_1, and the current at t_2, we have

$$\omega t_1 = \tfrac{1}{2}\pi, \quad \text{and} \quad \omega t_2 - \phi = \tfrac{1}{2}\pi,$$

i.e.

$$\phi = \omega(t_2 - t_1).$$

Between A and B the voltage is negative and the current positive so that the instantaneous power is negative in this region. From B to C, i.e. from t_2 to t_3, where $t_3 - t_1 = \dfrac{\pi}{\omega} = \dfrac{T}{2}$ and $t_2 - t_1 = \dfrac{\phi}{\omega}$, or $t_3 - t_2 = \dfrac{\pi - \phi}{\omega}$, both v and i are negative so that the instantaneous power is positive; and so on. The curve marked 'power' shows how the instantaneous power varies during a cycle. It has already been shown [cf. p. 510] that for one complete cycle

$$W = \tfrac{1}{2}\hat{V}\hat{I}T \cos \phi, \qquad [T = \text{period}]$$

so that

$$P = \frac{W}{T} = \tfrac{1}{2}\hat{V}\hat{I} \cos \phi = VI \cos \phi.$$

This equation signifies that the mean power (the so-called true power) of an alternating current is equal to the product of the r.m.s. voltage and the r.m.s. current and the cosine of the phase angle.

The quantity VI is known as the apparent power and from it the true power is obtained by multiplying by the power factor, $\cos \phi$.

Further note on the impedance of a circuit. If the instantaneous value of the voltage impressed upon a circuit is given by

$$v = \hat{V}_1 \cos \omega t + \hat{V}_2 \cos 2\omega t + \ldots$$

then the instantaneous current may be represented by

$$i = \hat{I}_1 \cos (\omega t - \phi_1) + \hat{I}_2 \cos (2\omega t - \phi_2) + \ldots$$

The r.m.s. values of the voltage and the current are given respectively by

$$V^2 = \tfrac{1}{2}(\hat{V}_1)^2 + \tfrac{1}{2}(\hat{V}_2)^2 + \ldots,$$

and

$$I^2 = \tfrac{1}{2}(\hat{I}_1)^2 + \tfrac{1}{2}(\hat{I}_2)^2 + \ldots$$

The quantity $\dfrac{V}{I}$ is called the ***impedance***, Z, of the circuit. Thus by measuring V and I a value for the impedance may be obtained. If, in addition, one measures the power, which is equivalent to

VI cos ϕ, where cos ϕ is the power factor, then ϕ is known. We may then write

$$R = Z\cos\phi, \quad \text{and} \quad X = Z\sin\phi,$$

where R and X are the effective values of the resistance and reactance of the circuit. To interpret R and X in terms of the constants of the components in the circuit is, in general, quite impossible.

Hot-wire instruments.—In these instruments the current to be measured, or a known fraction of it, is passed through a fine wire which expands due to the Joule effect in the wire. If the resistance and coefficient of expansion of the wire are constant, the heating and the expansion are each directly proportional to the square of the current. If the expansion is sufficiently large, it may be used to produce deflexions which are likewise proportional to the square of the current through the wire.

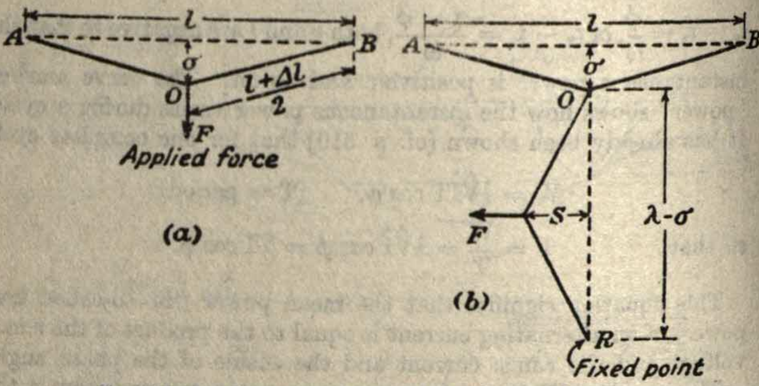


FIG. 15·15.—Sag-magnification in hot-wire instruments.

Since such instruments obey a square law, they are used to measure alternating currents (or voltages) as well as the corresponding d.c. quantities. Their great advantage lies in the fact that hot-wire instruments measure r.m.s. values of the current (or voltage) regardless of the wave-form or frequency. It should also be noted that they are not affected by stray magnetic fields.

In commercial instruments, the hot-wire is too short for the expansion to be used directly to produce a measurable deflexion; a magnifying device is used. Thus, let AB, Fig. 15·15 (a), be the hot-wire, of length l and let its expansion after reaching a steady temperature be Δl. Then if a downward force, F, is applied to the mid-point, O, of the wire, the sag σ is given by

$$\sigma = \sqrt{\left(\frac{l+\Delta l}{2}\right)^2 - \left(\frac{l}{2}\right)^2} = \sqrt{\frac{l.\Delta l}{2}},$$

if $(\Delta l)^2$ is negligible.

Now Δl is directly proportional to the square of the r.m.s. current, so that the sag is directly proportional to the r.m.s. current in the wire. The 'magnification' due to this arrangement is

$$\frac{\sqrt{\frac{l.\Delta l}{2}}}{\Delta l}=\sqrt{\frac{l}{2.\Delta l}}.$$

Further magnification may be obtained by utilizing the sag of the wire by means of which the horizontal wire is pulled downwards. Let λ be the length of this wire OR, Fig. 15·15 (*b*), and let it be pulled at its centre so that the 'slack' produced by the sag σ is taken up. Let S be the sag in the wire OR. Then

$$\mathrm{S}=\sqrt{\left(\frac{\lambda}{2}\right)^2-\left(\frac{\lambda-\sigma}{2}\right)^2}=\sqrt{\frac{2\lambda\sigma-\sigma^2}{4}}$$

$$\doteq\sqrt{\frac{\lambda\sigma}{2}}.$$

But $\sigma=\sqrt{\frac{l.\Delta l}{2}}.$

$$\therefore\ \mathrm{S}=\sqrt{\frac{\lambda}{2}\sqrt{\frac{l.\Delta l}{2}}}=\sqrt{\frac{\lambda}{2\sqrt{2}}}.\sqrt[4]{l.\Delta l},$$

i.e. $\mathrm{S}\propto\sqrt[4]{\Delta l}\propto\sqrt{\text{r.m.s. current}}.$

The above theory is incomplete in that it assumes that the wires are free from sag when the current is zero, but it shows how the requisite magnification may be obtained. The actual scale closely approaches the square-law scale, and is not unduly cramped when the current is small.

The structure of a hot-wire ammeter.—A hot-wire ammeter of the double-sag type is indicated in Fig. 15·16. AB is the hot wire, supported at both ends, and consists of a piece of platinum-iridium wire about 0·01 cm. in diameter. The above material is chosen since it does not oxidize readily and therefore does not deteriorate when subjected to high temperatures. A second thin wire, CD, made of phosphor-bronze, is attached to the centre of AB; to this second wire a fine silk thread, S, is attached. This passes round a small pulley, P, and then is attached to a spring, R, so that each portion of the system is kept in tension. A light aluminium pointer is attached to P. When the hot wire expands and the system is made taut by R, the pointer is deflected. Connexions are made to the terminals T_1 and T_2. The scale readings may be checked by passing known direct currents through the instrument. V is an aluminium vane attached to the pulley and it moves between the poles of a horse-shoe magnet M; in this way oscillation is prevented and, moreover, sudden changes in the current do not reveal themselves

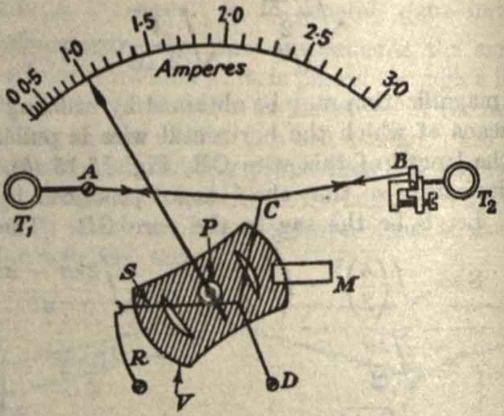


FIG. 15·16.—A hot-wire ammeter.

so quickly and consequently the wire is never subject to excessive stress.

Further remarks on hot-wire instruments. (*a*) ***Range.*** Hot-wire instruments are very limited as regards current-carrying capacity, since the diameter of the hot-wire is so small. For the range 0–0·5 A. a wire 0·01 cm. in diameter is suitable and the whole current traverses the wire. The range 0–5 A. is covered by using a wire 0·02 cm. in diameter; this is divided electrically into four equal segments connected in parallel so that only one quarter of the current being measured passes along any one segment. Fig. 15·17

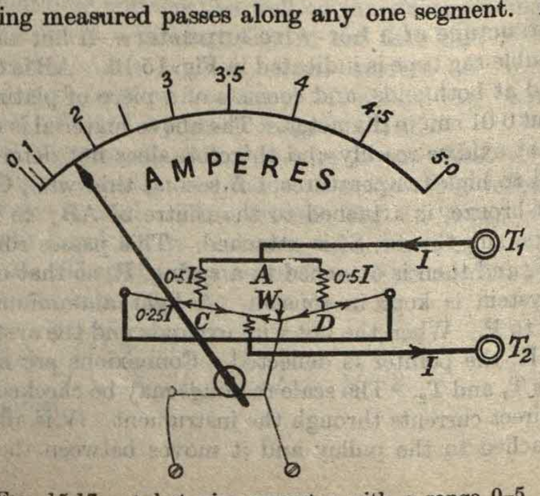


FIG. 15·17.—A hot-wire ammeter with a range 0–5 shows the essential features of the structure of such an instrument. The diagram reveals the fact that the current I divides at A into two equal portions and each of these is carried to the hot wire W by coils which exert little mechanical control on it; these currents enter W at the points C and D, and each divides into equal portions. Thus the four segments of the wire W are effectively in parallel and each carries a quarter of the current which is to be measured. The sag in W is used to produce a rotation of a suitably mounted needle as in the more simple forms of hot-wire instrument. For currents exceeding 5 A. a wire 0·02 cm. in diameter is used with a non-inductive low resistance connected across it.

When a hot-wire instrument is to be used as a voltmeter, the wire has a diameter of 0·04 cm. and this is arranged in series with a non-inductive resistance; the current required to give a full-scale deflexion is about 0·25 A., so that the power consumption is rather large.

(*b*) ***Power consumption.*** The power consumption in hot-wire instruments is high. With voltmeters, as indicated above, a rather large current is required to work the instrument.

(*c*) ***Disadvantages.*** The principal disadvantages of hot-wire instruments are that they are sluggish in action, have a cramped scale, and errors arise from the difference in temperature between the wire and the base of the instrument. In addition these instruments are fragile and cannot be overloaded. In this latter connexion it is found that a fuse is not an adequate protection.

Example.—A steady current I_0 is superimposed on an alternating current $\hat{I}\cos\omega t$. Prove that the r.m.s. current is $[I_0^2 + \frac{1}{2}(\hat{I})^2]^{\frac{1}{2}}$.

If i is the instantaneous current,

$$i^2 = [I_0 + \hat{I}\cos\omega t]^2.$$

$$\therefore \text{ Mean value of } i^2 = \frac{\int_0^T (I_0 + \hat{I}\cos\omega t)^2\,dt}{\int_0^T dt}, \qquad [T = \text{period}]$$

$$= \frac{1}{T}\Big[I_0^2 t + \frac{2I_0\hat{I}}{\omega}\sin\omega t\Big]_0^T + \tfrac{1}{2}(\hat{I})^2$$

$$= I_0^2 + 0 + \tfrac{1}{2}(\hat{I})^2.$$

$$\therefore \text{ r.m.s. current } = [I_0^2 + \tfrac{1}{2}(\hat{I})^2]^{\frac{1}{2}}.$$

The measurement of power.—The mean power, P, expended in watts in an alternating current circuit has been shown to be VI cos ϕ, where V and I are r.m.s. values and cos ϕ is the power factor. The maximum possible value of the power is VI and this is only the true power if $\phi = 0$. When ϕ is not zero, in order to find P it would be necessary to know the shapes of the voltage and the current curves and also ϕ. This could be done by means of some form of oscillograph and then the power P could be deduced. In practice, however, the mean power is best ascertained by using a wattmeter. The essential features and use of an electrodynamometer for the measurement of power have been described already, cf. pp. 316 to 329; no further remarks are necessary.

Example.—A loop of wire is in the form of a square of side a. It is rotated with constant angular speed ω about one side. If there is a uniform magnetic field with induction B perpendicular to this side, show that the mean rate at which work must be done to maintain the rotation is $\dfrac{\frac{1}{2}\omega^2 a^4 B^2 r}{(r^2 + \omega^2 l^2)}$, where r is the resistance and l the self inductance of the loop.

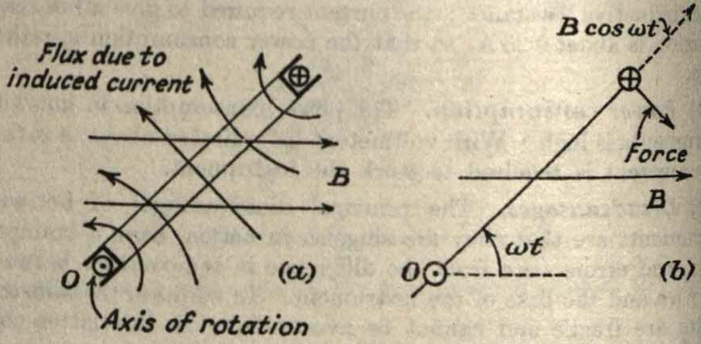


FIG. 15·18.

If the direction of rotation is anticlockwise and the induction B in the direction shown in Fig. 15·18 (*a*), then the direction of the induced current is as indicated. To determine the force on the vertical wire in which the current is directed downwards, let us consider Fig. 15·18 (*b*). Only the B cos ωt component of the induction will give rise to a force whose moment about O opposes the motion of the loop. This force is

$$B \cos \omega t \times i \times a.$$

$$\therefore \text{ Instantaneous couple} = a^2 B i \cos \omega t.$$

$$\text{But induced e.m.f.} = |\dot{\Phi}| = \left|\frac{d}{dt}(a^2 B \sin \omega t)\right|$$

$$= |a^2 B \omega \cos \omega t|.$$

The differential equation giving the instantaneous value of the current is

$$a^2 B\omega \cos \omega t = ri + l\frac{di}{dt},$$

i.e.

$$\frac{di}{dt} + \frac{r}{l}i = \frac{a^2 B\omega}{l} \cos \omega t.$$

The integrating factor is $\exp\left(\frac{r}{l}\right)t$ and using it we obtain

$$i = \frac{a^2 B\omega}{l}\cdot\frac{1}{\left(\frac{r^2}{l^2}+\omega^2\right)^{0\cdot5}}\cos(\omega t-\phi),$$

where $\tan\phi = \frac{\omega l}{r}$. Hence the instantaneous couple, due to the motion of the coil in the magnetic field, is

$$\frac{a^4 B^2\omega}{(r^2+\omega^2 l^2)^{0\cdot5}}\cos\omega t\cos(\omega t-\phi).$$

$$\therefore \text{ Mean couple} = \Gamma = \frac{1}{2}\frac{a^4 B^2\omega}{(r^2+\omega^2 l^2)^{0\cdot5}}\cos\phi$$

$$= \frac{1}{2}\frac{a^4 B^2\omega r}{(r^2+\omega^2 l^2)}.$$

The work done against this couple in one revolution, i.e. in time $T = \frac{2\pi}{\omega}$, is $\Gamma(2\pi)$, i.e. $\frac{\pi a^4 B^2\omega r}{(r^2+\omega^2 l^2)}$.

$\therefore$ Mean rate at which work is spent to maintain the motion of the coil is

$$\frac{\pi a^4 B^2\omega r}{(r^2+\omega^2 l^2)}\times\frac{1}{T} = \frac{\pi a^4 B^2\omega r}{r^2+\omega^2 l^2}\times\frac{\omega}{2\pi}$$

$$= \frac{1}{2}\frac{a^4 B^2\omega^2 r}{(r^2+\omega^2 l^2)}.$$

THREE-PHASE A.C. SYSTEMS

Before commencing any discussion of polyphase currents it is essential to distinguish by some convenient means between the two ends of an alternator winding, i.e. the ends of the coil in which an alternating voltage is produced. For instance, let AB and CD, Fig. 15·19 (*a*), represent the windings of two single-phase alternators; the frequencies must be equal, but not necessarily the e.m.fs. Let us now suppose that the generated e.m.fs. are in phase with each other so that when B is positive with respect to A, D is positive with respect to C. [N.B. In an external circuit connected to AB, for example, the current flows from B to A in that circuit when B is positive with respect to A. Thus the e.m.f. in the generator acts from A to B.] When we wish to refer to an e.m.f. the symbol E will be used, but when we refer to the voltage available at the terminals for use on an external circuit then the symbol V will be used. The ends of the coils are often marked by the letters S and F to denote respectively the 'start' and the 'finish' of the winding.

If the two coils are connected together as in Fig. 15·19 (*b*), the resultant e.m.f. will be the sum of the two component e.m.fs. If the separate e.m.fs. are not in phase then the resultant e.m.f. must be obtained by adding vectorially the two individual e.m.fs. If, however, B is connected to D as in Fig. 15·19 (*c*), then, in general, the resultant e.m.f. will be the vector difference of the two component e.m.fs.

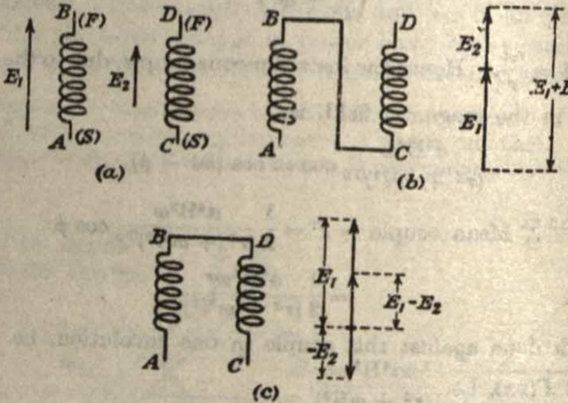


FIG. 15·19.—'Corresponding' and 'non-corresponding' ends of coils.

In engineering practice it is customary to call the ends A and C (or the ends B and D) the 'corresponding ends' of the two windings; ends such as A and D are said to be 'non-corresponding'.

The above discussion may be summarized as follows :—(*a*) When non-corresponding ends of two windings are joined together the resultant e.m.f. is the vector sum of the two separate e.m.fs. (*b*) When corresponding ends are joined, the resultant e.m.f. is the vector difference of the two separate e.m.fs.

Polyphase circuits.—When a single coil rotates with constant angular velocity in a uniform magnetic field, B, we have the simplest possible ***single-phase alternator***. The position of the coil at zero time is shown on the l.h.s. of Fig. 15·20 (*a*); the r.h.s. indicates how this e.m.f. varies with time. Suppose now that there is a second coil identical with the first, the two coils being coupled together mechanically so that their planes are mutually perpendicular at all times—cf. Fig. 15·20 (*b*). The e.m.fs. will be identical in amplitude and in frequency but, on account of their relative displacement, the maximum value of the e.m.f. in one coil occurs when that in the other is zero and vice versa. The two e.m.fs. are in quadrature and, with the usual notation, may be represented by

$$e_1 = \hat{\mathrm{E}} \cos \omega t, \quad \text{and} \quad e_2 = \hat{\mathrm{E}} \cos (\omega t - \tfrac{1}{2}\pi),$$

since at any given instant the e.m.f. in coil (2) is equal to the value of the e.m.f. in coil (1), at a time $\frac{1}{4}$T earlier, where T is the period of rotation. These two e.m.fs. are represented by two cosine curves displaced by a quarter of a period.

Such an arrangement of coils is known as a simple ***two-phase alternator*** and may be used to feed two single-phase circuits or a similar two-phase system, i.e. a piece of electrical apparatus having windings arranged in a similar manner.

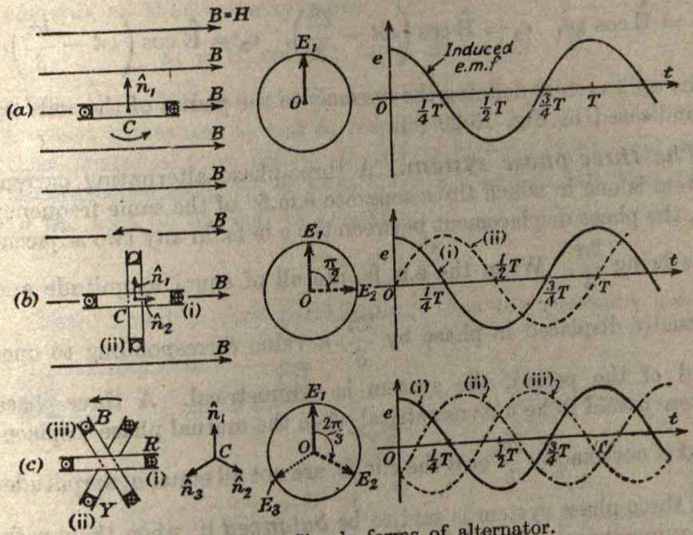


FIG. 15·20.—Simple forms of alternator.

Strictly speaking, in order to get the signs correct, in so far as the initial phase of the induced e.m.f. is concerned, the positions of the unit vectors which define the normal to the plane of each coil at its centre must be specified. If they are denoted by $\hat{n}_1$ and $\hat{n}_2$ respectively, the effective flux at time t through coil (1), Fig. 15·20 (*b*), is given by

$$\Psi_1 = \mathrm{AB}\cos(\vec{\mathrm{B}}, \hat{n}_1) = \mathrm{AB}\cos(\omega t + \tfrac{1}{2}\pi) = -\mathrm{AB}\sin\omega t,$$

where A is the effective cross-sectional area of the coil. Hence

$$e_1 = -\dot{\Psi}_1 = \mathrm{AB}\omega\cos\omega t = \hat{\mathrm{E}}\cos\omega t,$$

where $\hat{\mathrm{E}} = \mathrm{AB}\omega$.

Similarly, for coil (2) in Fig. 15·20 (*b*),

$$\Psi_2 = \mathrm{AB}\cos(\vec{\mathrm{B}}, \hat{n}_2) = \mathrm{AB}\cos\omega t.$$

$$\therefore\ e_2 = -\dot{\Psi}_2 = \mathrm{AB}\omega\sin\omega t = \mathrm{AB}\omega\cos(\omega t - \tfrac{1}{2}\pi)$$
$$= \hat{\mathrm{E}}\cos(\omega t - \tfrac{1}{2}\pi).$$

When the rotating system consists of three identical coils with their planes fixed at $\frac{2\pi}{3}$ radians apart, we have a simple ***three-phase alternator*** and the e.m.fs. in the three windings may be shown to be represented by the equations

$$e_1 = \hat{E} \cos \omega t, \quad e_2 = \hat{E} \cos \left(\omega t - \frac{2\pi}{3}\right), \quad e_3 = \hat{E} \cos \left(\omega t - \frac{4\pi}{3}\right),$$

if the unit vectors defining the normals to the planes of the coils are as indicated in Fig. 15·20 (*c*).

The three-phase system. A three-phase alternating current system is one in which three separate e.m.fs. of the same frequency act, the phase displacement between the e.m.fs. in any two adjacent coils being $\frac{2\pi}{3}$. When the e.m.fs. are all of equal magnitude and mutually displaced in phase by $\frac{2\pi}{3}$, a value corresponding to one-third of the period, the system is symmetrical. A three-phase system is said to be unsymmetrical when the mutual phase displacement is not exactly $\frac{2\pi}{3}$ or if the e.m.fs. are not all equal in amplitude.

A three-phase system is said to be ***balanced*** if, when the e.m.fs. are symmetrical, the load on each phase is exactly the same as regards current and power factor.

Some disadvantages of the single-phase system.—The earliest practical application of a single phase alternator was to raise to incandescence the filaments of electric lamps and for this purpose no objection can be raised against a single-phase a.c. supply. When, however, some years later a.c. motors were developed, it was found that they did not operate very satisfactorily on a single-phase supply. For example, the single-phase induction motor is not self-starting unless it is fitted with an auxiliary winding. By using two separate windings (or phases), with the currents in quadrature, induction motors become self-starting and, in addition, have a better efficiency and a higher power factor. The three-phase system is superior to the two-phase system and has been universally adopted for the generation and transmission of electrical energy.

Rotating magnetic fields.—The importance of these magnetic fields lies mainly in the fact that when they are used the design of electric motors becomes more simple, while the motors themselves become more robust. Such fields are most easily produced with the aid of polyphase alternating currents.

Suppose that C_1 and C_2, Fig. 15·21, are coils placed at right angles to each other and that they carry alternating currents differing in phase by ϕ, viz. $i_1 = \hat{I}_1 \cos \omega t$ and $i_2 = \hat{I}_2 \cos (\omega t - \phi)$, where the symbols have their usual meanings. The magnetic fields at O, the point of intersection of the axes of the coils, will be in phase with the currents so that we may write

$$h_1 = \hat{H}_1 \cos \omega t\,; \quad h_2 = \hat{H}_2 \cos (\omega t - \phi),$$

where $\hat{H}_1$ and $\hat{H}_2$ are the peak values of the fields whose instantaneous values are h_1 and h_2 respectively. Let OH_1 and OH_2 be these instantaneous fields. If OH is the resultant field at the instant considered, viz. h, $h^2 = {h_1}^2 + {h_2}^2$, and its direction relative to the axis of the coil C_1 is given

$$\tan \psi = \frac{h_2}{h_1}.$$

Both h and ψ vary periodically; thus the resultant magnetic field, which is completely defined by these two variables, rotates and at the same time varies in magnitude.

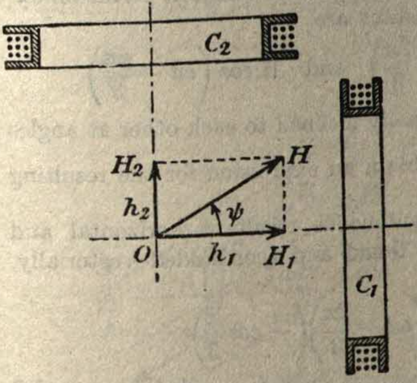


FIG. 15·21.—A rotating magnetic field.

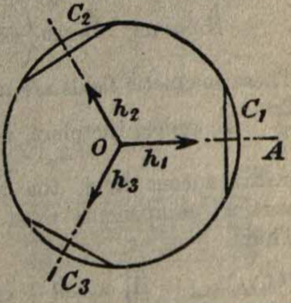


FIG. 15·22.—A rotating magnetic field produced from a three-phase a.c. supply.

The most important instance is when $\hat{H}_1 = \hat{H}_2 = \hat{H}$ (say), and $\phi = \frac{\pi}{2}$, i.e. the magnetic fields have equal amplitudes and are in quadrature. Under such conditions

$$h = \hat{H}\left[\cos^2 \omega t + \cos^2\left(\omega t - \frac{\pi}{2}\right)\right]^{\frac{1}{2}} = \hat{H},$$

and
$$\tan \psi = \frac{\cos\left(\omega t - \frac{\pi}{2}\right)}{\cos \omega t} = \tan \omega t.$$

Hence, when these conditions of amplitude and phase exist, the resultant field is constant in value and rotates with constant angular velocity ω. As t increases the value of ψ increases and so the rotation is anticlockwise. When the fields are given by

$$h_1 = \hat{H}\cos\left(\omega t - \frac{\pi}{2}\right) \quad \text{and} \quad h_2 = \hat{H}\cos\omega t,$$

it is found that

$$\tan\psi = \cot\omega t = \tan\left(\frac{\pi}{2} - \omega t\right).$$

Under such conditions ψ decreases with time, i.e. the rotation is clockwise.

A rotating magnetic field produced from a three-phase a.c. supply.—Let three identical coils be arranged with their centres at angular distance $\frac{2\pi}{3}$ along the circumference of a circle, as in Fig. 15·22, and suppose that each coil is supplied with one phase of a three-phase alternating current supply. As before, assuming sinusoidal wave-forms and cores of constant permeability, the magnetic fields at any instant are

$$\hat{H}\cos\omega t, \quad \hat{H}\cos\left(\omega t - \frac{2\pi}{3}\right) \quad \text{and} \quad \hat{H}\cos\left(\omega t - \frac{4\pi}{3}\right).$$

These magnetic fields are mutually inclined to each other at angles $\frac{2\pi}{3}$. In order, therefore, to obtain an expression for the resulting instantaneous field, the instantaneous resultant horizontal and vertical components must be found and then added vectorially. Thus

$$\begin{aligned}(h)_{\text{horizontal}} &= \hat{H}\left[\cos\omega t + \cos\left(\omega t - \frac{2\pi}{3}\right)\left(-\cos\frac{\pi}{3}\right)\right.\\ &\qquad\left. + \cos\left(\omega t - \frac{4\pi}{3}\right)\left(-\cos\frac{\pi}{3}\right)\right]\\ &= \tfrac{3}{2}\hat{H}\cos\omega t.\end{aligned}$$

Similarly,

$$\begin{aligned}(h)_{\text{vertical}} &= \hat{H}\left[0 + \cos\left(\omega t - \frac{2\pi}{3}\right)\sin\frac{\pi}{3} - \cos\left(\omega t - \frac{4\pi}{3}\right)\sin\frac{\pi}{3}\right]\\ &= \tfrac{3}{2}\hat{H}\sin\omega t.\end{aligned}$$

The resultant field is therefore $\frac{3}{2}\hat{H}$, and its inclination ψ to OA is given by

$$\tan\psi = \frac{\frac{3}{2}\hat{H}\sin\omega t}{\frac{3}{2}\hat{H}\cos\omega t} = \tan\omega t,$$

so that the field is constant in magnitude and rotates with constant angular velocity ω.

The couple acting on a small coil placed in a rotating magnetic field.—If has just been shown how a rotating magnetic field of constant magnitude may be produced in a region surrounding a fixed point O. If a metallic body is placed at O it will experience a couple which will tend to rotate the body in the same direction as the field itself rotates, for by Lenz's law the forces due to the magnetic effects between the field due to the induced currents and the original field, are such that they tend to prevent relative motion of the field and the conductor. It is impossible to obtain a general expression for this couple and we shall have to be content with one for the couple acting on a small coil, cf. Fig. 15·23 (*a*), whose centre is at O. Let NA be the effective area of the coil. Then if H is the field * rotating with angular velocity ω_1, the number of linkages associated with the coil at time t is

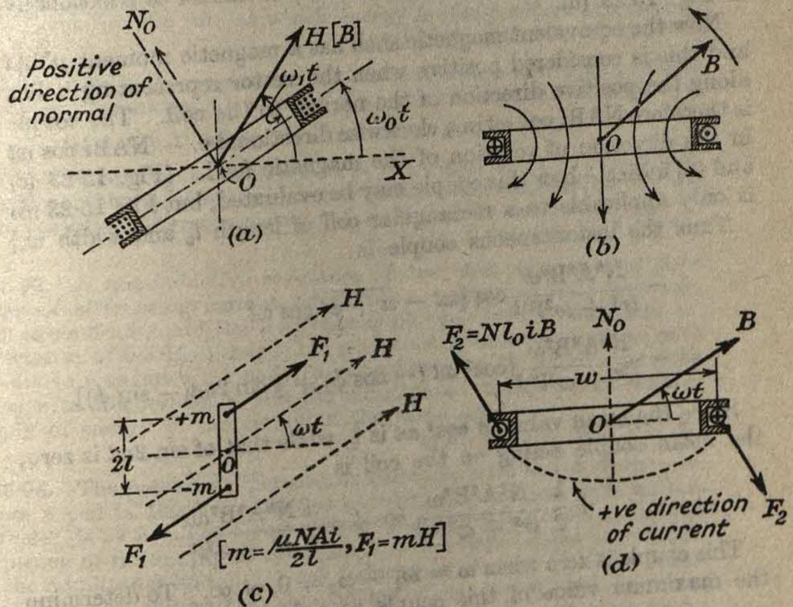


FIG. 15·23.

$$\Psi = \text{NAB} \sin (\omega_1 - \omega_0)t,$$

if we assume that the coil rotates with constant angular velocity ω_0 and $B = \mu H$, where μ is the permeability of the medium (generally air, so that $\mu = \mu_a \rightarrow 1$) in which the coil rotates. If $\omega_1 - \omega_0 = \omega$, the relative angular velocity of the magnetic field with respect to the coil, then e, the instantaneous induced e.m.f., is given by

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* Since the field is constant in magnitude it is convenient to denote it by H.

$$e = -\dot{\Psi} = -\mathrm{NAB}\omega \cos \omega t = \mathrm{NAB}\omega \cos (\omega t - \pi).$$

If r and l are respectively the resistance and the self inductance of the coil, the current through it at any instant is given by

$$i = \frac{\mathrm{NAB}\omega}{(r^2 + \omega^2 l^2)^{\frac{1}{2}}} \cos (\omega t - \pi - \phi),$$

where $\tan \phi = \frac{\omega l}{r}$. The direction of this current and the lines of magnetic induction to which it gives rise are shown conventionally in Fig. 15·23 (*b*).

Now the equivalent magnetic shell has a magnetic moment $\mu \mathrm{NA}i$ and this is considered positive when the vector representing it lies along the positive direction of the normal to the coil. The couple is therefore $\mathrm{NAB}i \cos \omega t$ in a clockwise direction, i.e. $-\mathrm{NAB}i \cos \omega t$ in the direction of rotation of the magnetic field. [Fig. 15·23 (*c*) and (*d*) indicate how this couple may be evaluated, but Fig. 15·23 (*d*) is only applicable to a rectangular coil of length l_0 and width w.]

Thus the instantaneous couple is

$$-\frac{\mathrm{N^2A^2B^2}\omega}{(r^2 + \omega^2 l^2)^{\frac{1}{2}}} \cos (\omega t - \overline{\pi + \phi}) \cos \omega t$$

$$= -\frac{\mathrm{N^2A^2B^2}\omega}{(r^2 + \omega^2 l^2)^{\frac{1}{2}}} [\cos^2 \omega t\, (-\cos \phi) + \tfrac{1}{2} \sin 2\omega t (-\sin \phi)].$$

Since the mean value of $\cos^2 \omega t$ is $\frac{1}{2}$, while that of $\sin 2\omega t$ is zero, the mean couple acting on the coil is

$$+\frac{1}{2} \frac{\mathrm{N^2A^2B^2}\omega}{(r^2 + \omega^2 l^2)^{\frac{1}{2}}} \cos \phi = \frac{1}{2} \frac{\mathrm{N^2A^2B^2}\omega r}{r^2 + \omega^2 l^2}.$$

This couple is zero when $\omega = \omega_1 - \omega_0 = 0$, or ∞. To determine the maximum value of this couple as a function of ω, let

$$\xi = \frac{\mathrm{N^2A^2B^2}\omega r}{r^2 + \omega^2 l^2} = \frac{k\omega}{r^2 + \omega^2 l^2} \text{ (say).}$$

Then $$\frac{d\xi}{d\omega} = k\left[\frac{1}{r^2 + \omega^2 l^2} - \frac{2\omega^2 l^2}{(r^2 + \omega^2 l^2)^2}\right],$$

and this is zero if $\omega l = r$. A second differentiation shows that when this condition is fulfilled, the mean couple, as a function of ω, is a maximum; its value is $\frac{1}{4} \frac{\mathrm{N^2A^2B^2}}{l}$ and since it occurs when $\omega l = r$, this means that $\phi = \frac{1}{4}\pi$.

Many practical applications have been made of the fact that closed coils experience a couple when placed in a rotating magnetic field, but the most important of these is the construction of electric motors. A system of closed coils, or a mass of metal, is mounted upon an axis and when the rotating field is in operation the system moves in virtue of the existence of the couple predicted above.

EXAMPLES XV

15·01. Explain the term *root-mean-square current.*

An e.m.f. of 100 volt r.m.s. at a frequency 50 cycle.sec.$^{-1}$ is applied to a circuit containing an inductance of 80 mH. and a resistance of 20 ohms in series. Calculate (*a*) the magnitude and phase angle of the current, (*b*) the rate at which energy is dissipated in the circuit. [(*a*) 3·12 A., 51° 28′, (*b*) 195 W.]

15·02. Derive an expression for the amplitude and phase of the current in a series circuit consisting of a resistance R, a self inductance L and a capacitance C, when an e.m.f. $\hat{V} \cos \omega t$ is applied.

If $\hat{V} = 30$ volt., $\omega = 20{,}000$ sec.$^{-1}$, $R = 4{\cdot}0$ ohm. and $L = 0{\cdot}05$ henry, calculate (*a*) the value of C which makes the circuit resonant and (*b*) the peak potential difference across the condenser at resonance. [(*a*) 0·05 μF., (*b*) 7500 V.]

15·03. A non-inductive resistance of 100 ohm. is connected to a source of alternating current, whose r.m.s. voltage is 200; its wave-form is sinusoidal and the frequency is 50 cycle.sec.$^{-1}$. Secondly, an inductance of 0·3 H. of negligible resistance is inserted in series with the above resistance. Finally, a capacitance of 20 μF. is placed in series with the resistance and the inductance. Calculate the rate of supply of energy to the circuit in the three instances. [400 W., 212 W., 281 W.]

15·04. The potential difference, as measured with an a.c. voltmeter, across a coil is 100 V., while the current, as measured with an a.c. ammeter, is 20 A. The ohmic resistance of the coil is 2 ohms and the frequency of the supply 50 cycle.sec.$^{-1}$. Calculate (*a*) the impedance, (*b*) the admittance and (*c*) the self inductance of the coil. Also determine the active and wattless components of the current, and the power factor of the coil. Construct, to scale, the voltage vector diagram. [5 Ω., 0·2 ohm.$^{-1}$, 14·6 mH., 8 A., 6 A., 0·40.]

15·05. A resistance of 100 Ω. and a condenser of capacitance 20 μF. are arranged in series and connected to a 240 volt. a.c. supply whose frequency is 60 cycle.sec.$^{-1}$. Calculate the current, power and power factor, also the potential drop across both the resistor and the condenser. Construct, to scale, the voltage vector diagram. [1·44 A., 207 W., 0·601, 144 V., 191 V.]

15·06. To the circuit given in the previous problem there is added a coil of self inductance 1·0 H. and negligible resistance. Discuss the effect of this change on the above characteristics. [0·91 A., 82·8 W., 0·379, 91 V., 121 V.]

15·07. A sinusoidal a.c. supply is applied to a non-inductive circuit consisting of a resistance and a condenser in series. Find an expression for the current finally produced. A condenser of capacitance 1 μF. and a resistor of resistance 50 Ω. are connected in series with an alternating supply of 100 V. amplitude and frequency 50 cycle.sec.$^{-1}$. The addition to the circuit of an inductive coil of negligible resistance is found to cause the current to increase. Explain this and find the value of the added self inductance to give maximum current. State also the r.m.s. value of this maximum current. [10·14 H., 1·41 A.]

15·08. Define *reactance, impedance.* Determine the r.m.s. value and phase of the current when the following are connected to a 240 V., 50-cycle.sec.$^{-1}$ supply: (*a*) a resistance of 500 Ω. in series with a condenser of capacitance 10 μF., and (*b*) an inductive coil of resistance 10 Ω. and self inductance 0·1 H.

[(*a*) 0·405 A., 32° 30′; (*b*) 7·28 A., 72° 20′.]

15·09. A sinusoidal voltage of 100 V. (r.m.s.) and a frequency of 100 cycle.sec.$^{-1}$ is applied to a coil and a condenser in series. Calculate (*a*) the r.m.s. current, (*b*) the voltage across the condenser and (*c*) the power factor. $R = 40\ \Omega$., $L = 0{\cdot}3$ H., $C = 5\ \mu$F.

[(*a*) 0·735 A., (*b*) 234 V., (*c*) 0·294.]

15·10. Explain the meaning of the term *power factor* in the case of an alternating current circuit.

Describe, and explain in general terms, a method for measuring the power in an alternating current circuit.

When a sinusoidal e.m.f. of 100 volts (r.m.s.) at a frequency of 100 cycle.sec.$^{-1}$ is applied to a circuit consisting of a condenser of 2 μF. capacitance in series with a resistance R, the current is 0·125 ampere (r.m.s.). Calculate the value of R and the power factor of the circuit. (G) [77·9 Ω., 0·0974.]

15·11. A long solenoid wound on a wooden former of cross-sectional area 20 cm.^{2} has 12 turns per unit length. Near to its centre there is closely wound a secondary coil of 4000 turns. Obtain a value for the mutual inductance of the system and indicate why the value so obtained may differ from the true value of the mutual inductance.

Describe and explain how the capacitance of a mica condenser may be determined in terms of the above mutual inductance. (G)

[12·1 mH.]

15·12. A 10 μF. condenser in series with a variable resistance R is connected to a 240 V., 50 cycle.sec.$^{-1}$, a.c. supply. Plot a graph to show how the r.m.s. current varies with R. Calculate the value of R for which the power taken from the mains will be a maximum and justify your calculations. [318·5 Ω., i.e. when $R = X$.]

15·13. An e.m.f. of 100 V. r.m.s. at a frequency 50 cycle.sec.$^{-1}$ is applied to an inductive coil of resistance 10 Ω. and self inductance 40 mH. Calculate (*a*) the magnitude and phase angle of the current, (*b*) the rate at which energy is dissipated in the circuit.

[(*a*) 6·23 A., 51° 27′; (*b*) 388 W.]

15·14. Two factory buildings are supplied from the same a.c. mains. The first factory takes a load of 250 A. at a power factor of 0·75. The two factories together have a load of 400 A. and a power factor of 0·62. Assuming that the current lags in each instance calculate the load and power factor for the second factory. [160·4 A., 0·374.]

15·15. Two 100 volt., 100 watt. lamps are connected in series with a condenser and a 240 volt., 60 cycle.sec.$^{-1}$ supply. What is the value of the capacitance of the condenser when the lamps operate under normal working conditions? If a third identical lamp is added to the system find the potential drop across each lamp. [Assume that the resistance of each lamp is independent of the voltage across it.]
[20 μF., 73·1 V.]

15·16. A 240 volt., single phase generator, has a current carrying capacity of 100 A. What is the maximum rate at which energy can be taken from this generator by a system with a power factor of 0·64? How is this maximum affected if the power factor of the system is raised to a value 0·92? [15·4 kW., 22·1 kW.]

15·17. Two single-phase generators, A and B, of the same frequency, generate e.m.fs. of 600 V. and 400 V. respectively, B lagging behind A by $\cos^{-1} \frac{1}{8}$. These two generators may be connected in series by joining together one pair of either (*a*) corresponding terminals, or (*b*) non-corresponding terminals. Find the resultant e.m.f. in each instance.
[600 V., 825 V.]

15·18. A three-phase, star-connected generator supplies a delta-connected induction motor, the line voltage being 420 V. The current in each line is 30 A. Calculate values for the phase-voltage of the generator and the current in each winding of the motor.
[243 V., 17·3 A.]

15·19. The r.m.s. value of the phase e.m.f. in a three-phase alternator is 242 V. Determine the instantaneous values of the e.m.fs. in the Y and B phases when that in the R phase is 100 V. positive and (*a*) increasing, (*b*) decreasing.
[(*a*) 286 V., − 386 V.; (*b*) − 386 V., 286 V.]

15·20. Derive an expression, in terms of the readings of two wattmeters, for the power factor of a balanced three-phase load.

The input to a three-phase, star-connected, induction motor is 32 kW. and this is measured by the two wattmeter method. The ratio of the two readings is 1·5. If the line voltage is 420, find the reading of each wattmeter, the power factor and the current in each winding of the motor. If the power factor for the above motor had been (*a*) 0·5, (*b*) 0·4, obtain values for the current and the readings of the two wattmeters.
[19·2 kW., 12·8 kW., 0·945, 46·5 A.; (*a*) 88·0 A., 32 kW., 0; (*b*) 110 A., 28·1 kW., − 3·90 kW.]

15·21. A circulating water pump delivers 2 × 10^{5} lb. of water per minute against a head of 20 ft. It is driven by a three-phase, 3000 V. induction motor. Calculate the current input to the motor if the power factor is 0·90 and the efficiencies of the motor and pump are 0·88 and 0·82 respectively.

Explain with the aid of a diagram how the power input to the above motor could be measured by means of two wattmeters. Calculate a value for the reading shown by each wattmeter.
[26·2 A., 80·2 kW., 45·1 kW.]

15·22. A coil possessing both resistance and inductance is connected in parallel with a condenser to a source of alternating current. Find the condition that the current from the source shall be in phase with the charge on the condenser, and show that the impedance of the circuit is then equivalent to a pure resistance.

$\left[\omega^2 = \frac{L - CR^2}{CL^2}\right.$. N.B.—Special cases :—(*a*) No solution if $CR^2 > L$, (*b*) solution for d.c. if $L = CR^2$, and (*c*) if $R = 0$ (never in practice), $\left.\omega = \frac{1}{\sqrt{LC}}\right]$.

15·23. Derive an expression for the amplitude and phase of the current in a series circuit consisting of a resistance R, a self inductance L, and a capacitance C, when an e.m.f. $\hat{V} \cos \omega t$ is applied

If $\hat{V} = 30$ volt., $\omega = 20{,}000$ sec.$^{-1}$, $R = 4{\cdot}0$ ohm. and $L = 0{\cdot}05$ H., calculate (*a*) the value of C which makes the circuit resonant and (*b*) the peak potential difference across the condenser at resonance.

$\Big[C = 0{\cdot}05\ \mu$F., 7500 V. lagging by $\frac{\pi}{2}$. N.B.—Across the inductance the p.d. is $\frac{\vec{V}}{Z}(j\omega L)$ and, at resonance, this is equal in magnitude to that across the condenser but leads the voltage by $\frac{\pi}{2}$. The phase difference between the p.ds., across condenser and inductance is π.]

15·24. A self inductance 1 H. is connected in parallel with a variable capacitance C. The combination is then placed in series with another inductance of 2 H. and an a.c. supply of angular frequency ω. Assuming that both inductances have negligible resistances investigate and comment upon the change in the value of the impedance operator for the circuit as C is increased from zero. Show your results on a suitable graph.

[$C = 0$, $Z = 3j\omega$; $C = \omega^{-2}$, $Z = \infty$;
$C = 1{\cdot}5\omega^{-2}$, $Z = 0$; $C = \infty$, $Z = 2j\omega$.]

15·25. An inductive reactance X_L is connected in parallel with a capacitive reactance X_C. If $X_L < X_C$ show that the impedance of the combination is $X_L X_C \div (X_C - X_L)$.

15·26. Two coils whose resistances are R_1 and R_2 and whose self inductances are L_1 and L_2 respectively, are connected in parallel and fed from a potential source of pure sine wave-form. Prove that the currents in the two coils are in phase with each other if the mutual inductance between them is

$$\frac{R_1 L_2 - R_2 L_1}{R_1 - R_2}.$$

CHAPTER XVI

ALTERNATING CURRENTS TREATED BY THE ROTATING VECTOR METHOD

VECTORS AND VECTOR OPERATORS

Definition of a unit vector.—A unit vector is one which is completely represented by a straight line, one unit in length measured in some definite direction in space.

The symbol $\vec{\nu}$ is used to denote a unit vector whose direction lies in the plane of the paper and which is parallel to the lower edge of the paper, its sense being from left to right.

Derivation of a vector from a unit vector.—Starting with some unit vector $\vec{\nu}$, represented by ON, Fig. 16·01 (*a*), its length

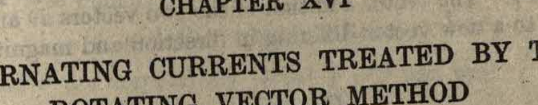


FIG. 16·01.—Derivation of a vector from a unit vector.

can be increased *a* times without altering its direction, where *a* is any positive real number not necessarily integral. The result is a vector $\overrightarrow{\mathrm{OA}}$, cf. Fig. 16·01 (*b*), such that $\overrightarrow{\mathrm{OA}} = a\vec{\nu}$. Thus $a\vec{\nu}$ is a vector parallel to $\vec{\nu}$ and its magnitude is *a* times that of $\vec{\nu}$. Thus *a* may be regarded as an operator the effect of which on $\vec{\nu}$ is to increase the length of the unit vector *a* times without altering its direction.

Rotation of a vector in a given plane.—It has just been shown that the scalar coefficient *a* can be regarded as an operator, the effect of which is to increase the magnitude of its operant, in this instance the unit vector $\vec{\nu}$, *a* times without changing its direction. The natural complement of such an operator is one whose effect is to alter the direction of its operant without altering its magnitude. The ***operator j*** is defined as one whose effect is to rotate its operant through an angle $\frac{1}{2}\pi$ in a counterclockwise direction, without altering its magnitude. Hence if $\overrightarrow{\mathrm{OA}} = \vec{\nu}$, $\overrightarrow{\mathrm{CD}} = j\vec{\nu}$ the lengths of both OA and CD being unity, cf. Fig. 16·02, then CD is perpendicular to OA.

Since $j^2 = jj$, it follows at once that $j^2 = -1$.

The vector $a\vec{v} + jb\vec{v}$, or $(a + jb)\vec{v}$. Now $a\vec{v}$ is a vector of length a units in a direction parallel to that of $\vec{v}$. The vector $j\vec{v}$ is of unit length and is in a direction normal to that of $\overrightarrow{OA}$. Thus $bj\vec{v}$ or $jb\vec{v}$ is a vector b units in length and drawn in a direction perpendicular to that of $\vec{v}$. The vector addition of the two vectors $a\vec{v}$ and $jb\vec{v}$ will give rise to a new vector differing in direction and magnitude from both $a\vec{v}$ and $jb\vec{v}$. This addition is illustrated in Fig. 16·03. If $\vec{r} = a\vec{v} + jb\vec{v}$, then $r = (a^2 + b^2)^{\frac{1}{2}}$, and $\theta = \tan^{-1}\frac{b}{a}$, where θ is the angle indicated. Thus $\vec{r}$ is a vector of magnitude $+ (a^2 + b^2)^{\frac{1}{2}}$, and whose direction makes with $\vec{v}$ an angle $\theta = \tan^{-1}\frac{b}{a}$.

FIG. 16·02.—Rotation of a vector in a given plane.

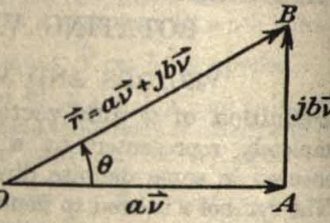


FIG. 16·03.—The vector $(a + jb)\vec{v}$.

The operator $a + jb$. The two operations on the unit vector $\vec{v}$ involved in the preceding paragraph, i.e. a multiplication by a plus the result of a multiplication of $\vec{v}$ by b with a positive rotation through $\frac{\pi}{2}$, can conveniently be grouped together into one expression, since both have the same operant, i.e.

$$\vec{r} = a\vec{v} + jb\vec{v} = (a + jb)\vec{v}.$$

Thus by operating on $\vec{v}$ with $(a + jb)$, a vector $\vec{r}$, of magnitude $+ (a^2 + b^2)^{\frac{1}{2}}$ and a slope relative to $\vec{v}$ given by $\theta = \tan^{-1}\frac{b}{a}$ is obtained. In other words, the effect of the operator $a + jb$ has been to multiply the magnitude of $\vec{v}$ by $r = (a^2 + b^2)^{\frac{1}{2}}$ and to rotate it counterclockwise through an angle $\theta = \tan^{-1}\frac{b}{a}$ — cf. Fig. 16·03.

It must be noted that the quantities r and θ depend only on a and b, and not on $\vec{v}$. Thus, if $\overrightarrow{OQ} = \vec{z}$ is a vector of magnitude z whose direction makes an angle ϕ with that of $\vec{v}$, as shown in Fig. 16·04, then $(a+jb)\vec{z}$ is a vector of magnitude rz making an angle $(\phi+\theta)$ with $\vec{v}$, where

$$r = +(a^2+b^2)^{\frac{1}{2}}, \quad \text{and} \quad \theta = \tan^{-1}\frac{b}{a}.$$

Alternative form for (a + jb). From Fig. 16·03 we have $a = r\cos\theta$, $b = r\sin\theta$.

$$\therefore\ a + jb = r(\cos\theta + j\sin\theta),$$

where r and θ are respectively $+(a^2+b^2)^{\frac{1}{2}}$ and $\tan^{-1}\left(\frac{b}{a}\right)$.

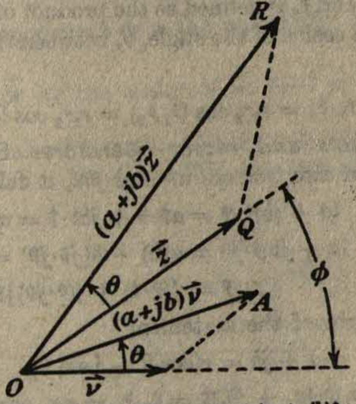


FIG. 16·04.—The vector $(a+jb)\vec{z}$.

Since $$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and $$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta + j\sin\theta = 1 + \frac{(j\theta)}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots,$$

so that from analogy with the series form of exp (x), viz.

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

the expression we have obtained for $\cos\theta + j\sin\theta$ may be written $\exp(j\theta)$.

Hence $$a + jb = r\exp(j\theta).$$

From this it follows that

$$\frac{1}{a+jb} = \frac{1}{r}\exp(-j\theta),$$

i.e. the effect of this operator on $\vec{v}$ is to multiply the unit vector by $\frac{1}{r}$ and to rotate it clockwise through an angle θ.

To evaluate $\frac{a + jb}{c + jd}$, where a, b, c and d are real, we write it in the form $\frac{r_1 \exp(j\theta_1)}{r_2 \exp(j\theta_2)}$, where $r_1{}^2 = a^2 + b^2$ and $\tan\theta_1 = \frac{b}{a}$, etc. Then the expression becomes $\frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} \exp j(\theta_1 - \theta_2)$.

The scalar product of two vectors. The scalar product $\vec{r}_1 \cdot \vec{r}_2$ of two vectors $\vec{r}_1$ and $\vec{r}_2$ is defined as the product of their magnitudes multiplied by the cosine of the angle, θ, between their positive directions, i.e.

$$\vec{r}_1 \cdot \vec{r}_2 = r_1 r_2 \cos(\widehat{\vec{r}_1, \vec{r}_2}) = r_1 r_2 \cos\theta.$$

Scalar products and vector operators. Since $(a + jb)\vec{v}$ is really the sum of two vectors $a\vec{v}$ and $jb\vec{v}$, it follows that

$$(a + jb)\vec{v} \cdot \vec{v} = a\vec{v} \cdot \vec{v} + jb\vec{v} \cdot \vec{v} = a + 0 = a,$$

and

$$(a + jb)\vec{v} \cdot j\vec{v} = a(0) + b(j\vec{v} \cdot j\vec{v}) = b.$$

$$\therefore \vec{r} = \{(\vec{r} \cdot \vec{v}) + j(\vec{r} \cdot j\vec{v})\}\vec{v}$$

is a vectorial form of the statement

$$a + jb = r(\cos\theta + j\sin\theta).$$

Also $\quad (a_1 + jb_1)\vec{v} \cdot (a_2 + jb_2)\vec{v} = \vec{r}_1 \cdot \vec{r}_2 = r_1 r_2 \cos(\theta_1 - \theta_2),$

while $\quad (a_1 + jb_1)(a_2 + jb_2)\vec{v} = r_1 r_2 \exp\{j\,\theta_1 + \theta_2)\}\vec{v}.$

VARIABLE VECTORS

Vector functions of time.—Up to this stage of our investigations vectors have been regarded as constant quantities; the investigations must now be extended to certain vectors of which the magnitude or the direction, or both, are functions of some independent variable, such as time.

If r alone varies with time, so that $r = r(t)$, then

$$\vec{r} = [r(t)]\vec{v},$$

is a vector parallel to $\vec{v}$ at all times but whose magnitude varies with the time. For such a vector

$$\vec{r} \cdot \vec{v} = r(t).$$

If $\vec{r} = [r(t)] \exp(j\theta)\vec{v}$, where $\theta =$ constant, we have a vector inclined at an angle θ to $\vec{v}$ at all times, but whose magnitude varies with time.

If r is constant and $\theta = \theta(t)$, then $\vec{r} = r \exp[j\theta(t)]\vec{v}$ is a vector of constant magnitude but whose slope referred to $\vec{i}$ varies with time. For such a vector $\vec{r}\cdot\vec{i} = r\cos\theta$.

If, in addition to $r(t) =$ constant, $\theta(t) = \omega t$, where ω is a constant, then

$$\vec{r} = r\exp(j\omega t)\vec{i}$$

is a vector of constant magnitude r, rotating with constant angular velocity ω.

Derivative of a vector: differentiation.—Suppose that the vector $\vec{r}$ is a continuous and single-valued function of a scalar variable t. Then to each value of t there corresponds only one value of $\vec{r}$; and as t varies continuously so does $\vec{r}$. Let OP, Fig. 16·05, be the vector $\vec{r}$ at the time t and OP′ the vector $\vec{r} + \overrightarrow{\delta r}$ at time $t + \delta t$. The increment $\overrightarrow{\delta r}$ is the vector PP′ and the quotient $\frac{\overrightarrow{\delta r}}{\delta t}$, of the vector $\overrightarrow{\delta r}$ by the scalar δt, is itself a vector. If δt is small then $\overrightarrow{\delta r}$ will also be small. As $\delta t \to 0$, the point P′ tends to coincide with P, i.e. the chord PP′ becomes the tangent at P.

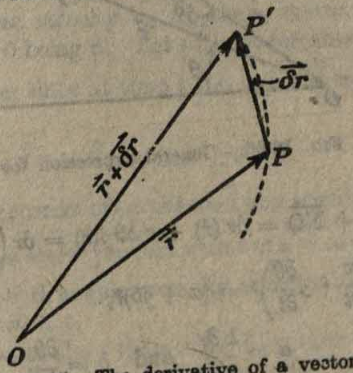

P'
$\overrightarrow{\delta r}$
P
$\vec{r} + \overrightarrow{\delta r}$
$\vec{r}$
O

FIG. 16·05.—The derivative of a vector.

The limiting value of $\frac{\overrightarrow{\delta r}}{\delta t}$, as δt tends to zero, is a vector whose direction is the limiting direction of $\overrightarrow{\delta r}$, i.e. it is a vector directed along the tangent at P. The limiting value of $\frac{\overrightarrow{\delta r}}{\delta t}$, when it exists, is called the derivative or differential coefficient of $\vec{r}$ with respect to t, and is denoted by $\frac{d}{dt}(\vec{r})$, i.e. $\frac{d\vec{r}}{dt}$. [Partial derivatives $\frac{\partial\vec{r}}{\partial t}$, $\frac{\partial\vec{r}}{\partial\theta}$, etc., also occur.]

General expression for $\frac{\partial \vec{r}}{\partial t}$.—From Fig. 16·06, in which $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{OQ} = \vec{r} + \overrightarrow{\delta r}$ and N is the projection of Q on OP produced, we have

$$PN = \delta r.$$

$$\therefore \overrightarrow{PN} = \delta r(\hat{r}), \quad \text{or} \quad \hat{r}\,\delta r.$$

Also
$$NQ = r\,\delta\theta.$$

As a vector, therefore,

$$\overrightarrow{NQ} = r\,\delta\theta\, j\left(\frac{\vec{r}}{r}\right), \quad \text{or} \quad r\,\delta\theta\, j(\hat{r}),$$

where $\frac{\vec{r}}{r}$, or $\hat{r}$, is a unit vector along the direction of $\vec{r}$, so that $j\hat{r}$ is a unit vector parallel to NQ.

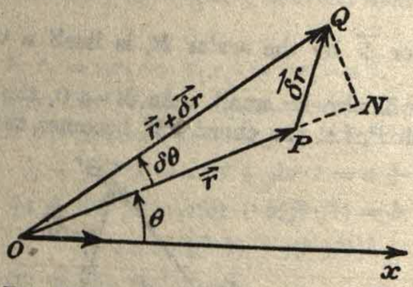

Q
$\vec{r} + \overrightarrow{\delta r}$
$\overrightarrow{\delta r}$
N
P
δθ
$\vec{r}$
θ
O
x

FIG. 16·06.—General expression for $\frac{\partial \vec{r}}{\partial t}$.

$$\therefore \overrightarrow{\delta r} = \overrightarrow{PN} + \overrightarrow{NQ} = \delta r\,(\hat{r}) + r\,\delta\theta\, j(\hat{r}) = \delta r\left(\frac{\vec{r}}{r}\right) + r\,\delta\theta\, j\left(\frac{\vec{r}}{r}\right)$$

$$\therefore \frac{\partial \vec{r}}{\partial t} = \left(\frac{1}{r}\frac{\partial r}{\partial t} + j\frac{\partial \theta}{\partial t}\right)\vec{r} = (a + jb)\vec{r},$$

where
$$a = \frac{1}{r}\frac{\partial r}{\partial t} \quad \text{and} \quad b = \frac{\partial \theta}{\partial t}.$$

For vectors of constant magnitude rotating with constant angular velocity ω, $a = 0$, $b = \omega$. In such an instance therefore

$$\frac{\partial \vec{r}}{\partial t} = \frac{d\vec{r}}{dt} = j\omega\vec{r}.$$

In general, for such vectors,

$$\frac{d^n\vec{r}}{dt^n} = (j\omega)^n\vec{r}.$$

Since $\frac{d\vec{r}}{dt} = j\omega\vec{r}$, it follows that $\vec{r} = \int \overrightarrow{dr} = j\omega \int \vec{r}\,dt.$

$$\therefore \int \vec{r}\,dt = \frac{1}{j\omega}\vec{r},$$

a result showing that the processes of differentiation and integration are reciprocal.

THE APPLICATION OF VECTOR ANALYSIS TO ALTERNATING CURRENT CIRCUITS

Introduction.—The use of the complex vector notation in alternating current theory has for its object to provide for a.c. quantities (current, voltage, impedance, etc.) symbols which not only represent them completely, but which can be manipulated in every way like d.c. quantities, so that the techniques developed for dealing with d.c. circuits may be used with a.c. circuits. In proceeding to develop these ideas it must be noted that the circular frequency ω and the form of the function (containing a cosine term) are implicit; never mix frequencies or sines with cosines.

The representation of the instantaneous value of an alternating current as a scalar product.—Let $\vec{\mathrm{I}}$ be a vector of constant magnitude $\hat{\mathrm{I}}$, rotating in a positive (counter-clockwise) sense with constant angular velocity ω, its slope, relative to the unit vector $\mathfrak{p}$, at time $t = 0$ being ϕ. Let i be the instantaneous current $\hat{\mathrm{I}} \cos (\omega t + \phi)$. Then since at time t the vector $\vec{\mathrm{I}}$ makes an angle $(\omega t + \phi)$ with $\mathfrak{p}$,

$$i = \vec{\mathrm{I}} \cdot \mathfrak{p} = \hat{\mathrm{I}} \exp j(\omega t + \phi)\mathfrak{p} \cdot \mathfrak{p}.$$

Thus an alternating current of amplitude $\hat{\mathrm{I}}$ and angular frequency ω can be represented as a scalar product with $\mathfrak{p}$ of a vector $\vec{\mathrm{I}}$ of constant magnitude $\hat{\mathrm{I}}$ rotating with angular velocity ω, the initial position of $\vec{\mathrm{I}}$ making an angle ϕ with $\mathfrak{p}$.

[If ever it is desired to show the essential features of $\vec{\mathrm{I}}$ as regards the circular frequency ω and the initial phase difference ϕ, we may write $\vec{\mathrm{I}} = \vec{\mathrm{I}}_{\omega,\phi}$, but normally the subscripts ω and ϕ are understood. This is especially so when we are dealing with voltages, for then ϕ is usually taken as zero, and $\vec{\mathrm{V}} \cdot \mathfrak{p}$ is employed to denote the instantaneous value of the impressed potential difference.]

Current and potential relationships in vectorial form.—(*a*) ***Pure resistance* R.** Suppose that an alternating p.d. $\hat{\mathrm{V}} \cos \omega t$ is applied to a non-inductive resistance R—cf. Fig. 16·07 (*a*). Then i, the instantaneous value of the current, is given by

$$\mathrm{R}i = v \quad \text{or} \quad \mathrm{R}\vec{\mathrm{I}} \cdot \mathfrak{p} = \vec{\mathrm{V}} \cdot \mathfrak{p}.$$

$$\therefore \vec{\mathrm{V}} = \mathrm{R}\vec{\mathrm{I}}$$

and the voltage and current are in phase. The initial positions of the rotating vectors $\vec{\mathrm{I}}$ and $\vec{\mathrm{V}}$ are shown in Fig. 16·07 (*b*); at time t they are as indicated in Fig. 16·07 (*c*).

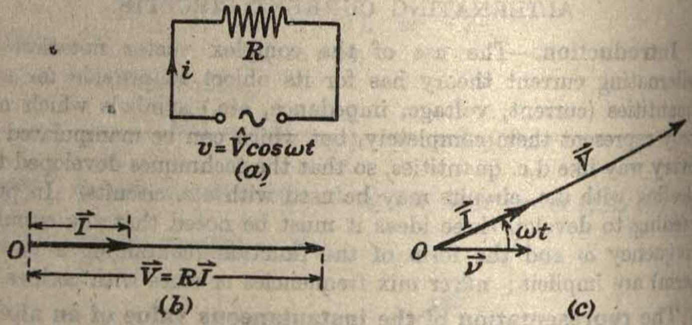


FIG. 16·07.—Current through a resistance, fed from an a.c. supply of sinusoidal wave-form.

(*b*) ***Inductive resistance* (R *and* L *in series*).** Let an alternating potential difference be established across a coil which possesses both resistance, R, and self inductance, L—cf. Fig. 16·08 (*a*). The

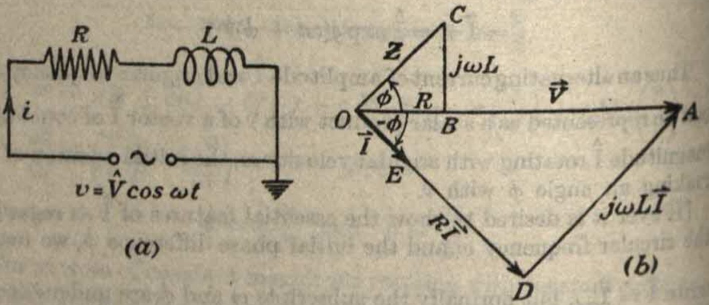


FIG. 16·08.—Current through an inductive resistance [R and L in series].

differential equation representing the relation between the instantaneous values of the current and voltage is

$$v = \mathrm{R}i + \mathrm{L}\frac{di}{dt} \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

Let $v = \hat{\mathrm{V}}\cos\omega t = \hat{\mathrm{V}}\exp(j\omega t)\mathfrak{p}\cdot\mathfrak{p} = \vec{\mathrm{V}}\cdot\mathfrak{p}$; also let $i = \vec{\mathrm{I}}\cdot\mathfrak{p}$, where $\vec{\mathrm{I}}$ has to be determined.

The equation (i) may be written

$$\vec{V}\cdot\mathfrak{p} = \hat{V}\exp(j\omega t)\mathfrak{p}\cdot\mathfrak{p} = R\vec{I}\cdot\mathfrak{p} + j\omega L\,\vec{I}\cdot\mathfrak{p}.$$

$$\therefore\ \vec{I} = \frac{\hat{V}\exp(j\omega t)\mathfrak{p}}{R + j\omega L} = \frac{\vec{V}}{Z},$$

where $Z = R + j\omega L$ and is known as the ***vector impedance*** or ***impedance operator*** for the impedance $Z = \sqrt{R^2 + \omega^2L^2}$.

Let $\dfrac{1}{R + j\omega L} = \dfrac{R - j\omega L}{R^2 + \omega^2L^2} = \alpha - j\beta = \rho\exp(-j\phi)$.

Then $\alpha = \dfrac{R}{R^2 + \omega^2L^2}$, and $\beta = \dfrac{\omega L}{R^2 + \omega^2L^2}$,

so that $\rho = \dfrac{1}{\sqrt{R^2 + \omega^2L^2}}$, and $\tan\phi = \dfrac{\omega L}{R}$.

$$\therefore\ \vec{I} = \frac{\hat{V}\exp(j\omega t)\mathfrak{p}}{R + j\omega L} = \frac{\hat{V}}{\sqrt{R^2 + \omega^2L^2}}\exp(j\omega t)\exp[(-j\phi)]\mathfrak{p}$$

$$= \frac{\hat{V}}{\sqrt{R^2 + \omega^2L^2}}[\exp j(\omega t - \phi)]\mathfrak{p}.$$

$$\therefore\ i = \vec{I}\cdot\mathfrak{p} = \vec{I}_{\omega,\phi}\cdot\mathfrak{p}.$$

Hence $\vec{I}$ is a vector of magnitude $\hat{I} = \dfrac{\hat{V}}{\sqrt{R^2 + \omega^2L^2}}$ rotating with angular velocity ω and whose angular position at time $t = 0$ is

$$-\tan^{-1}\left(\frac{\omega L}{R}\right),$$

i.e. the phase of the current is always behind that of the impressed voltage by an amount ϕ.

[ωL is termed the ***inductive reactance*** of the coil.]

The vector diagram, strictly at time $t = 0$, although this restriction is seldom necessary since the diagram may be considered to rotate with uniform angular velocity in the plane of the diagram and about O, is shown in Fig. 16·08 (*b*).

OB is drawn horizontally equal to R, while BC, normal to OB and of length ωL, is $j\omega L$. Thus $OC = Z = R + j\omega L$.

If $\vec{V}$ is drawn horizontally as OA, $\vec{I}$ is represented by a straight line OE of length $\dfrac{\hat{V}}{(R^2 + \omega^2L^2)^{\frac{1}{2}}}$ drawn at angle ϕ below OA, where

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) = B\hat{O}C.$$

If, from A we draw a perpendicular to OE to meet OE produced in D, then

$$\overrightarrow{\mathrm{OD}} + \overrightarrow{\mathrm{DA}} = \overrightarrow{\mathrm{OA}} = \vec{\mathrm{V}}.$$

But $\vec{\mathrm{V}} = \mathrm{R}\vec{\mathrm{I}} + j\omega \mathrm{L}\vec{\mathrm{I}}$, and since $\overrightarrow{\mathrm{OD}}$ and $\overrightarrow{\mathrm{DA}}$ are mutually perpendicular,

$$\overrightarrow{\mathrm{OD}} = \mathrm{R}\vec{\mathrm{I}}, \quad \text{and} \quad \overrightarrow{\mathrm{DA}} = j\omega\mathrm{L}\vec{\mathrm{I}}.$$

Thus $\overrightarrow{\mathrm{OD}}$ and $\overrightarrow{\mathrm{DA}}$ are the voltage vectors giving the potential differences across R and L respectively.

(*c*) ***Imperfect condenser, i.e.* R *and* C *in series.*** An imperfect condenser may be regarded as a capacitance C in series with a resistance R. Such a condenser, when connected to an a.c. supply, is shown conventionally in Fig. 16·09 (*a*). In the usual way, we have

$$v = \mathrm{R}i + \frac{q}{\mathrm{C}} = \mathrm{R}i + \frac{1}{\mathrm{C}}\int i\,dt.$$

$$\therefore\ \vec{\mathrm{V}}\cdot\not{p} = \mathrm{R}\vec{\mathrm{I}}\cdot\not{p} + \frac{1}{j\omega\mathrm{C}}\,\vec{\mathrm{I}}\cdot\not{p},$$

or

$$\vec{\mathrm{V}} = \mathrm{R}\vec{\mathrm{I}} + \frac{1}{j\omega\mathrm{C}}\vec{\mathrm{I}} = \left(\mathrm{R} - \frac{j}{\omega\mathrm{C}}\right)\vec{\mathrm{I}},$$

where $\vec{\mathrm{I}}$ has to be determined. As before, let

$$\mathrm{R} - \frac{j}{\omega\mathrm{C}} = \alpha - j\beta = \rho \exp(-j\phi) = \mathbf{Z}.$$

Then

$$\rho = \sqrt{\mathrm{R}^2 + \frac{1}{\omega^2\mathrm{C}^2}}, \quad \text{and} \quad \tan\phi = \frac{1}{\omega\mathrm{CR}}.$$

$$\therefore\ \vec{\mathrm{I}} = \frac{\vec{\mathrm{V}}}{\mathbf{Z}} = \frac{\hat{\mathrm{V}}(\exp j\omega t)(\exp j\phi)\not{p}}{\rho}$$

$$= \frac{\hat{\mathrm{V}}}{\sqrt{\mathrm{R}^2 + \dfrac{1}{\omega^2\mathrm{C}^2}}}[\exp j(\omega t + \phi)]\not{p}.$$

$$\therefore\ i = \vec{\mathrm{I}}\cdot\not{p} = \frac{\hat{\mathrm{V}}}{\sqrt{\mathrm{R}^2 + \dfrac{1}{\omega^2\mathrm{C}^2}}}\cos(\omega t + \phi), \quad \text{where } \tan\phi = \frac{1}{\omega\mathrm{CR}}.$$

To construct the vector diagram, OA, Fig. 16·09 (*b*), is drawn

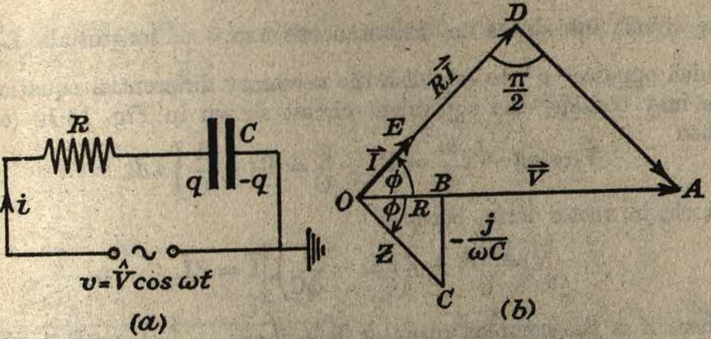


FIG. 16·09.—Current through an imperfect condenser [R and C in series].

horizontally to represent $\vec{V}$. Then OB is made equal to R while BC is drawn vertically downwards to denote $-\frac{j}{\omega C}$. Then

$$OC = Z = R - \frac{j}{\omega C}.$$

Also $\vec{I} = \frac{\vec{V}}{Z}$, so that OE is made equal to $\frac{\hat{V}}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}$, and is in advance of OA by an angle ϕ. Then, as before,

$$OD = R\vec{I}, \quad \text{and} \quad DA = -\frac{j}{\omega C}\vec{I}.$$

$\left[\frac{1}{\omega C}\right.$ is termed the ***capacitive reactance*** of the condenser.$\left.\right]$

(*d*) ***Resistance, inductance and capacitance in series.*** Let a resistance R, an inductance L, and a capacitance C be arranged

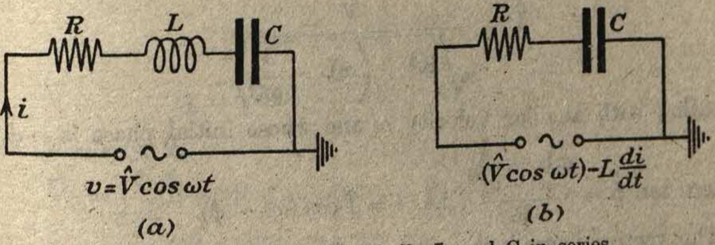

R
L
C
i
$v = \hat{V}\cos\omega t$
(a)
R
C
$(\hat{V}\cos\omega t) - L\frac{di}{dt}$
(b)

FIG. 16·10.—Current through R, L and C in series.

in series as shown in Fig. 16·10 (*a*) and connected to an alternating supply of sinusoidal wave-form. Since the presence of the coil L in the circuit introduces an instantaneous e.m.f. of magnitude $\mathrm{L}\frac{di}{dt}$ which opposes 'v', to establish the necessary differential equation we may consider the equivalent circuit shown in Fig. 16·10 (*b*). Then

$$\hat{\mathrm{V}}\cos\omega t - \mathrm{L}\frac{di}{dt} = \mathrm{R}i + \frac{q}{\mathrm{C}} = \mathrm{R}i + \frac{1}{\mathrm{C}}\int i\,dt,$$

which, in vector form, becomes

$$\vec{\mathrm{V}} = \left[\mathrm{R} + j\left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)\right]\vec{\mathrm{I}} = \bar{\mathrm{Z}}\vec{\mathrm{I}},$$

where $\bar{\mathrm{Z}}$ is the complex operator $\mathrm{R} + j\left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)$, or $\mathrm{R} + j\mathrm{X}$.

Let $v = \vec{\mathrm{V}}\cdot\vec{p} = \hat{\mathrm{V}}(\exp j\omega t)\vec{p}\cdot\vec{p}$, and $i = \vec{\mathrm{I}}\cdot\vec{p}$, where $\vec{\mathrm{I}}$ has to be determined. Then

$$\hat{\mathrm{V}}(\exp j\omega t)\vec{p}\cdot\vec{p} = \left(\mathrm{R} + j\omega\mathrm{L} + \frac{1}{j\omega\mathrm{C}}\right)\vec{\mathrm{I}}\cdot\vec{p}.$$

$$\therefore\ \vec{\mathrm{I}} = \frac{\hat{\mathrm{V}}}{\mathrm{R} + \hat{\jmath}\left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)}(\exp j\omega t)\vec{p}$$

$$= \hat{\mathrm{V}}\frac{\mathrm{R} - j\left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)}{\mathrm{R}^2 + \left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)^2}(\exp j\omega t)\vec{p}.$$

Let

$$\mathrm{R} - j\left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right) = \alpha - j\beta = \rho\exp(-\ j\phi).$$

Then
$$\vec{\mathrm{I}} = \frac{\hat{\mathrm{V}}}{\sqrt{\mathrm{R}^2 + \left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)^2}}[\exp j(\omega t - \phi)]\vec{p}$$

so that $\vec{\mathrm{I}}$ (really $\vec{\mathrm{I}}_{\omega,\phi}$) is a vector of magnitude

$$\hat{\mathrm{I}} = \frac{\hat{\mathrm{V}}}{\sqrt{\mathrm{R}^2 + \left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)^2}}$$

rotating with angular velocity ω and whose initial phase is $-\phi$, where $\tan\phi = \dfrac{\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}}{\mathrm{R}}$, i.e. $i = \hat{\mathrm{I}}\cos(\omega t - \phi)$.

The vector diagram is constructed in the usual manner and is shown in Fig. 16·11 (*a*) for the case where $\omega\mathrm{L} > \frac{1}{\omega\mathrm{C}}$, i.e. the phase of the current is always behind that of the impressed voltage.

If DE > BD, ϕ is negative and the current leads the voltage, cf. Fig. 16·11 (*b*).

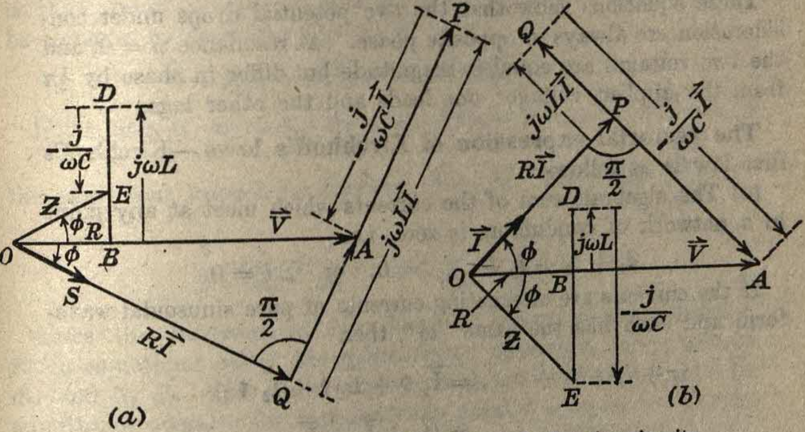


FIG. 16·11.—Vector diagrams for two R–L–C series circuits.

If DE = DB, ϕ is zero, the current and voltage are in phase, $\mathfrak{Z}$ is real and equal to R. This is the case of *series resonance.*

On the potential drops across the inductance and the capacitance in an R–L–C circuit fed from an a.c. supply.—If $\vec{V}_L$ is the vector denoting the potential drop across the inductance, since the impedance operator for an inductance is given by $\mathfrak{Z}_L = j\omega L$, we have

$$\begin{aligned}\vec{V}_L &= j\omega L\hat{I}[\exp j(\omega t - \phi)]\mathfrak{p}\\ &= (0 + j\omega L)\,\hat{I}[\exp j(\omega t - \phi)]\mathfrak{p}\\ &= \sqrt{0^2 + \omega^2 L^2}\,\hat{I}\left[\exp j\left(\omega t - \phi + \frac{\pi}{2}\right)\right]\mathfrak{p}\\ &= \omega L\hat{I}\left[\exp j\left(\omega t - \phi + \frac{\pi}{2}\right)\right]\mathfrak{p}.\end{aligned}$$

$$\therefore\ v_L = \omega L\hat{I}\cos\left(\omega t - \phi + \frac{\pi}{2}\right).$$

Similarly, $\mathfrak{Z}_C = 1 \div j\omega C$, and hence

$$\vec{V}_C = \frac{1}{j\omega C}\hat{I}[\exp j(\omega t - \phi)]\mathfrak{p} = -\frac{j}{\omega C}\hat{I}[\exp j(\omega t - \phi)]\mathfrak{p}$$

$$= \sqrt{0^2 + \frac{1}{\omega^2 C^2}}\,\hat{I}\left[\exp j\left(\omega t - \phi - \frac{\pi}{2}\right)\right]\mathfrak{p}$$

$$= \frac{1}{\omega C}\hat{I}\left[\exp j\left(\omega t - \phi - \frac{\pi}{2}\right)\right]\mathfrak{p}.$$

$$\therefore\ v_C = \frac{1}{\omega C}\hat{I}\cos\left(\omega t - \phi - \frac{\pi}{2}\right).$$

These equations show that the two potential drops under consideration are always in opposite phase. At resonance $\phi = 0$, and the two voltages are equal in magnitude but differ in phase by $\frac{1}{2}\pi$ from the applied voltage; one leads and the other lags.

The vectorial expression of Kirchhoff's laws.—Kirchhoff's first law is as follows:

(*a*) The algebraic sum of the currents which meet at any point in a network of conductors is zero, i.e.

$$i_1 + i_2 + i_3 + \ldots = 0, \quad \text{or} \quad \Sigma i = 0.$$

If the currents are alternating currents of pure sinusoidal waveform and each has the same 'ω', then

$$\begin{aligned} i_1 + i_2 + i_3 + \ldots &= \vec{I}_1\cdot\vec{p} + \vec{I}_2\cdot\vec{p} + \vec{I}_3\cdot\vec{p} + \ldots \\ &= (\vec{I}_1 + \vec{I}_2 + \vec{I}_3 + \ldots)\cdot\vec{p}, \end{aligned}$$

and since this is true at all times

$$\sum_{k=1}^{n}\vec{I}_k = 0.$$

The vectorial form of statement for Kirchhoff's first law is accordingly:—***In any network of conductors carrying alternating currents of sinusoidal waveform and each having the same 'ω', the vector sum of the vectors representing the currents which meet at any point is zero.***

(*b*) The second law as modified is :—***In any closed loop in a network of conductors, each with a vector characteristic of the type $\vec{V} = \text{Ƶ}\vec{I}$ and carrying alternating currents of sinusoidal waveform and constant 'ω', $\Sigma(\text{Ƶ}\vec{I}) = \Sigma\vec{E}$, where $\Sigma\vec{E}$ is the vector sum of the vector e.m.fs. in the loop.***

The equivalent vector impedance for impedances in series and in parallel.—If $Z_1, Z_2, Z_3 \ldots Z_n$ are n impedances in series, then if $\vec{I}$ is the current vector when the applied voltage is $\vec{V}\cdot\vec{p}$, we have

$$\vec{V} = \text{Ƶ}_1\vec{I} + \text{Ƶ}_2\vec{I} + \ldots + \text{Ƶ}_n\vec{I}.$$

If Ƶ is the equivalent vector impedance then $\text{Ƶ}\vec{I} = \vec{V}$. Comparing these two expressions for $\vec{V}$, we have,

$$\text{Ƶ} = \text{Ƶ}_1 + \text{Ƶ}_2 + \text{Ƶ}_3 + \ldots + \text{Ƶ}_n = \sum_{r=1}^{n}\text{Ƶ}_r.$$

When the n impedances are arranged in parallel, let $\vec{\mathrm{I}}_1, \vec{\mathrm{I}}_2, \ldots$ be the current vectors. If $\vec{\mathrm{V}}$ is the potential vector, and this will be common to each impedance, we have

$$\vec{\mathrm{V}} = \mathbb{Z}_1\vec{\mathrm{I}}_1 = \mathbb{Z}_2\vec{\mathrm{I}}_2 = \ldots = \mathbb{Z}_r\vec{\mathrm{I}}_r = \ldots$$

If the current vector for the current from the source is $\vec{\mathrm{I}}$ and $\mathbb{Z}$ the equivalent impedance operator, $\vec{\mathrm{V}} = \mathbb{Z}\vec{\mathrm{I}}$. Since $\vec{\mathrm{I}} = \sum_{r=1}^{n} \vec{\mathrm{I}}_r$,

$$\frac{1}{\mathbb{Z}} = \frac{1}{\mathbb{Z}_1} + \frac{1}{\mathbb{Z}_2} + \cdots \frac{1}{\mathbb{Z}_n} = \sum_{r=1}^{n} \frac{1}{\mathbb{Z}_r}.$$

Since the reciprocal of impedance is known as ***admittance***, $\mathbb{Z}^{-1}$ is sometimes called the ***admittance operator***, and it will be denoted by $\mathbb{Y}$. The above expression shows that the equivalent admittance operator for impedances in parallel is equal to the sum of the admittance operators for the separate impedances.

Example.—A series circuit consists of a resistance R, a self inductance L, and a capacitance C, to which an e.m.f. $\hat{\mathrm{V}} \cos \omega t$ is applied. If $\hat{\mathrm{V}} = 30$ volt., $\omega = 20{,}000$ sec.$^{-1}$, R = 4·0 ohm. and L = 0·050 henry, calculate (*a*) the value of C which makes the circuit resonant and (*b*) the peak value of the potential difference across the condenser at resonance.

The current and phase angle are given by the expressions

$$\hat{\mathrm{I}} = \frac{\hat{\mathrm{V}}}{\sqrt{\mathrm{R}^2 + \left(\omega \mathrm{L} - \dfrac{1}{\omega \mathrm{C}}\right)^2}}; \quad \tan \phi = \frac{\omega \mathrm{L} - \dfrac{1}{\omega \mathrm{C}}}{\mathrm{R}},$$

where ϕ is the angle by which the current lags.

The condition for current resonance is $\omega \mathrm{L} = \dfrac{1}{\omega \mathrm{C}}$ and this gives

$$\mathrm{C} = (\omega^2 \mathrm{L})^{-1} = \frac{1}{4 \times 10^8 \times 0{\cdot}05}\mathrm{F} = 0{\cdot}050\ \mu\mathrm{F}.$$

If $\vec{\mathrm{I}}$ is the current vector, the potential drop across the condenser is

$$\vec{\mathrm{I}}\left(-\frac{j}{\omega \mathrm{C}}\right) = \frac{\vec{\mathrm{V}}}{\mathbb{Z}}\left\{0 - j\left(\frac{1}{\omega \mathrm{C}}\right)\right\}.$$

At resonance $\mathbb{Z} = \mathrm{R}$ and hence the potential drop across the capacitor is

$$\frac{\hat{\mathrm{V}} \exp(j\omega t)}{\mathrm{R}}\left\{0 - j\left(\frac{1}{\omega \mathrm{C}}\right)\right\}\hat{\mathbf{v}}.$$

Let $$0 - j\left(\frac{1}{\omega \mathrm{C}}\right) = \mathrm{A}(\cos \phi - j \sin \phi).$$

Then $$\phi = \frac{\pi}{2}, \quad \text{and} \quad A = \frac{1}{\omega C}.$$

$$\therefore \text{ p.d. vector} = \frac{\hat{V}}{R\omega C}\left[\exp j\left(\omega t - \frac{\pi}{2}\right)\right]\vec{v}.$$

Thus the peak value of the p.d. is

$$\frac{\hat{V}}{R\omega C} = \frac{30 \times 10^6}{4 \times 2 \times 10^4 \times 5 \times 10^{-2}}$$
$$= 7500 \text{ volts.}$$

[N.B. Across the inductance the potential drop is $\frac{\vec{V}}{\mathbf{Z}}\{j\omega L\}$ and at resonance this is equal in magnitude to that across the condenser but it leads the applied voltage by $\frac{1}{2}\pi$. The phases of the potential drops across the condenser and across the inductance at resonance therefore differ by π, i.e. they are in ***antiphase***.]

An inductive coil shunted by a condenser.—Suppose an inductive coil (R, L) is shunted by a condenser of capacitance C. [This system represents a coil with self capacitance.]

Suppose the combination, cf. Fig. 16·12 (*a*), is fed from an a.c. supply with circular frequency ω; let $\mathbf{Z}_1$ be the impedance operator for the coil, i.e. $\mathbf{Z}_1 = R + j\omega L$, while $\mathbf{Z}_2$ is that for the condenser, i.e. $\mathbf{Z}_2 = \frac{1}{j\omega C}$. The equivalent operator, $\mathbf{Z}$, for the combination is given by

$$\frac{1}{\mathbf{Z}} = \frac{1}{R + j\omega L} + j\omega C = \frac{1 + j\omega CR - \omega^2 LC}{R + j\omega L}.$$

$$\therefore \mathbf{Z} = \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega CR}\cdot\frac{(1 - \omega^2 LC) - j\omega CR}{(1 - \omega^2 LC) - j\omega CR}$$

$$= \frac{R + j\omega[L(1 - \omega^2 LC) - CR^2]}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}.$$

If we write $\mathbf{Z} = R_0 + j\omega L_0$, the effective resistance of the coil is

$$R_0 = \frac{R}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

while the effective inductance is

$$L_0 = \frac{L(1 - \omega^2 LC) - CR^2}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}.$$

The impedance of the combination is given by

$$Z = \sqrt{R_0^2 + \omega^2 L_0^2}.$$

To evaluate this we substitute values for R_0 and L_0, and write

$$Z^2[(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2]^2$$
$$= R^2 + \omega^2\{(1 - \omega^2 LC)^2 L^2 - 2LCR^2(1 - \omega^2 LC) + C^2 R^4\}$$

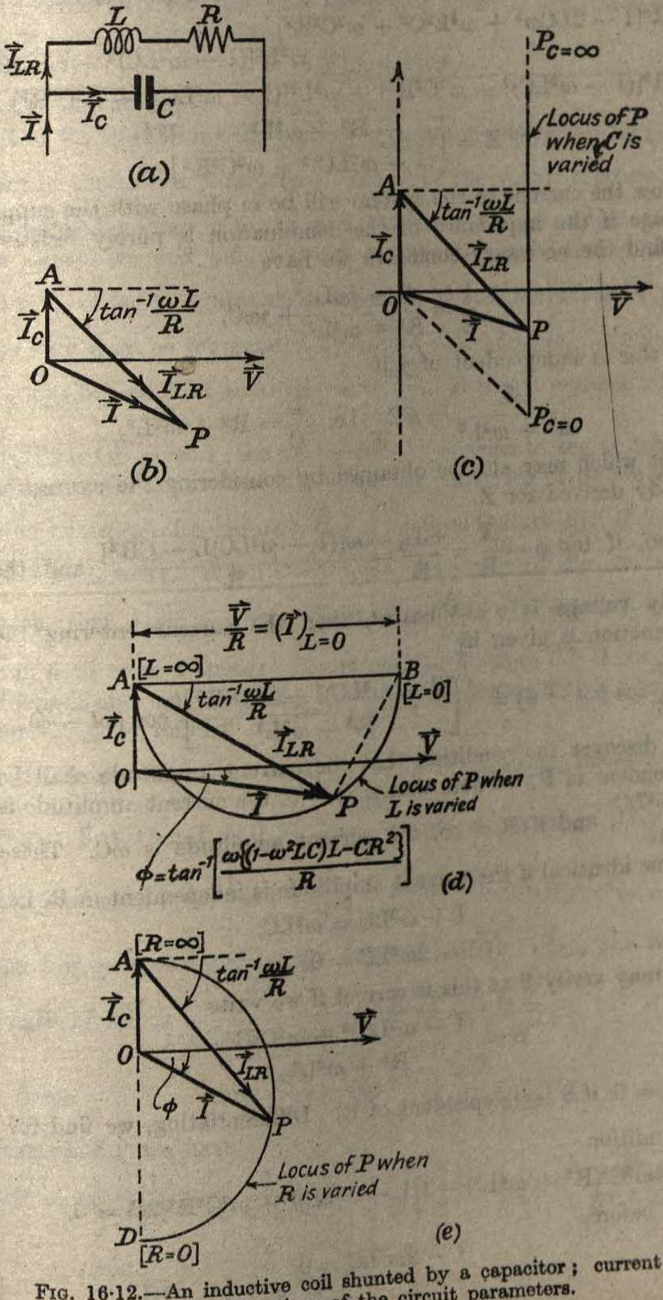

L
R
$\vec{I}_{LR}$
$\vec{I}_C$
C
$\vec{I}$
(a)
A
$\tan^{-1}\frac{\omega L}{R}$
$\vec{I}_C$
O
$\vec{V}$
$\vec{I}_{LR}$
$\vec{I}$
P
(b)
$P_{C=\infty}$
Locus of P when C is varied
A
$\tan^{-1}\frac{\omega L}{R}$
$\vec{I}_C$
$\vec{I}_{LR}$
O
$\vec{V}$
$\vec{I}$
P
$P_{C=0}$
(c)
$\frac{\vec{V}}{R} = (\vec{I})_{L=0}$
A
[L=∞]
B
[L=0]
$\tan^{-1}\frac{\omega L}{R}$
$\vec{I}_C$
$\vec{I}_{LR}$
$\vec{V}$
O
$\vec{I}$
P
Locus of P when L is varied
$\phi = \tan^{-1}\left[\frac{\omega\{(1-\omega^2 LC)L - CR^2\}}{R}\right]$
(d)
[R=∞]
A
$\tan^{-1}\frac{\omega L}{R}$
$\vec{I}_C$
$\vec{V}$
O
$\vec{I}_{LR}$
ϕ
$\vec{I}$
P
Locus of P when R is varied
D
[R=0]
(e)

FIG. 16·12.—An inductive coil shunted by a capacitor; current vectors as functions of the circuit parameters.

It should be noted also that the power factor is unity if $\phi = 0$, i.e. $\tan\phi = 0$, which gives

$$(1 - \omega^2 LC)L = CR^2,$$

and is zero if $\tan\phi = \infty$, i.e. $R = 0$.

Now let us examine the current vectors for this system as functions of the circuit parameters. Let $\vec{V}$ be the vector representing the applied voltage, while $\vec{I}_C$ and $\vec{I}_{LR}$ are the current vectors for the capacitive and inductive arms, respectively, of the network. Now $\vec{I}_C = j\omega C\vec{V}$ is always in advance of and normal to $\vec{V}$ and therefore may be represented by $\overrightarrow{OA}$ in Fig. 16·12 (*b*); similarly, $\vec{I}_{LR}$ lags behind $\vec{V}$ at an angle $\tan^{-1}\dfrac{\omega L}{R}$. If this is represented by $\overrightarrow{AP}$, then $\overrightarrow{OP}$ gives $\vec{I}$, the vector for the current to the system.

Now let C vary; when $C = 0$, i.e. the condenser plates are separated by an infinite distance, $[\vec{I}_C]_{C=0} = 0$, so that, cf. Fig. 16·12 (*c*), $\vec{I} = \overrightarrow{OP}_{C=0} = \vec{I}_L$. When $C = \infty$, i.e. the plates of the condenser touch, then $[\vec{I}_C]_{C=\infty} = \infty$. Thus the locus of P when C varies from 0 to ∞ is a straight line normal to $\vec{V}$, since $\overrightarrow{AP}$ is constant in length and moves parallel to itself; the locus extends upwards from $P_{C=0}$ to infinity.

If L varies from 0 to ∞, $\vec{I}_C$ or $\overrightarrow{OA}$, Fig. 16·12 (*d*) remains constant. But when $L = 0$, $\vec{I}_{L=0} = \dfrac{\vec{V}}{R} = \overrightarrow{AB}$, a vector of length $\dfrac{V}{R}$ and parallel to $\vec{V}$, for $\left[\tan^{-1}\dfrac{\omega L}{R}\right]_{L=0} = 0$. When $L = \infty$, $\vec{I}_{L=\infty} = 0$, and this is represented by the point A. If $\overrightarrow{AP} = \vec{I}_{LR}$, the actual length of $\overrightarrow{AP}$ is $\dfrac{\hat{V}}{\sqrt{R^2 + \omega^2 L^2}}$; that of $\overrightarrow{AB}$ is $\dfrac{\hat{V}}{R}$.

Since
$$\tan^{-1}\frac{\omega L}{R} = \cos^{-1}\frac{R}{\sqrt{R^2 + \omega^2 L^2}},$$
from ΔABP we have
$$BP^2 = (\hat{V})^2\left[\frac{1}{R^2} + \frac{1}{R^2 + \omega^2 L^2} - 2.\frac{1}{R\sqrt{R^2 + \omega^2 L^2}}.\frac{R}{\sqrt{R^2 + \omega^2 L^2}}\right]$$
$$= (\hat{V})^2\left[\frac{1}{R^2} - \frac{1}{R^2 + \omega^2 L^2}\right],$$

i.e. $A\hat{P}B = \frac{1}{2}\pi$. Thus the locus of P is a semicircle on AB as diameter.

Fig. 16·12 (*e*) indicates the details of the procedure necessary to show that, when R is varied, the locus of P is a semicircle on AD as diameter, where $\overrightarrow{AD}$ is normal to $\overrightarrow{V}$ and of length $\dfrac{\hat{V}}{\omega L}$.

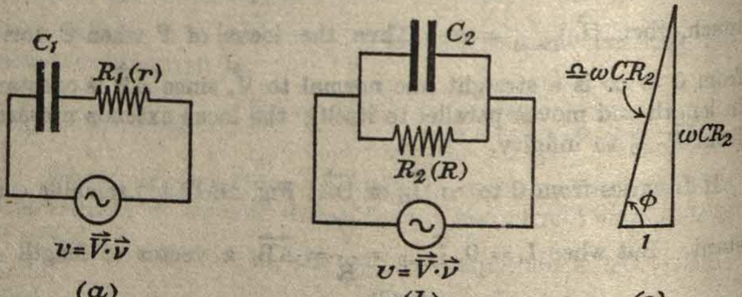
Power loss in an imperfect condenser.—The power loss in a condenser depends on the nature of the dielectric and upon the frequency. If the imperfect condenser is represented by a pure capacitance C with a resistance R_1 ***in series,*** Fig. 16·13 (*a*), the fundamental differential equation for the circuit is

$$v = R_1\frac{dq}{dt} + \frac{q}{C}.$$

If p is the instantaneous value of the power, we have since $i = \dot{q}$,

$$p = vi = R_1\left(\frac{dq}{dt}\right)^2 + \frac{1}{C}q\frac{dq}{dt}.$$

The first term is always positive whereas the second is as often negative as it is positive during a cycle so that no power is dissipated in the pure capacitance C; all is dissipated in R_1. To verify these statements when



(*a*) (*b*) (*c*)

FIG. 16·13.—The power factor of a condenser.

$$v = \hat{V}\cos\omega t \quad\text{and}\quad i = \hat{I}\cos(\omega t + \phi),$$

we have

$$p\,\delta t = \delta W = v\,\delta q = \frac{q}{C}\delta q.$$

Hence P_C, the mean power in the capacitance, is given by

$$P_C = \oint\frac{q}{C}dq \div \int_0^T dt,$$

where $\oint$ denotes that the summation is to be carried out for one complete cycle whose period is T. Thus

$$P_C = \frac{1}{TC}\int_0^T q\,dq = 0,$$

since the charge on the condenser is the same at each end of a complete cycle.

Similarly, for P_R, the mean power in R_1, we have

$$P_R = \frac{1}{T}R_1(\hat{I})^2\int_0^T \cos^2(\omega t + \phi)dt = \frac{R_1(\hat{I})^2}{2} = R_1 I^2,$$

where I is the r.m.s. current.

$$\therefore\ P = P_C + P_R = R_1 I^2.$$

The imperfect or 'leaky' condenser may also be represented by a pure capacitance with a large resistance R_2 ***in parallel,*** cf. Fig. 16·13 (*b*). If the impressed voltage is $v = \hat{V}\cos\omega t = \vec{V}\cdot\vec{\nu}$, the impedance operator $\mathbf{Z}$ is given by

$$\frac{1}{\mathbf{Z}} = \frac{1}{R_2} + \frac{1}{\left(\dfrac{1}{j\omega C}\right)} = \frac{1}{R_2} + j\omega C.$$

$$\therefore\ \mathbf{Z} = \frac{R_2}{1 + j\omega CR_2}.$$

$$\therefore\ \vec{I} = \frac{\vec{V}}{R_2}(1 + j\omega CR_2) = \frac{\hat{V}}{R_2}\exp(j\omega t).\sqrt{1 + \omega^2C^2R_2{}^2}\,[\exp(j\phi)]\vec{\nu},$$

where $\tan\phi = \omega CR_2$.

$$\therefore\ i = \frac{\hat{V}\sqrt{1 + \omega^2C^2R_2{}^2}}{R_2}\cos(\omega t + \phi),$$

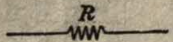
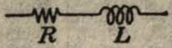
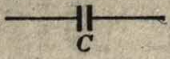
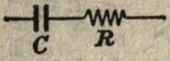
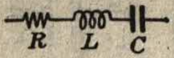
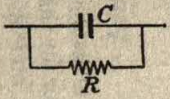
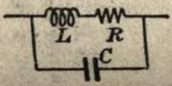
and the power factor is $\cos\phi$, which, cf. Fig. 16·13 (*c*), is given by $\cos\phi \simeq \dfrac{1}{\omega CR_2}$, since R_2 is very large for any condenser which is usable. The power is

$$VI\cos\phi = \frac{V^2}{Z}\cdot\frac{1}{\sqrt{1 + \omega^2C^2R_2{}^2}}$$

$$\simeq V^2R_2{}^{-1},\ \text{since } Z = R_2 \div \sqrt{1 + \omega^2C^2R_2{}^2}.$$

If R_1 is a real resistance in series with an ideal condenser C, the power factor for that circuit is ωCR_1, i.e. it increases with frequency. For a real and large resistance R_2 in parallel with an ideal condenser, C, the power factor is $\dfrac{1}{\omega CR_2}$ and this decreases with

Table of Impedance Operators

Circuit element		Z	$i = \hat{I}\cos(\omega t - \phi)$ ϕ
	R	R	0
	$R + j\omega L$	$(R^2 + \omega^2 L^2)^{\frac{1}{2}}$	$\tan^{-1}\dfrac{\omega L}{R}$
	$\dfrac{1}{j\omega C}\left(\text{or } -\dfrac{j}{\omega C}\right)$	$\dfrac{1}{\omega C}$	$-\dfrac{\pi}{2}$
	$R + \dfrac{1}{j\omega C}$	$\left[R^2 + \dfrac{1}{\omega^2 C^2}\right]^{\frac{1}{2}}$	$-\tan^{-1}\dfrac{1}{\omega CR}$
	$R + j\left(\omega L - \dfrac{1}{\omega C}\right)$	$\left[R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2\right]^{\frac{1}{2}}$	$\tan^{-1}\dfrac{\omega L - \dfrac{1}{\omega C}}{R}$
	$\dfrac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$	$\dfrac{R}{[1 + \omega^2 C^2 R^2]^{\frac{1}{2}}}$	$-\tan^{-1}\omega CR$
	$\dfrac{R + j\omega[(1 - \omega^2 LC)L - CR^2]}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$	$\left[\dfrac{R^2 + \omega^2 L^2}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}\right]^{\frac{1}{2}}$	$\tan^{-1}\dfrac{\omega[(1 - \omega^2 LC)L - CR^2]}{R}$

Now
$$\frac{R_1}{R_2}+\frac{C_2}{C_1}=1.$$

$$\therefore\ \frac{C_2}{C_1}=1-\frac{R_1}{R_2}=1-\frac{1}{1+\omega^2C_2^2R_2^2}=\frac{\omega^2C_2^2R_2^2}{1+\omega^2C_2^2R_2^2},$$

or
$$\frac{1}{\omega^2C_1^2R_1^2}\cdot\frac{1}{1+\dfrac{1}{\omega^2C_1^2R_1^2}}=\frac{1}{1+\omega^2C_1^2R_1^2}.$$

Thus $C_1 \neq C_2$, strictly, but the difference is very minute for any condenser which is likely to be used. For example, if $C_2 = 1\ \mu F.$, $R_2 = 10^6\ \Omega$.—a really bad condenser—then at $f = 50$ sec.$^{-1}$, or $\omega = 314$ sec.$^{-1}$,

$$C_1 = C_2\left[1+\frac{1}{314^2.10^{-12}.10^{12}}\right] \doteqdot C_2.$$

For all condensers which are likely to be used in practice R_1 is small and R_2 large; accordingly one often writes $R_1 = r$, and $R_2 = R$.

Since with a high-class condenser $\phi \longrightarrow \frac{1}{2}\pi$, in order to state the magnitude of the imperfection it is found convenient to introduce an angle δ, such that $\delta = \frac{1}{2}\pi - \phi$. δ is known as the ***loss angle.*** Since $\tan\phi = \frac{1}{\omega Cr}$, $\tan\delta = \omega Cr$, and this equals $\cos\phi$, the power factor.

Similarly,
$$\tan\delta = \frac{1}{\omega CR}.$$

On the maximum power which can be dissipated in a load in series with a generator.—(*a*) ***A d.c. generator.*** Let V be the e.m.f. of the generator and R_1 its internal resistance; it is shown diagrammatically in Fig. 16·14 (*a*). If R_2 is the resistance of the load, the current I is given by $I=\frac{V}{R_1+R_2}$, and the potential difference across the load is IR_2.

$\therefore$ Power dissipated in load is

$$I(IR_2)=\frac{V^2R_2}{(R_1+R_2)^2}=P \text{ (say)}.$$

Now
$$\frac{dP}{dR_2}=V^2\left[\frac{(R_1+R_2)^2-2R_2(R_1+R_2)}{(R_1+R_2)^4}\right]$$
$$=0 \text{ for a maximum or a minimum.}$$

$$\therefore\ R_2 = R_1,$$

and a second differentiation shows that, when this condition is satisfied, the rate of dissipation of energy in the load is a maximum.

(*b*) ***An a.c. generator.*** The system is indicated symbolically in Fig. 16·14 (*b*). Then

$$\vec{I} = \frac{\vec{V}}{(R_1 + R_2) + j(X_1 + X_2)}.$$

$$\therefore\ i = \frac{\hat{V}}{\sqrt{(R_1 + R_2)^2 + (X_1 + X_2)^2}} \cos(\omega t - \phi),$$

where $$\tan\phi = \frac{X_1 + X_2}{R_1 + R_2}.$$

$$\therefore\ I = \frac{V}{\sqrt{(R_1 + R_2)^2 + (X_1 + X_2)^2}}.$$

The r.m.s. current is a maximum as a function of R_1 when $R_1 \rightarrow 0$, and similarly for R_2; it is a maximum as a function of $(X_1 + X_2)$ when $X_1 + X_2 = 0$, i.e. the condition for series resonance.

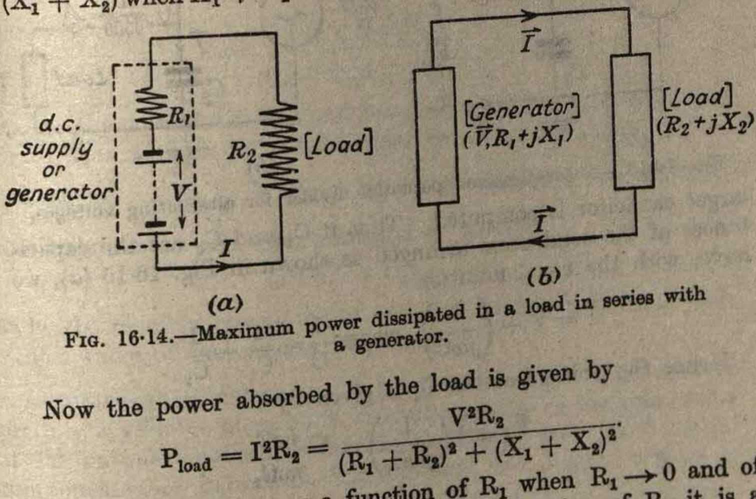


FIG. 16·14.—Maximum power dissipated in a load in series with a generator.

Now the power absorbed by the load is given by

$$P_{\text{load}} = I^2R_2 = \frac{V^2R_2}{(R_1 + R_2)^2 + (X_1 + X_2)^2}.$$

This is a maximum as a function of R_1 when $R_1 \rightarrow 0$ and of $(X_1 + X_2)$ when $(X_1 + X_2) \rightarrow 0$. As a function of R_2 it is a maximum when $R_2 = \sqrt{R_1{}^2 + (X_1 + X_2)^2}$. To prove this, write $R_1 = R$, $R_2 = r$. Then

$$P_{\text{load}} = \frac{V^2r}{(R + r)^2 + (X_1 + X_2)^2}.$$

$$\therefore\ \frac{dP}{dr} = V^2\left[\frac{(R + r)^2 + (X_1 + X_2)^2 - 2r(R + r)}{\{(R + r)^2 + (X_1 + X_2)^2\}^2}\right].$$

This is zero if

$$r^2 = R^2 + (X_1 + X_2)^2, \quad \text{i.e.} \quad R_2{}^2 = R_1{}^2 + (X_1 + X_2)^2.$$

These equations give for the instantaneous voltages at X and Y,

$$v_x = \tfrac{1}{2}\hat{V}\cos\omega t,$$

and

$$v_y = \left\{\frac{1}{1 + j\omega CR}\cdot\frac{1 - j\omega CR}{1 - j\omega CR}\right\}\hat{V}\,[\exp(j\omega t)]_{\vec{v}.\vec{v}}$$

$$= \frac{\hat{V}}{\sqrt{1 + \omega^2C^2R^2}}\cos(\omega t - \phi),$$

where $\tan\phi = \omega CR$. Hence, after expanding the cosine term,

$$v_y - v_x = \frac{\hat{V}}{\sqrt{1 + \omega^2C^2R^2}}\left[\frac{1}{\sqrt{1 + \omega^2C^2R^2}}\cos\omega t + \frac{\omega CR}{\sqrt{1 + \omega^2C^2R^2}}\sin\omega t\right] - \tfrac{1}{2}\hat{V}\cos\omega t$$

$$= \frac{\hat{V}}{1 + \omega^2C^2R^2}[\{1 - \tfrac{1}{2}(1 + \omega^2C^2R^2)\}\cos\omega t + \omega CR\sin\omega t],$$

which, if written in the form $A\cos\psi\cos\omega t + A\sin\psi\sin\omega t$, gives

$$A = \tfrac{1}{2}\hat{V}, \text{ and } \tan\psi = \frac{2\tan\phi}{1 - \tan^2\phi} = \tan 2\phi, \text{ i.e. } \psi = 2\phi.$$

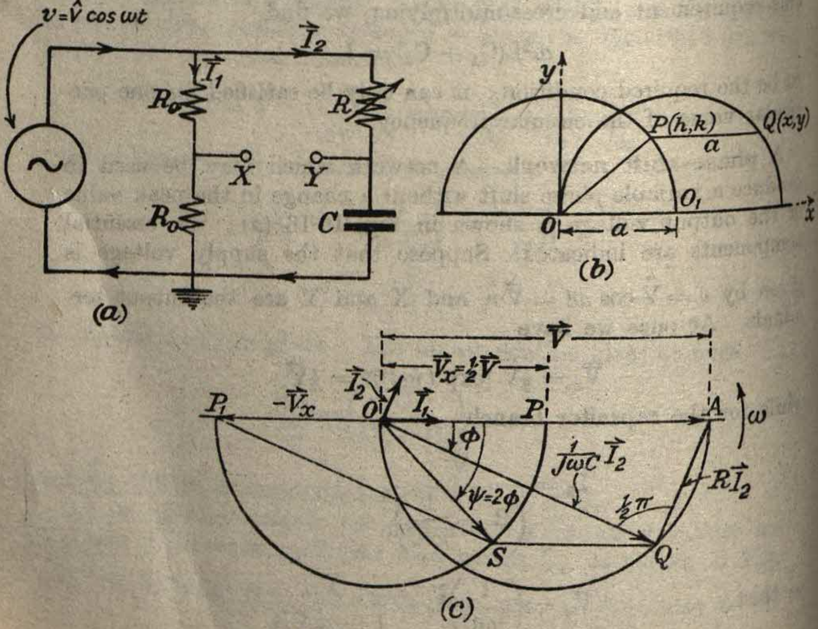


FIG. 16·16.—A phase-shift network.

capacitance alone is varied. Then

$$I^2 = \frac{V^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

If the value of the current is plotted against the corresponding value of the capacitance, the well-known ***response curves*** (each for a particular value of R) are obtained; a typical curve is shown in Fig. 16·17 (*b*). The maximum value of the r.m.s. current is $I_0 = \frac{V}{R}$ and this occurs when the capacitance has the value C_0, which satisfies the condition $\omega^2 LC_0 = 1$. Moreover, the value of I_0 increases as $R \rightarrow 0$ and at the same time the curve exhibits a sharper peak. The sudden increase in the value of the current to a maximum is utilized in radio-frequency measurements to show that the condition of current-resonance has been attained. In such work it is essential that the peak should be narrow.

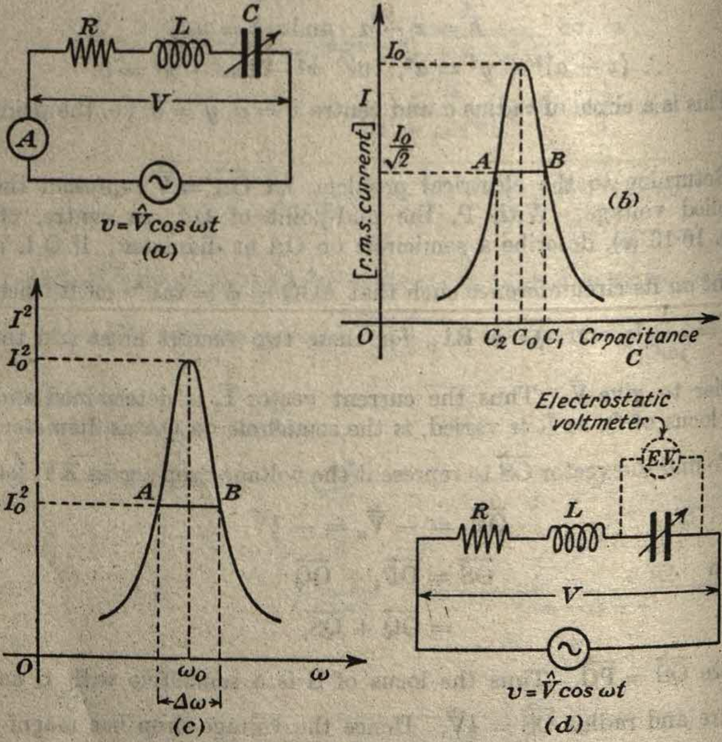


FIG. 16·17.—Series resonance.

the ***quality factor*** or ***magnification***. Thus

$$Q = \frac{\omega_0 L}{R} = \frac{\omega_0}{\alpha},$$

where $\alpha = \frac{R}{L}$. Since δ, the decrement per cycle, is $\frac{1}{2}\alpha T$, we have $Q = \frac{\pi}{\delta}$.

Since $C_1 + C_2 \doteqdot 2C_0$, equation (ii) may be written in the form

$$\frac{C_1 + C_2}{C_1 - C_2} = Q = \frac{\pi}{\delta}, \qquad . \quad . \quad . \quad \text{(iii)}$$

and its value now lies in the fact that C_1 and C_2 may each always be observed with greater accuracy than C_0, since C_1 and C_2 correspond to points on the sides of the response curve where the slope is steep.

Equation (iii) also permits us to make an accurate determination of δ, and hence of α, which is such that $\delta = \frac{1}{2}\alpha T = \alpha\left(\frac{\pi}{\omega_0}\right)$.

If the constants, R, L and C of the circuit under investigation are kept fixed, the ***resonance curve*** is obtained by examining the variation of the current with the frequency of the supply. Then

$$I = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}},$$

or $$I^2 = \frac{K^2\omega^2}{(\omega_0^2 - \omega^2)^2 + \alpha^2\omega^2}, \qquad \left[\text{By dividing throughout by } L^2.\right]$$

where K is a constant, viz. VL^{-1}.

Let I_0 be the current at resonance. Then $I_0 = \frac{K\omega_0}{\alpha\omega_0} = \frac{V}{R}$.

In order that $\frac{I_0^2}{I^2}$ may be equal to 2, let the frequency of the supply be such that ω is $\omega_0 \pm \frac{1}{2}\Delta\omega$. Then

$$I^2 \doteqdot \frac{K^2\omega_0^2}{(\omega_0\Delta\omega)^2 + (\alpha\omega_0)^2},$$

or $$\frac{I_0^2}{I^2} = 2 = \frac{(\Delta\omega)^2 + \alpha^2}{\alpha^2}.$$

$$\therefore \Delta\omega = \alpha.$$

Since I^2R is the power in the circuit we may call the (I^2, ω) curve, shown in Fig. 16·17 (*c*), a 'power spectrum' and the above analysis shows that the 'half-power' bandwidth, $\Delta\omega$, is equal to the attenuation constant α. Thus we have the following definitions for the selectivity Q.

and this corresponds to a minimum value of y since when $\dfrac{dy}{dC} = 0$ $\dfrac{d^2y}{dC^2}$ is positive.

[Thus, for the voltage-resonance we no longer have the simple relation

$$\omega^2 LC_0 = 1.$$

The resonance point is now affected by the value of the resistance R.]

Now

$$\omega^2 LC_0 = \frac{\omega^2 L^2}{R^2 + \omega^2 L^2} = 1 - \frac{R^2}{R^2 + \omega^2 L^2} = 1 - \cos^2\phi = 1 - \sin^2\delta,$$

where ϕ is the phase angle and δ the loss angle of a resistance R in series with an inductance L.

To obtain $[V_C]_0$, the potential difference across the condenser at resonance, we have

$$\left\{\left[V_C\right]_0\right\}^2 = \frac{V^2}{\omega^2 R^2\left(\dfrac{L}{R^2 + \omega^2 L^2}\right)^2 + \left\{\omega^2 L . \dfrac{L}{R^2 + \omega^2 L^2} - 1\right\}^2}$$

$$= \frac{V^2(R^2 + \omega^2 L^2)}{R^2} = \frac{V^2}{\cos^2\phi}.$$

As in the case of current-resonance, to any other value of V_C there will correspond two values of the capacitance C, one on either side of the resonance peak. When $\dfrac{\left\{\left[V_C\right]_0\right\}^2}{{V_C}^2} = 2$, combining the equations we have obtained we get

$$\frac{\left\{\left[V_C\right]_0\right\}^2}{{V_C}^2} = 2 = \frac{[\omega^2 R^2 C^2 + (\omega^2 LC - 1)^2][R^2 + \omega^2 L^2]}{R^2}.$$

$$\therefore \frac{2R^2}{R^2 + \omega^2 L^2} = C^2\omega^2(R^2 + \omega^2 L^2) - 2\omega L\omega C + 1.$$

$$\therefore \omega C = \frac{\omega L \pm R}{R^2 + \omega^2 L^2} = \omega C_0 \pm \frac{R}{R^2 + \omega^2 L^2}.$$

$$\therefore C_0 = \tfrac{1}{2}(C_1 + C_2),$$

where C_1 and C_2 are the two values of C given by the preceding equation.

Such a relationship is valid for all values of $\dfrac{\left\{\left[V_C\right]_0\right\}^2}{{V_C}^2}$, so that in this instance the response curve is symmetrical.

It can be shown that Z is then a maximum and, as we have seen already, when C has this value the power factor of the combination

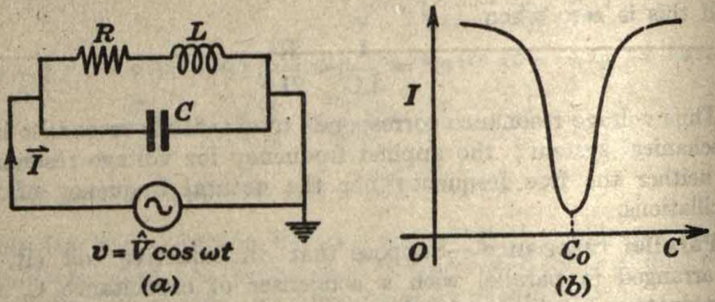


Fig. 16·18.—Parallel resonance.

is unity. The variation of the current with C in the neighbourhood of C_0 is shown in Fig. 16·18 (*b*).

Also
$$Z_{\text{max.}} = [Z]_{C=C_0} = \frac{R^2 + \omega^2 L^2}{R} = \frac{Z_L{}^2}{R}.$$

If L is the variable, Z is a maximum when $\omega^2 L^2 C - L - CR^2 = 0$ and the power factor is not unity.

If ω is the variable, Z is a maximum when

$$\omega = \sqrt{\frac{\sqrt{2R^2\left(\frac{C}{L}\right) + 1}}{LC} - \frac{R^2}{L^2}}$$

and again the power factor is not unity.

Now the combination under discussion is said to show ***parallel resonance*** when the power factor is unity. This, as shown above, occurs when

$$L - C(R^2 + \omega^2 L^2) = 0, \quad \text{or} \quad \omega^2 = \frac{1}{LC} - \frac{R^2}{L^2} = \omega_0{}^2.$$

Under such conditions the impedance of the combination is

$$\sqrt{\frac{R^2 + \frac{L}{C} - R^2}{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)C^2R^2 + \left(1 + \frac{R^2C}{L} - 1\right)^2}} = \frac{L}{CR} = Z_0 \text{ (say)}.$$

This particular value of the impedance is purely resistive * and is known as the ***dynamic resistance*** of the combination.

---

* $\left[\frac{L}{R}\right] = [T]$, $[CR] = [T]$; hence $\left[\frac{L}{RC}\right] = [R]$.

The selectivity of a parallel resonant circuit.—The manner in which the impedance of the combination, discussed in the previous paragraph, varies with the circular frequency ω is shown in Fig. 16·19. At resonance

$$\omega = \omega_0 \quad \text{and} \quad [Z]_{\omega=\omega_0} = Z_{\text{max.}} = \frac{L}{CR}.$$

Now the selectivity of the combination is that quantity which expresses its ability to present a high impedance at the resonant frequency and a much lower impedance at other frequencies. Thus the more pronounced the peakiness of the curve, the greater will be the selectivity.

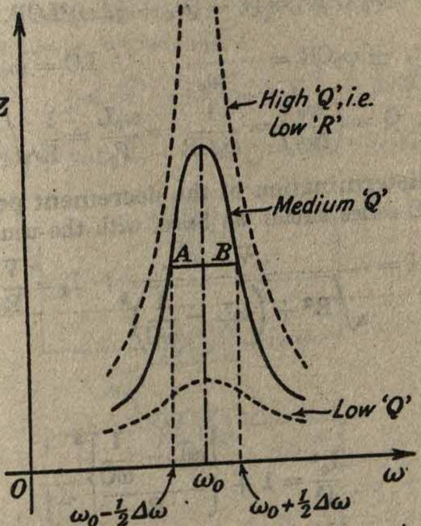


FIG. 16·19.—The impedance of a parallel resonant circuit and its variation with circular frequency.

Let AB ($= \Delta\omega$) be the width of the resonance curve when the impedance is $\frac{1}{\sqrt{2}} \times$ its maximum value, Z_0. Then Q, the selectivity, is defined as $\frac{\omega_0}{\Delta\omega}$. In practice, $R \to 0$ and thus

$$Z_{R\to 0} = \frac{j\omega L}{(1 - \omega^2 LC) + j\omega CR},$$

or $$Z_{R\to 0} = \frac{\omega L}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}}. \qquad \text{[cf. p. 560.]}$$

At $\omega = \omega_0$, $Z = Z_0 = \frac{L}{RC}$. If $\omega = \omega_0 \pm \frac{1}{2}\Delta\omega$, the terms ωL and

$\omega^2C^2R^2$ will not differ appreciably from ω_0L and $\omega_0{}^2C^2R^2$ but the term $(1 - \omega^2LC)$ will no longer be zero. It will have undergone a considerable change. Thus we may write, when $R \rightarrow 0$,

$$Z = \frac{\omega_0L}{\sqrt{(1-\omega^2LC)^2 + \omega_0{}^2C^2R^2}}.$$

If $\omega = \omega_0 + \frac{1}{2}\Delta\omega$ when $Z = \frac{1}{\sqrt{2}}Z_0$, we have

$$\omega_0{}^2C^2R^2 = [1 - (\omega_0 + \tfrac{1}{2}\Delta\omega)^2LC]^2.$$

$$\therefore \pm\, \omega_0CR = -\frac{\Delta\omega}{\omega_0}, \qquad \because\ LC = \omega_0{}^{-2}.$$

$$\therefore\ Q = \left(\frac{\omega_0}{\Delta\omega}\right) = \frac{1}{\omega_0CR} = \frac{\omega_0L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

Practical determination of the decrement per cycle (δ).— For an R–L–C series circuit we have, with the usual notation,

$$I = \frac{V}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}; \quad I_0 = \frac{V}{R}.$$

Hence

$$\frac{I_0{}^2}{I^2} = 1 + \left\{\frac{\omega L - \dfrac{1}{\omega C}}{R}\right\}^2.$$

$$\therefore\ \frac{\omega L - \dfrac{1}{\omega C}}{R} = \pm\sqrt{\frac{I_0{}^2}{I^2} - 1} = \pm\, m \text{ (say)} \qquad \text{(i)}$$

A resonance curve is obtained by plotting $\frac{I}{I_0}$ against $\frac{\omega}{\omega_0}\left(\text{or } \frac{f}{f_0}\right)$. In this curve the same values of the curve ordinate, and therefore of m, are given by two values of ω, viz., ω_1 and ω_2, one on either side of ω_0; each satisfies equation (i), which may be written

$$\omega^2 \mp \frac{mR\omega}{L} - \frac{1}{LC} = 0.$$

$$\therefore\ \omega_1,\ \omega_2 = \frac{1}{2}\left\{\pm\frac{mR}{L} + \sqrt{\frac{m^2R^2}{L^2} + \frac{4}{LC}}\right\}.$$

$$\therefore\ \omega_1 - \omega_2 = mRL^{-1}.$$

$$\therefore\ \text{‘Width’ of curve} = \frac{f_1}{f_0} - \frac{f_2}{f_0} = \frac{\omega_1 - \omega_2}{\omega_0} = \frac{mR}{L}\cdot\frac{1}{2\pi f_0} = \frac{m}{\pi}\delta.$$

$\therefore\ \delta = \frac{\pi}{m} \times$ width of the resonance curve at a height $\frac{I}{I_0}$ given b

$$m = \sqrt{\frac{I_0^2}{I^2} - 1}. \qquad \text{[Cf. Ex. 14·28, p. 496.]}$$

A simple air-transformer.—We may represent such a simple transformer as in Fig. 16·20. The values of the various components are indicated. If $v = \hat{V}\,[\exp(j\omega t)]\vec{v}\cdot\vec{v} = \vec{V}\cdot\vec{v}$ is the applied voltage and Z_1 and Z_2 denote the vector impedances $R_1 + j\omega L_1$ and $R_2 + j\omega L_2$ respectively, we have

$$\vec{V} = (R_1 + j\omega L_1)\vec{I}_1 + j\omega M\vec{I}_2 = Z_1\vec{I}_1 + j\omega M\vec{I}_2,$$

$$0 = (R_2 + j\omega L_2)\vec{I}_2 + j\omega M\vec{I}_1 = Z_2\vec{I}_2 + j\omega M\vec{I}_1.$$

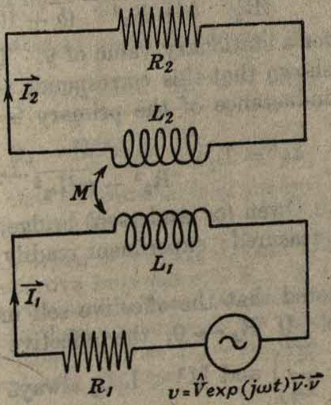


FIG. 16·20.—The theory of a simple air-transformer.

Eliminating $\vec{I}_2$ from the first equation by means of the second, we get

$$\vec{I}_1 = \frac{\vec{V}}{Z_1 + \dfrac{\omega^2 M^2}{Z_2}} = \frac{\vec{V}}{Z_1'},$$

which leads to

$$\vec{I}_2 = -\frac{j\omega M\vec{V}}{Z_1 Z_2 + \omega^2 M^2} = \frac{\vec{V}}{Z_2'},$$

where Z_1' and Z_2' are the effective impedance operators for the primary and secondary circuits.

Now $$Z_1' = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2}.\frac{R_2 - j\omega L_2}{R_2 - j\omega L_2}$$

$$= R_1 + \frac{\omega^2 M^2}{R_2^2 + \omega^2 L_2^2} R_2 + j\omega\left[L_1 - \frac{\omega^2 M^2}{R_2^2 + \omega^2 L_2^2} L_2\right].$$

The effective resistance of the primary is therefore

$$R_1' = R_1 + \frac{\omega^2 M^2}{R_2^2 + \omega^2 L_2^2} R_2.$$

Now $R_1' - R_1$ is large and reaches a maximum value for a certain value of R_2, viz. $R_2 = \omega L_2$. To prove this, we write

$$R_1' - R_1 = y = \frac{aR_2}{b + R_2^2} \text{ (say).}$$

$$\therefore \frac{dy}{dR_2} = \frac{a}{b + R_2^2} - \frac{2aR_2^2}{(b + R_2^2)^2},$$

and this is zero for a stationary value of y. Then $R_2 = b^{\frac{1}{2}} = \omega L_2$, and it is easily shown that this corresponds to a maximum value.

The effective inductance of the primary is

$$L_1' = L_1 - \frac{\omega^2 M^2}{R_2^2 + \omega^2 L_2^2}.L_2.$$

By means of an Owen (or Anderson) bridge, both $R_1' - R_1$ and $L_1' - L_1$ can be measured; experiment readily confirms the theory here developed.

It should be noted that the effective self inductance can never become negative. If $R_2 \rightarrow 0$, the effective inductance of the primary is $\left(L_1 - \frac{M^2}{L_2}\right)$ and $M^2 < L_1 L_2$ always [cf. p. 238]. Thus $\vec{I}_1$ always lags behind $\vec{V}$ although the amount of this lag decreases as $R_2 \rightarrow 0$ and the coupling becomes more tight.

Example.—A circuit consists of two branches in parallel, the first containing an inductive coil of resistance R_1 and a self inductance L, the second a condenser of capacitance C in series with a resistance R_2. Find the conditions that when a pure sine-form alternating voltage is applied to it, the circuit may have a constant impedance at all frequencies.

The effective impedance operator for the two impedances in parallel is given by

$$\frac{1}{Z} = \frac{1}{R_1 + j\omega L} + \frac{1}{R_2 + \frac{1}{j\omega C}} = \frac{R_1 + R_2 + j\left(\omega L - \frac{1}{\omega C}\right)}{R_1 R_2 + \frac{L}{C} + j\left(\omega L R_2 - \frac{R_1}{\omega C}\right)},$$

so that Z is of the form $\frac{a + jf(\omega)}{p + j\phi(\omega)}$, where a, p, f and ϕ are real. Now such a fraction is independent of ω if its first differential coefficient with respect to ω vanishes, i.e.

$$0 = \frac{jf'(\omega)\{p + j\phi(\omega)\} - j\phi'(\omega)\{a + jf(\omega)\}}{[p + j\phi(\omega)]^2}.$$

Hence

$$jpf'(\omega) - f'(\omega)\phi(\omega) - ja\phi'(\omega) + \phi'(\omega)f(\omega) = 0.$$

Equating in turn the real and the imaginary parts we get

$$f'(\omega)\phi(\omega) = \phi'(\omega)f(\omega), \quad \text{and} \quad pf'(\omega) = a\phi'(\omega),$$

i.e.

$$\frac{p}{\phi(\omega)} = \frac{a}{f(\omega)}.$$

Thus, in the present example,

$$(R_1 + R_2)\left(\omega LR_2 - \frac{R_1}{\omega C}\right) = \left(R_1R_2 + \frac{L}{C}\right)\left(\omega L - \frac{1}{\omega C}\right),$$

which leads to

$$\omega^2LC\left(R_2^2 - \frac{L}{C}\right) = R_1^2 - \frac{L}{C},$$

and this can only be satisfied at all frequencies if

$$R_2^2 - \frac{L}{C} = 0, \quad \text{and} \quad R_1^2 - \frac{L}{C} = 0,$$

i.e.

$$R_1 = R_2 \quad \text{and} \quad \frac{L}{R_1} = CR_2.$$

These conditions are that the resistances of the two branches as well as their time constants must be equal.

Alternative method. When $\omega = 0$, no current flows in the capacitive branch and the effective impedance is R_1. If $\omega \rightarrow \infty$, no current flows in the inductive branch so that the effective impedance is R_2. Thus one condition that the impedance of the combination shall be independent of the frequency is that $R_1 = R_2 = R$ (say).

Hence for the general case we have

$$\frac{1}{Z} = \frac{1}{R} = \frac{1}{R + j\omega L} + \frac{1}{R - \frac{j}{\omega C}}$$

$$= \frac{2R + j\omega L - \frac{j}{\omega C}}{R^2 + \frac{L}{C} + j\left(\omega LR - \frac{R}{\omega C}\right)} \cdot \frac{R^2 + \frac{L}{C} - j\omega LR + \frac{jR}{\omega C}}{R^2 + \frac{L}{C} - j\omega LR + \frac{jR}{\omega C}}.$$

$$\therefore \frac{1}{R} = \frac{2R^3 + \omega^2L^2R + \frac{R}{\omega^2C^2} + j\left(-\omega LR^2 + \frac{R^2}{\omega C} + \frac{\omega L^2}{C} - \frac{L}{\omega C^2}\right)}{R^4 + \frac{L^2}{C^2} + \omega^2L^2R^2 + \frac{R^2}{\omega^2C^2}}.$$

Considering first the real part and cross-multiplying, we have

$$R^4 + \frac{L^2}{C^2} + \omega^2L^2R^2 + \frac{R^2}{\omega^2C^2} = 2R^4 + \omega^2L^2R^2 + \frac{R^2}{\omega^2C^2}.$$

$$\therefore R^2 = \frac{L}{C}, \quad \text{or} \quad \frac{L}{R} = CR.$$

The imaginary part is zero and leads to $\left(\omega L - \frac{1}{\omega C}\right)\left(\frac{L}{C} - R^2\right) = 0$ and if $L = CR^2$, this is independent of ω.

Oscillations in inductively coupled circuits.—Let a circuit of resistance R_1, self inductance L_1 and capacitance C_1 be inductively coupled to a second circuit with parameters R_2, L_2 and C_2. Further let M be the mutual inductance of the two coils and suppose, cf. Fig. 16·21 (*a*), an e.m.f. $v = \hat{V} \cos \omega t$ is applied to the first circuit. Then with the usual notation

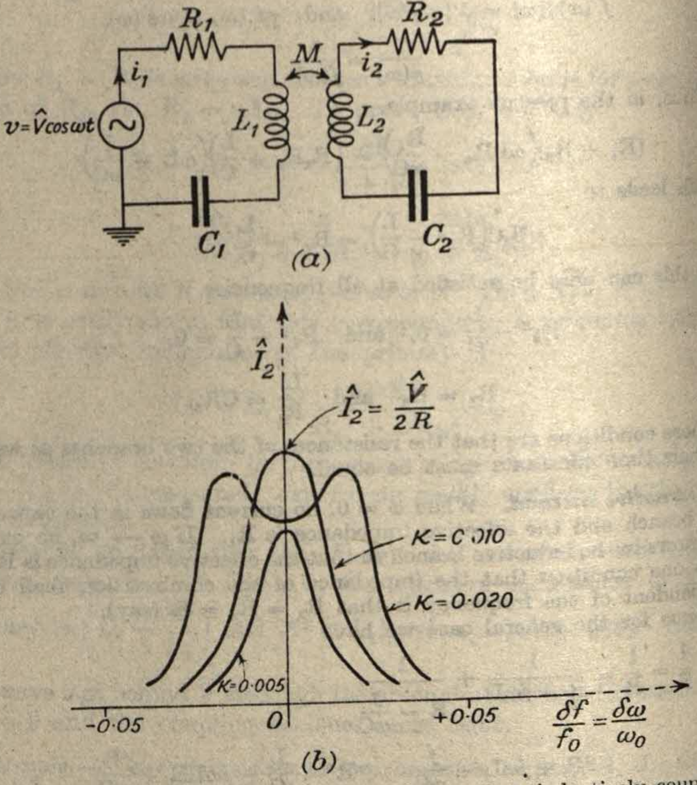

R_1
R_2
M
i_1
i_2
$v = \hat{V}\cos\omega t$
L_1
L_2
C_1
C_2
(a)
$\hat{I}_2$
$\hat{I}_2 = \frac{\hat{V}}{2R}$
K=C 010
K=0·020
K=0·005
-0·05
0
+0·05
$\frac{\delta f}{f_0} = \frac{\delta\omega}{\omega_0}$
(b)

FIG. 16·21.—Current resonance in the secondary of an inductively coupled circuit system.

$$\vec{V} = \left\{R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right\}\vec{I}_1 + j\omega M\vec{I}_2,$$

and
$$0 = \left\{R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right\}\vec{I}_2 + j\omega M\vec{I}_1.$$

These equations may be written

$$\vec{V} = Z_1\vec{I}_1 + Z\vec{I}_2, \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

and

$$0 = Z_2\vec{I}_2 + Z\vec{I}_1, \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

where the Z's are the appropriate impedance operators. Solving for $\vec{I}_1$ and $\vec{I}_2$ we find

$$\vec{I}_1 = \frac{Z_2}{Z_1Z_2 - Z^2}\vec{V}, \quad \text{and} \quad \vec{I}_2 = -\frac{Z}{Z_1Z_2 - Z^2}\vec{V}.$$

These expressions give the current vectors so that the values of their amplitudes and initial phases can be written down but their explicit forms are too involved to be of much physical importance. To proceed, however, let us write

$$Z_1 = R_1 + jX_1, \quad \text{and} \quad Z_2 = R_2 + jX_2.$$

To eliminate $\vec{I}_2$ from equation (i) we use

$$\vec{I}_2 = \frac{-j\omega M\vec{I}_1}{R_2 + jX_2},$$

and obtain

$$\vec{V} = (R_1 + jX_1)\vec{I}_1 + \frac{M(j\omega M)(-j\omega M\vec{I}_1)}{R_2 + jX_2}$$

$$= \left(R_1 + jX_1 + \frac{\omega^2M^2}{R_2 + jX_2}\right)\vec{I} = Z'\vec{I}_1,$$

where Z' is the effective impedance operator for the primary circuit. But

$$\frac{\omega^2M^2}{R_2 + jX_2} = \frac{\omega^2M^2(R_2 - jX_2)}{R_2^2 + X_2^2},$$

so that as a result of the coupling the effective resistance of the primary is increased by $\frac{\omega^2M^2R_2}{R_2^2 + X_2^2}$; its effective reactance is decreased by $\frac{\omega^2M^2X_2}{R_2^2 + X_2^2}$.

When the current in the secondary circuit is a maximum we have a state of affairs in which the transfer of energy from the primary to the secondary is also a maximum; the system is then said to have *optimum coupling*.

To determine the value of M when the coupling has its optimum value let us examine the particular but nevertheless very importan case in which each circuit is resonant to a supply with circula frequency ω_0. Then $X_1 = X_2 = 0$, and we have, if the supply has a circular frequency ω,

$$\vec{I}_2 = \frac{-j\omega M\vec{V}}{R_2\left(R_1 + \omega^2\dfrac{M^2}{R_2}\right)},$$

or

$$i_2 = \frac{(0 - j\omega M)\hat{V}\,[\exp\,(j\omega t)]\,\vec{\nu}\cdot\vec{\nu}}{R_2\left(R_1 + \omega^2\dfrac{M^2}{R_2}\right)}$$

$$= \frac{\omega M\hat{V}\cos\,(\omega t - \tfrac{1}{2}\pi)}{R_2\left(R_1 + \omega^2\dfrac{M^2}{R_2}\right)} = \hat{I}_2\cos\,(\omega t - \tfrac{1}{2}\pi).$$

To examine how $\hat{I}_2$ varies with M, let us write $u = \dfrac{M}{R_1R_2 + \omega^2M^2}$. Then $\dfrac{du}{dM}$ is zero when

$$\omega M = \sqrt{R_1R_2},$$

and a second differentiation shows that this value for M is such that the transfer of energy between two mutually coupled tuned circuits is a maximum.

If, in addition, the resistances R_1 and R_2 are each equal to R, then optimum coupling occurs when $\omega M = R$. In such an instance let k_0 be the coupling coefficient. Then

$$k_0 = \left(\frac{M}{L}\right)_{\omega M = R} = \frac{R}{\omega L} = \frac{1}{Q},$$

where Q is the quality factor, cf. p. 569.

Again, considering two coupled tuned circuits each with resistance R, the reactive components of the impedance operators for the primary and secondary will vanish if

$$j\left(\omega L - \frac{1}{\omega C}\right)\vec{I}_1 + j\omega M\vec{I}_2 = 0,$$

and

$$j\left(\omega L - \frac{1}{\omega C}\right)\vec{I}_2 + j\omega M\vec{I}_1 = 0.$$

These equations give

$$\frac{\omega L - \dfrac{1}{\omega C}}{\omega M} = \frac{\omega M}{\omega L - \dfrac{1}{\omega C}},$$

and since $LC = \dfrac{1}{{\omega_0}^2}$, where ω_0 is the resonant circular frequency,

$$L = \frac{1}{{\omega_0}^2C}.$$

Hence
$$\left(\frac{1}{\omega_0{}^2C} - \frac{1}{\omega^2C}\right)^2 = \omega^4M^2.$$

i.e.
$$\left(\frac{\omega^2}{\omega_0{}^2} - 1\right) = \pm\,\omega^2MC = \pm\,\omega^2kLC$$
$$= \pm\,\frac{\omega^2}{\omega_0{}^2}k.$$
$$\therefore\ \omega = \frac{\omega_0}{\sqrt{1+k}} \text{ or } \frac{\omega_0}{\sqrt{1-k}}.$$

The coupled circuit is therefore resonant to the two frequencies f_1 and f_2, where
$$f_1 = \frac{f_0}{\sqrt{1+k}}, \quad\text{and}\quad f_2 = \frac{f_0}{\sqrt{1-k}}.$$

The band of frequencies which is passed by the system we are considering is therefore
$$\frac{f_0}{\sqrt{1-k}} - \frac{f_0}{\sqrt{1+k}}.$$

When $k \rightarrow 0$, i.e. the coupling is loose, the above band width becomes
$$f_0[(1 + \tfrac{1}{2}k) - (1 - \tfrac{1}{2}k)] = kf_0.$$

Coupled circuits play an important part in the reception of radio signals when it is desired to accept a narrow band of frequencies and reject any frequencies outside it. The ideal response curve would therefore have a rectangular outline. In practice, by using tuned circuits with somewhat more than optimum coupling a resonance curve with steep sides is obtained. The general behaviour cf the current in the secondary circuit is shown in Fig. 16·21 (*b*). When the circuits are overcoupled, ($\omega M > R$), the secondary current amplitude shows two maxima; with optimum coupling (i.e. $\omega M = R$), $\hat{I}_2 = \dfrac{\hat{V}}{2R}$, and the transfer of energy to the secondary is a maximum. The secondary circuit is then considered as 'matched' to the primary.

Example.—The admittance operator, ¥, for a certain R, L circuit is 0·04 − *j*0·08. Obtain values for the resistance and inductive reactance of the circuit if the components are (*a*) in series, (*b*) in parallel.

(*a*) We have, since admittance is the reciprocal of impedance,
$$\text{Ƶ} = [\text{¥}]^{-1} = (0{\cdot}04 - j0{\cdot}08)^{-1} = \frac{0{\cdot}04 + j0{\cdot}08}{(0{\cdot}0016 + 0{\cdot}0064)}$$
$$= \frac{0{\cdot}04 + j0{\cdot}08}{0{\cdot}0080} = 5{\cdot}0 + j10{\cdot}0.$$

Thus the circuit would consist of a resistance of 5·0 ohm. in series with an inductance whose reactance is 10·0 ohm.

(*b*)
$$Y = Z^{-1} = \frac{1}{R} + \frac{1}{jX} = G - jB,$$

where G is the conductance, and B the susceptance, viz. the reciprocal of the reactance.

Now $\qquad Y = 0{\cdot}04 - j0{\cdot}08.$

$\therefore\ G = 0{\cdot}04$ ohm.$^{-1}$ and $B = 0{\cdot}08$ ohm.$^{-1}$,

i.e. $\qquad R = 25$ ohm. and $X = 12{\cdot}5$ ohm.

In passing, it is of interest to see whether or not a condenser could be found which would effectively replace the inductance. If C is the capacitance of this condenser when in series with R then

$$Z = R + \frac{1}{j\omega C} = R - \frac{j}{\omega C},$$

and since $Z = 5{\cdot}0 + j10{\cdot}0$, no value of C with a physical meaning satisfies the conditions.

Similarly a combination with C in parallel with R cannot be found.

Example.—Fig. 16·22 shows two arrangements of a self inductance and two capacitors; the values of the components are shown. It is required to determine the values of the components in (*b*) in terms of

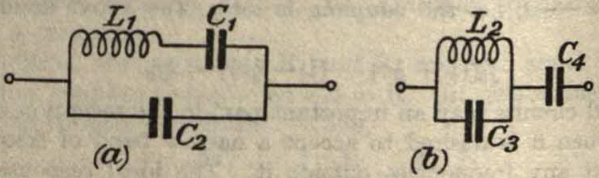


FIG. 16·22.

those in (*a*) when the two systems have identical impedance operators at all frequencies.

If Z_1 is the impedance operator for the first system, we have

$$\frac{1}{Z_1} = \frac{1}{j\omega L_1 + \dfrac{1}{j\omega C_1}} + j\omega C_2 = \frac{j\omega C_1}{1 - \omega^2 L_1 C_1} + j\omega C_2.$$

Let Z' be the impedance operator for L_2 in parallel with C_3. Then

$$\frac{1}{Z'} = \frac{1}{j\omega L_2} + j\omega C_3, \quad \text{or} \quad Z' = \frac{j\omega L_2}{1 - \omega^2 L_2 C_3}.$$

Hence for (*b*)

$$Z_2 = \frac{j\omega L_2}{1 - \omega^2 L_2 C_3} + \frac{1}{j\omega C_4}.$$

If the operators are identical, then $\dfrac{1}{Z_1}.Z_2 = 1$,

i.e.

$$\left(\frac{j\omega C_1}{1 - \omega^2 L_1 C_1} + j\omega C_2\right)\left(\frac{j\omega L_2}{1 - \omega^2 L_2 C_3} + \frac{1}{j\omega C_4}\right) = 1.$$

Hence, multiplying out and then removing denominators, we find

$$- \omega^2 L_2 C_1 - \omega^2 L_2 C_2 + \omega^4 L_1 L_2 C_1 C_2 + \frac{C_1}{C_4} - \omega^2 L_2 \frac{C_1 C_3}{C_4}$$
$$+ \frac{C_2}{C_4}(1 - \omega^2 L_1 C_1 - \omega^2 L_2 C_3 + \omega^4 L_1 L_2 C_1 C_3)$$
$$= 1 - \omega^2 L_1 C_1 - \omega^2 L_2 C_3 + \omega^4 L_1 L_2 C_1 C_3.$$

Arranging this equation in the form $\alpha + \beta\omega^2 + \gamma\omega^4 = 0$, the coefficients α, β and γ must vanish if the stated conditions are to be satisfied.

Hence

$$C_1 + C_2 = C_4,$$
$$L_2[C_1 + C_2] = L_1 C_1\left[1 - \frac{C_2}{C_4}\right],$$

i.e.

$$L_2 = \frac{L_1}{\left(1 + \frac{C_2}{C_1}\right)^2},$$

and

$$C_3\left(\frac{C_2}{C_4} - 1\right) = - C_2,$$

or

$$C_3 = C_2\left(\frac{C_1}{C_1 + C_2}\right).$$

EXAMPLES XVI

16·01 (*a*). Evaluate the sum, product and quotient of $(1 + j3)$ and $(2 + j)$.

[$5 \exp j\phi$, where $\tan\phi = \frac{4}{3}$; $\sqrt{50} \exp j\phi$, where $\tan\phi = -7$; $1 + j$.]

(*b*). Prove that $\sqrt{3 + j4} = \pm 5^{\frac{1}{4}}(0{\cdot}895 + 0{\cdot}446j) = \pm(2 + j)$.

(*c*). Show that one value of j^j is $\exp(-\frac{1}{2}\pi)$. [Hint: Use

$$j = \cos\tfrac{1}{2}\pi + j\sin\tfrac{1}{2}\pi = \exp j(\tfrac{1}{2}\pi),$$

and take logarithms.]

Also show that one value of $\ln(-1)$ is $j\pi$.

16·02. A 240 volt a.c. supply is connected to a resistance of 500 ohm. and a 8 μF. condenser. If the frequency of the supply is (*a*) 50 cycle.sec.$^{-1}$, (*b*) 100 cycle.sec.$^{-1}$, calculate the current and phase angle in each instance. If the condenser were replaced by a 3 H. inductance calculate the new values of the current and phase angle for the 50 cycle.sec.$^{-1}$ supply.

[0·376 A., 38° 30′ (lead), 0·446 A., 21° 42′ (lead);
0·225 A., 62° 0′ (lag), 0·123 A., 75° 9′ (lag).]

16·03. An inductive coil tuned by a condenser C is loosely coupled to a high frequency generator. The values of C to produce resonance at 10^6 cycle.sec.$^{-1}$ and $1{\cdot}25 \times 10^6$ cycle.sec.$^{-1}$ are 700 pF. and 350 pF., respectively. Find the self capacitance of the coil. [272 pF.]

16·04. A circuit consists of an inductance of 0·80 H., a capacitance 12 μF. and resistance of 50 ohm., all in series. If the frequency of the supply is 50 cycle.sec.$^{-1}$, calculate the r.m.s. voltage required to pass a current of 3 amp. through the circuit. Calculate the potential difference across both the inductance and the condenser.

[156 V., 754 V., 796 V.]

16·05 (*a*). A 5 μF. condenser and a 10 ohm. coil are connected in series to a 10 V. a.c. supply of frequency $\frac{100}{2\pi}$ cycle.sec.$^{-1}$. Calculate the power supplied.

[0·25 mW.]

(*b*). A condenser of capacitance C is in parallel with a resistance R and the combination is fed from a source of potential difference whose instantaneous value is $v = \hat{V}\cos\omega t$. Show that the current from the source is

$$i = \hat{V}\sqrt{\frac{1}{R^2} + \omega^2C^2}\,.\cos(\omega t + \phi),$$

where $\tan\phi = \omega CR$. Show also that the power is V^2R^{-1}.

16·06. An alternating electromotive force whose e.m.f. is sinusoidal is applied to a non-inductive circuit consisting of a resistance and a capacitance in series. Find an expression for the current finally produced.

A condenser of capacitance 1 μF. and a resistance of 50 ohm. are connected in series with an alternating supply of 100 volt. amplitude and frequency 50 cycle.sec.$^{-1}$. The addition to the circuit of an inductive coil of negligible resistance is found to cause the current to increase. Explain this and find the value of the added self inductance to give maximum current. State also the value of this maximum current.

[10·14 H., I = 1·414 A.]

16·07. A small electric furnace, operated from a 240 V., 50 cycle.sec.$^{-1}$ supply has a power consumption of 4 kW. at unity power factor. If the supply voltage is raised to 400 V., calculate the reactance which must be placed in series with the furnace so that it may operate under the specified conditions. If the reactance takes the form of (*a*) a pure inductance, (*b*) an ideal condenser, calculate their values. What is then the power factor? [9·6 Ω., 0·0306 H., 333 μF., 0·83.]

16·08. The losses in a condenser can be represented as the effect of a series or a shunt resistance. Obtain expressions for the relationships between the two equivalent resistances, and between the corresponding capacitances.

The product of the equivalent series resistance and capacitance of a leaky condenser at a certain frequency is 2×10^{-9} sec. and the power factor 0·001. At what frequency was the measurement made, and what is the capacitance of the condenser if the leak is in fact due to a shunt resistance of 1 megohm? (S)

[$f = 7{\cdot}95 \times 10^4$ cycle.sec.$^{-1}$, 0·002 μF.]

16·09. A generator for which $v = \hat{V}\cos\omega t$ is connected at time t_0 to an inductive reactor (R, L). Find the current in the circuit at a time $t(> t_0)$ and prove that the current will be free from all transients if t_0 is such that

$$\omega t_0 = \frac{\pi}{2} + \tan^{-1}\frac{\omega L}{R}.$$

16·10. An e.m.f. of 240 V. r.m.s. value at 50 cycle.sec.$^{-1}$ is applied to a circuit containing a capacitance of 5 μF. in series with a resistance of 1000 ohms. Find the rate at which energy is dissipated in the circuit. Also find the power factor. [40·9 watt., 0·84.]

16·11. Explain what is meant by the root-mean-square value of an alternating current.

When an ideal condenser and a resistance are connected in parallel to a 100 volt. z.f. supply a current of 1 amp. flows. When, instead, a 100 volt a.c. supply at a frequency of 50 cycle.sec.$^{-1}$ is used, the current is 2 amp. Find (*a*) the value of the resistance, (*b*) the capacitance of the condenser. [100 ohm. 55 μF.]

16·12. Find the r.m.s. value of the resultant current in a wire which carries simultaneously a direct current of 1 A. and a sinusoidal one for which $\hat{I} = 5$ A. [3·67 A.]

16·13. Compare the heating effects of two current waves of equal peak values, the wave-form of the first being sinusoidal while that of the second is rectangular. [1 : 2.]

16·14. Calculate (i) the maximum value, (ii) the r.m.s. value of each of the following quantities :—(*a*) $240 \cos \omega t$, (*b*) $2(m + n) \cos (\omega t - \frac{1}{2}\pi)$, (*c*) $4 \cos \omega t - 3 \sin \omega t$.

[(i) 240, $2(m + n)$, 5 ; (ii) 170, $(m + n)\sqrt{2}$, 3·54.]

16·15. A voltage $v = 10 \cos \omega t$ is applied to a circuit which contains in series a moving-coil ammeter, a hot-wire ammeter and a rectifier. The circuit offers infinite resistance to the flow of current in one direction and a non-inductive resistance of 5 Ω. to the flow in the other direction. Calculate values for the readings of the two instruments. [0·64 A., 1·0 A.]

16·16. Calculate a value for the reading which will be indicated by a hot-wire ammeter through which passes an alternating current whose wave-form is that of an isosceles triangle with a peak value of 20 A. [11·54 A.]

16·17. A solenoid, 1 metre in length, has a cross-sectional area of 10 cm.^{2} and 2000 turns of wire. If the wire is wound uniformly on a wooden core calculate an approximate value for the self inductance of the solenoid.

If a secondary with 5000 turns is wound closely on the solenoid near to its mid-section, calculate a value for the mutual inductance of the combination.

Describe and explain a method, in which use is made of a standard condenser, for investigating the accuracy of your calculations. [5·03 mH., 12·6 mH.]

16·18. A non-inductive resistance is connected in series with a coil across a 240 V., $\dfrac{200}{\pi}$ cycle.sec.$^{-1}$ supply. The current is 2 amp. and the potential differences across the resistance and the coil are 80 V. and 180 V. respectively. Obtain values for the power factor, resistance and inductance of the coil. Illustrate your answer by means of a vector diagram. [0·65, 58·7 Ω., 0·170 H.]

16·19. The current in one circuit is $\hat{I} \cos \omega t$. A neighbouring circuit consists of a coil of self inductance L and negligible resistance in series with a condenser C. If M is the mutual inductance between the circuits show that the instantaneous charge on the condenser is

$$\frac{\omega MC\hat{I}}{(\omega^2 LC - 1)} \sin \omega t.$$

16·20. A 1·01 μF. condenser is in series with a resistance R and the combination is fed from a supply for which V = 100 volt., and $\omega = 1000$ radian.sec.$^{-1}$. What must be the value of R in order that the current may be 0·10 A.? Show that the power factor is 0·140. What self inductance, substituted for the condenser, would produce the same current and what would then be the power factor?

[140 ohm., 0·99 H., 0·140.]

16·21. Explain what is meant by the dynamic impedance of a parallel resonant circuit. A circuit consisting of a condenser in parallel with a coil of self inductance 1 mH. and resistance 20 ohm. is connected to an oscillator whose output has a pure sine-wave form of frequency 0·05 megacycle.sec.$^{-1}$. Determine (*a*) the value of the capacitance of the condenser at which the current drawn from the source is in phase with the applied voltage and (*b*) the magnitude of this in-phase current if the output voltage of the oscillator has a r.m.s. value of 5 V. [Take $\pi^2 = 10$.]

[0·01 μF., 1 mA.]

16·22. A simple parallel circuit consists of a resistance R in parallel with a reactance X. Prove that the impedance of the circuit is given by $RX(R^2 + X^2)^{-0\cdot5}$, and the power factor by $X(R^2 + X^2)^{-0\cdot5}$. A resistance and a reactance, when connected in series across a 240 V. a.c. supply, take a current of 4 A. and the power factor is 0·6. When the resistance and reactance are connected in parallel, across the same supply, calculate values for the current from the generator and the power factor of the combination. [8·33 A., 0·8.]

16·23. A resistance of 40 Ω. is joined in parallel with a reactance of 30 Ω. across an a.c. supply. Obtain values for the resistance and reactance which, when connected in series with the same supply, would take the same current and have the same power factor.

[14·4 Ω., 19·2 Ω.]

16·24. A low-resistance choke may be regarded as equivalent to a pure inductance L_1 in series with a resistance R_1, or a pure inductance L_2 in parallel with a resistance R_2. Prove that $L_1 \doteqdot L_2$ and $R_2 \doteqdot \frac{\omega^2 L_1^2}{R_1}$.

16·25. A non-inductive coil of resistance 20 ohm. is placed in series with a resistance R which is also in series with an ideal condenser of capacitance C. The whole system is fed from a 50 cycle.sec.$^{-1}$ supply and the potential drop across the 20 ohm. coil is found to be 20 V.; across the R–C combination it is 50 V. and across the whole system it is 64 V. Find the power factor of the R–C combination and values for R and C.

[0·6, 30 Ω., 807 μF.]

16·26. A single-phase a.c. supply of r.m.s. voltage 200 and for which $\omega = 1000$ radian.sec.$^{-1}$ is used to feed a resistance of 100 ohm. in parallel with a 10 μF. condenser. Calculate values for the power in the load and its power factor. How would you verify your calculations experimentally?

[400 watt., 0·71.]

16·27. A series circuit consists of a resistance of 10 ohms, an inductance 5 mH. and a condenser whose capacitance is 800 pF. A sinusoidal e.m.f. of *amplitude* 5 volts is connected across the above series combination and its frequency may be varied over a wide range.

Calculate values for the frequencies at which the amplitude of the current will be (*a*) a maximum, (*b*) one-half its maximum value.

[(*a*) $7\cdot962 \times 10^4$ cycle.sec.$^{-1}$,
(*b*) $7\cdot985 \times 10^4$ and $7\cdot930 \times 10^4$ cycle.sec.$^{-1}$.]

16·28. If a voltage, whose instantaneous value is given by $v = 100 \cos \omega t + 10 \cos 3\omega t$, is applied to a resistance of 5 Ω. in series with an inductance of 0·1 H., and $\omega = 100$ radian.sec.$^{-1}$, calculate a value for the power factor which, in such an instance, is defined as the mean power divided by the product of the r.m.s. voltage and the r.m.s. current. [0·445.]

16·29. A 2 b.h.p. motor runs off a 240 V., 50 cycle.sec.$^{-1}$ supply. Its efficiency is 0·72 and it operates at a power factor of 0·6. Determine the capacitance of that ideal condenser which must be placed in parallel with the motor so that the power factor is raised to unity. [153 μF.]

[N.B.—The *brake horse-power* (*b.h.p.*) is the (mechanical) work done by the machine per second. Dividing this by the efficiency gives the (electrical) energy put into the machine per second; this is VI cos ϕ. The current taken by the motor lags behind the applied voltage and may be resolved into two components, one in phase and the other in quadrature. An ideal condenser in parallel with the machine will take a current with a phase $\frac{\pi}{2}$ in advance of the applied voltage and its capacitance may be selected so that this current neutralizes the lagging component of the current taken by the machine. This will relieve the mains which will only have to supply the in-phase component.]

16·30. If two coils with self inductance L_1 and L_2 respectively, and negligible resistance, are arranged in parallel, and if M is the mutual inductance of the system, show that its equivalent self inductance is $\dfrac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$.

16·31. Two low-resistance coils each with a self inductance of 0·04 H. are wound on a common core, the coefficient of coupling being ± 1. If the coils are connected in series show that the effective self inductance is either 0·16 H. or zero. When theo oils are connected in parallel prove that the self inductance of the system is 0·04 H. or zero.

16·32. A condenser of capacitance C is shunted by a coil of self inductance L and negligible resistance. In series with this combination is a resistor, R, and the whole is fed from an a.c. supply, $v = \hat{V} \cos \omega t$. Prove that the periodic part of the current supplied is

$$\frac{\hat{V}(1 - \omega^2 LC) \cos (\omega t - \phi)}{[\omega^2 L^2 + R^2(1 - \omega^2 LC)^2]^{0.5}},$$

where $\tan \phi = \dfrac{\omega L}{R(1 - \omega^2 LC)}$.

16·33. If an e.m.f. $V_0 + \hat{V} \cos \omega t$ is impressed upon an inductive coil, R, L, prove that the mean power in the coil is

$$\frac{V_0^2}{R} + \frac{1}{2} \frac{(\hat{V})^2}{\sqrt{R^2 + \omega^2 L^2}} \cos \phi,$$

where $\tan \phi = \omega L R^{-1}$, and hence equal to

$$\frac{1}{Z}\left[\frac{V_0^2}{\cos \phi} + \tfrac{1}{2}(\hat{V})^2 \cos \phi\right],$$

where Z is the impedance of the coil.

16·34. A condenser C, a resistance R and a self inductance L occupy the arms AB, BC and CA respectively of a delta network. Show that the impedance operators for the equivalent star network have values $Z_a = \frac{L}{CZ}$, $Z_b = \frac{R}{j\omega CZ}$ and $Z_c = \frac{j\omega LR}{Z}$, where $Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$.

16·35. The arms OA, OB and OC of a T-network consist of a self inductance L_1, a self inductance L_2 and a resistance R respectively. Show that the impedance operators for the equivalent delta network are given by $Z_1 = R\left(1 + \frac{L_2}{L_1}\right) + j\omega L_2$, $Z_2 = R\left(1 + \frac{L_1}{L_2}\right) + j\omega L_1$, and $Z_3 = -\frac{\omega^2 L_1 L_2}{R} + j\omega(L_1 + L_2)$.

CHAPTER XVII

ALTERNATING CURRENT BRIDGES AND THE MEASUREMENT OF SOME CIRCUIT PARAMETERS

The four-branch impedance network.—The theory of the Wheatstone bridge for the comparison of resistances, using direct currents, has already been discussed; the theory of a.c. bridges is perhaps even more important, for such bridges enable accurate comparisons of almost all types of impedance to be made. A typical impedance bridge is shown, symbolically, in Fig. 17·01.

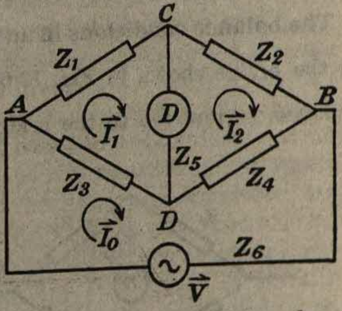


FIG. 17·01.—A four-branch impedance network.

The arms are composed of impedances and the cell and galvanometer of the familiar d.c. bridge are replaced by an a.c. generator and a suitable detector. This may be a vibration galvanometer or telephone; sometimes a transformer or amplifier is used to connect the detector to the bridge so that the sensitivity may be increased.

When the bridge is out of balance let $\vec{I}_1$, $\vec{I}_2$ and $\vec{I}_0$ be the vectors for the cyclic currents in the loops indicated. Then $\vec{I}$, the current vector for the current through the detector, is given by

$$\vec{I} = \vec{I}_1 - \vec{I}_2,$$

and, as on p. 163 for a z.f. bridge, it may be shown that

$$\vec{I} = \frac{\mathbf{Z}_1\mathbf{Z}_4 - \mathbf{Z}_2\mathbf{Z}_3}{\Delta}\vec{V},$$

where $\Delta = \begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_5, & -\mathbf{Z}_5, & -\mathbf{Z}_3 \\ -\mathbf{Z}_5, & \mathbf{Z}_2 + \mathbf{Z}_4 + \mathbf{Z}_5, & -\mathbf{Z} \\ -\mathbf{Z}_3, & -\mathbf{Z}_4, & \mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_6 \end{vmatrix},$

and the applied voltage is $v = \vec{V}\cdot\mathfrak{p}$.

If the bridge is balanced, $\vec{I} = 0$, i.e.

$$\mathbf{Z}_1\mathbf{Z}_4 = \mathbf{Z}_2\mathbf{Z}_3.$$

This symbolic condition for the bridge to be balanced may be written in the form

$$Z_1Z_4 \exp j(\phi_1 + \phi_4) = Z_2Z_3 \exp j(\phi_2 + \phi_3).$$

Now when two vectors are equal, they are equal in magnitude and coincident in phase, so that

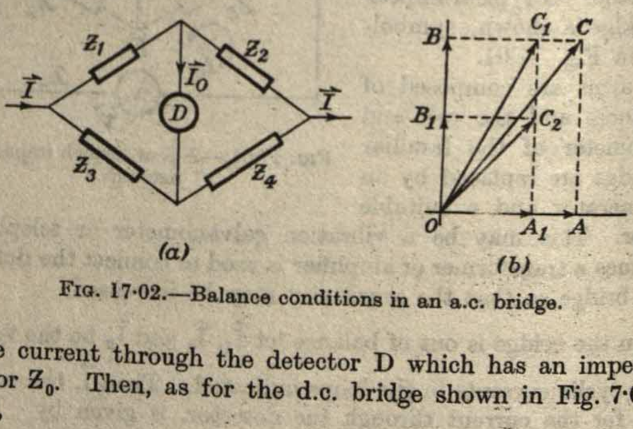
$$Z_1Z_4 = Z_2Z_3,$$

and

$$\phi_1 + \phi_4 = \phi_2 + \phi_3.$$

These are the conditions for the bridge to be balanced, and when they are fulfilled there is zero current in the detector at all times.

The balance conditions in an a.c. bridge.—For the impedances in the bridge shown in Fig. 17·02 (*a*) let $\vec{I}$ be the vector for the constant current fed to the bridge. Let $\vec{I}_0$ be the current vector for the current through the detector D which has an impedance operator $\mathbf{Z}_0$. Then, as for the d.c. bridge shown in Fig. 7·04 (*b*), p. 164,



(*a*) (*b*)

FIG. 17·02.—Balance conditions in an a.c. bridge.

$$\vec{I}_0 = \frac{\mathbf{Z}_1\mathbf{Z}_4 - \mathbf{Z}_2\mathbf{Z}_3}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3, & \mathbf{Z}_0 \\ \mathbf{Z}_2 + \mathbf{Z}_4, & -\mathbf{Z}_2 - \mathbf{Z}_4 - \mathbf{Z}_0 \end{vmatrix}} \vec{I}$$

$$= \frac{(\mathbf{Z}_1\mathbf{Z}_4 - \mathbf{Z}_2\mathbf{Z}_3)}{\Delta} . \vec{I} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where Δ represents the denominator in the previous fraction. The expression for Δ is very cumbersome, but when the bridge is nearly balanced we may regard Δ as a constant and, writing each $\mathbf{Z}$ in the form $R + jX$, expand $Z_1Z_4 - Z_2Z_3$. Thus

$$\vec{I_0} = \frac{\vec{I}}{\Delta}\cdot\{(R_1 + jX_1)(R_4 + jX_4) - (R_2 + jX_2)(R_3 + jX_3)\}$$

$$= \frac{\vec{I}}{\Delta}[(R_1R_4 - X_1X_4 - R_2R_3 + X_2X_3) + j(X_1R_4 + X_4R_1 - X_2R_3 - X_3R_2)] \qquad \text{(ii).}$$

Hence $\vec{I_0}\cdot\vec{v}$ may be regarded as the sum of two currents which are in phase quadrature. The phases of these currents relative to the applied current are complicated since they involve Δ, but here they are not required.

For most a.c. bridges the two balance conditions are contained separately in the two components on the r.h.s. of equation (ii); each component must vanish. In well-designed bridges, such as Owen's, Schering's etc., the contents of the brackets in equation (ii) can be made zero independently and the effect of successive adjustments to secure balance may be examined graphically. Thus if OA and OB, Fig. 17·02 (*b*), represent the two small currents, which are in quadrature when the bridge is nearly balanced, the resultant current which determines the width of the band in a vibration galvanometer or the intensity of the sound in the telephones, is represented by OC, the diagonal of the rectangle OACB. If OA is reduced to OA_1, the resultant falls to OC_1, which is less than OC; if OB is now brought down to OB_1, the resultant falls still further to OC_2. In practice a few alternate adjustments, made in this way, are sufficient to secure zero current through the detector. An illustration and extension of the meaning of this analysis will be given later, cf. p. 601.

de Sauty's bridge for the comparison of capacitances.—This bridge in its most elementary form is shown in Fig. 17·03 (*a*). When the bridge is balanced, let $\vec{I_1}$ and $\vec{I_2}$ be the vectors for the currents in the branches AHB and AKB respectively. The conditions for balance are given by

$$\frac{\dfrac{1}{j\omega C_1}}{\dfrac{1}{j\omega C_3}} = \frac{R_2}{R_4},$$

$$\text{i.e.} \qquad \frac{R_2}{R_4} = \frac{C_3}{C_1},$$

so that there is only one condition of balance.

If ϕ_1 and ϕ_2 are the phase differences for the currents in and AKB when the bridge is balanced, we have

$$\tan\phi_1 = \frac{1}{\omega C_1 R_2}, \quad \text{and} \quad \tan\phi_2 = \frac{1}{\omega C_3 R_4}.$$

Since $C_1R_2 = C_3R_4$, it follows at once that $\phi_1 = \phi_2$.

The vector diagram for the bridge is interesting and instructive. It is shown in two parts in Fig. 17·03 (*b*).

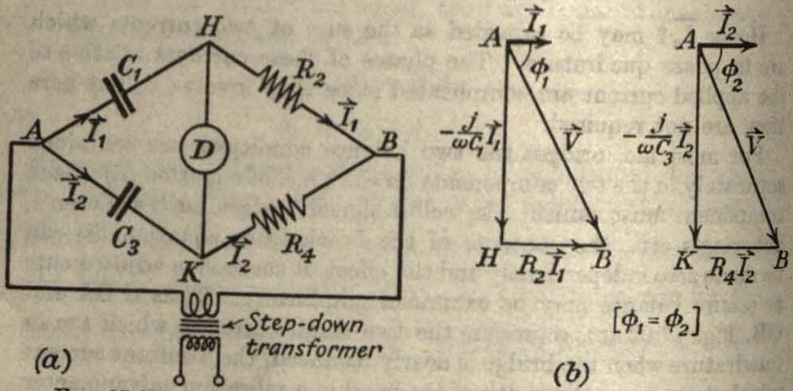


FIG. 17·03.—de Sauty's bridge and its associated vector diagram.

The vector potential difference between A and B is the same whether we go via H or K. Hence $R_2\vec{I}_1 = R_4\vec{I}_2$ and $\frac{\vec{I}_1}{C_1} = \frac{\vec{I}_2}{C_3}$; these equations give $\frac{R_2}{R_4} = \frac{C_3}{C_1}$.

de Sauty's bridge with imperfect condensers.—An imperfect condenser may be represented by a resistance, *r*, in series with a pure capacitance C. If each condenser is imperfect, the de Sauty bridge is as shown in Fig. 17·04 (*a*). The conditions for balance are

$$\frac{r_1 + \frac{1}{j\omega C_1}}{r_3 + \frac{1}{j\omega C_3}} = \frac{R_2}{R_4},$$

i.e.

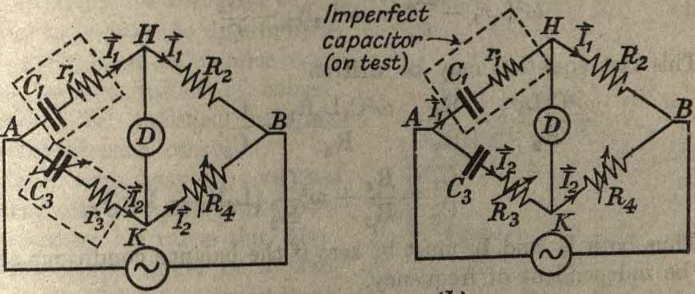
$$R_4 r_1 + \frac{R_4}{j\omega C_1} = R_2 r_3 + \frac{R_2}{j\omega C_3}.$$

Equating (*a*) real parts and (*b*) imaginary parts, we have, as the required conditions

$$\frac{r_1}{r_3} = \frac{R_2}{R_4} = \frac{C_3}{C_1}.$$

Such a bridge is difficult to balance.

Wien's series-resistance bridge.—If the imperfect capacitor in the arm AK of de Sauty's bridge with imperfect condensers is replaced by a capacitor C_3 with a low power factor, e.g. a high-class mica condenser, arranged in series with a known adjustable resistor R_3, we have, cf. Fig. 17·04 (*b*), WIEN's series-resistance bridge for determining the capacitance and power factor of a capacitor, i.e. of the one shown in the arm AH. The balance conditions require



(*a*) (*b*)

FIG. 17·04 (*a*).—de Sauty's bridge with imperfect condensers. (*b*) Wien's series-resistance bridge.

$$\frac{r_1 + \dfrac{1}{j\omega C_1}}{R_2} = \frac{R_3 + \dfrac{1}{j\omega C_3}}{R_4}.$$

Thus $$\frac{r_1}{R_2} = \frac{R_3}{R_4}, \quad \text{and} \quad C_1 = \frac{C_3 R_4}{R_2}.$$

These conditions may be satisfied (though not quite independently) by varying in turn R_3 and R_4.

The power factor required is

$$\omega C_1 r_1 = \omega \frac{C_3 R_4}{R_2} \cdot \frac{R_3 R_2}{R_4}$$

$$= \omega C_3 R_3,$$

i.e. the power factor for the components in the arm AK is eq to that of the capacitor on test.

If the resistors R_2 and R_4 have (residual) self inductances L_2 and L_4, the balance conditions becom

$$\frac{r_1 + \dfrac{1}{j\omega C_1}}{R_2 + j\omega L_2} = \frac{R_3 + \dfrac{1}{j\omega C_3}}{R_4 + j\omega L_4}.$$

Cross-multiplying and proceeding in the usual way, we find as the conditions for balance

$$\frac{L_2}{C_3} = \frac{L_4}{C_1} + r_1R_4 - R_2R_3,$$

and
$$\omega^2L_4r_1 - \frac{R_4}{C_1} = \omega^2L_2R_3 - \frac{R_2}{C_3}.$$

This last equation may be written

$$\frac{\omega^2C_1L_4r_1}{R_2} - \frac{R_4}{R_2} = \frac{\omega^2C_1L_2R_3}{R_2} - \frac{C_1}{C_3},$$

or
$$\frac{C_1}{C_3} = \frac{R_4}{R_2} + \omega^2\frac{C_1}{R_2}\{L_2R_3 - L_4r_1\}.$$

Thus both L_2 and L_4 must be zero if the balance conditions are to be independent of frequency.

Maxwell's bridge for the comparison of self inductances.— Maxwell's bridge for the comparison of self inductances is arranged as shown in Fig. 17·05. The impedance operators for the two inductance arms are $R_1 + j\omega L_1$ and $R_2 + j\omega L_2$, so that the bridge is balanced provided

$$\frac{R_1 + j\omega L_1}{R_3} = \frac{R_2 + j\omega L_2}{R_4},$$

i.e.
$$R_1R_4 + j\omega L_1R_4 = R_2R_3 + j\omega L_2R_3.$$

Equating (*a*) real parts and (*b*) imaginary parts, the *two* conditions for balance are

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}, \quad \text{and} \quad \frac{L_1}{L_2} = \frac{R_3}{R_4}.$$

At this stage it is convenient to examine the effect of mutual inductance, M, between the coils L_1 and L_2. If $\vec{I}_1$ and $\vec{I}_2$ are the current vectors for the arms AHB and AKB of the balanced bridge, we have

$$(R_1 + j\omega L_1 + j\omega M)\vec{I}_1 = R_3\vec{I}_2,$$

and
$$(R_2 + j\omega L_2 + j\omega M)\vec{I}_1 = R_4\vec{I}_2.$$

Eliminating $\vec{I}_1$ and $\vec{I}_2$ from these equations, we have

$$R_1R_4 + j\omega R_4(L_1 + M) = R_2R_3 + j\omega R_3(L_2 + M),$$

so that the required conditions are

$$R_1R_4 = R_2R_3,$$

and
$$\frac{L_1 + M}{R_4} = \frac{L_2 + M}{R_3}.$$

If R_3 is made equal to R_4 and L_2 is an adjustable known self inductance, then $L_1 = L_2$ and the balance point is unaffected by mutual inductance between the coils; in general, however, care must be taken to arrange the coils L_1 and L_2 so that their mutual inductance is negligible.

Although the theory of the bridge is simple, the bridge is seldom used in practice to determine the self inductance of a coil in terms of a standard inductance since the two conditions of balance cannot be satisfied independently of each other.

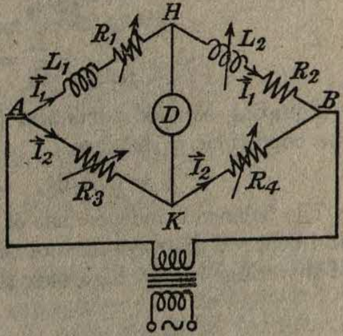


FIG. 17·05.—Maxwell's bridge for the comparison of self-inductances.

If it were easy to construct a variable self inductance over a considerable range then Maxwell's bridge would be a practical possibility. Consequently inductances are generally measured in terms of resistance and capacitance; one of the best known bridges is due to OWEN (1914).

Let us consider the three electrical quantities, resistance, R, self inductance, L, and capacitance, C. The comparison of any one of these with a standard of the same kind would be a very simple procedure if each of the four arms of the bridge contained one electrical quantity only. The current in any arm would be exactly in phase or in quadrature with the voltage across that arm. Only one extra condition would have to be satisfied to secure zero current in the detector at any frequency. This simplicity is, of course, in effect fully attained in the determination of resistance when d.c. is used, as any inductance present is inoperative when the current is steady. We have already seen that in the comparison of two capacitances by means of de Sauty's bridge with a.c. the condition is very nearly complied with, though for accuracy it is necessary to treat an absorptive condenser as a pure capacitance in series with a small resistance.

When we come to the measurement of self inductance, however, we have in every case to reckon with the associated resistance of the coil under test. The electrically simple bridge is no longer possible; the simple relation between the phases of the voltage and current disappear and two conditions of balance appear in place of one. This limits the choice of the method, for, to be practicable, the bridge must be such that the two conditions of balance may be satisfied without interfering one with the other.

Owen's bridge for the measurement of self inductance.—In this bridge * these independent conditions are readily obtained and it is suitable for the measurement of self inductances from a few micro-henries upwards. The bridge is arranged as in Fig. 17·06. When the bridge is balanced

$$\frac{R_1 + j\omega L}{R_3 + \dfrac{1}{j\omega C_3}} = \frac{R_2}{\dfrac{1}{j\omega C_4}}.$$

$$\therefore \quad \frac{R_1}{j\omega C_4} + \frac{L}{C_4} = R_2R_3 + \frac{R_2}{j\omega C_3}.$$

Equating (*a*) real parts and (*b*) imaginary parts we obtain, as the conditions of balance,

$$L = C_4R_2R_3, \quad \text{and} \quad R_1C_3 = R_2C_4.$$

The balance is independent of the frequency of the current used, and hence of the wave-form of the applied voltage. The sensitivity of the bridge is very high over the range 100–1000 cycle.sec.$^{-1}$.

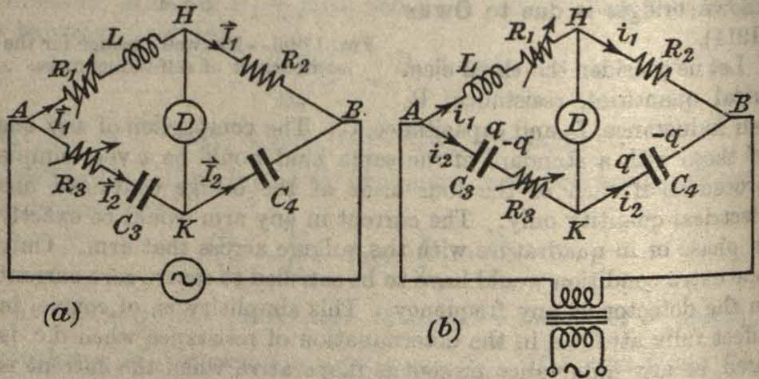


FIG. 17·06.—Owen's bridge for the measurement of self inductance.

C_4 should be a standard mica-condenser and it is advisable to use a mica-condenser for C_3 although its value is not used in the calculation. If $C_3 = C_4$ $(= \frac{1}{3}\ \mu F.) = C$ (say), the balance conditions become

$$L = CR_2R_3, \quad \text{and} \quad R_1 = R_2.$$

Owen has shown that even with limited apparatus the range of this bridge is very wide. Thus with $C_1 = C_2 = 0{\cdot}3\ \mu F.$ and values of R_2 from 1 to 200 Ω, self inductances from 2 $\mu H.$ to 0·5 H. can be determined with accuracy. The bridge is extremely convenient and possesses the great practical advantage that a balance is obtained by adjusting resistances only. It should also be noted that the only inductance used is the one on test and therefore all mutual reaction between neighbouring inductances is avoided.

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* *Proc. Phy. Soc.*, **27**, 39, 1914.

At this stage it is convenient to examine the full meaning of the balance conditions derived on p. 595. Substituting the values

$$Z_1 = R_1 + j\omega L, \quad Z_2 = R_2, \quad Z_3 = R_3 + \frac{1}{j\omega C_3} \text{ and } Z_4 = \frac{1}{j\omega C_4},$$

in equation (i), p. 594, we find for an Owen bridge,

$$\vec{I}_0 = \frac{\vec{I}}{\Delta}\left[\left(\frac{L}{C_4} - R_2R_3\right) + j\left(\frac{R_2}{\omega C_3} - \frac{R_1}{\omega C_4}\right)\right],$$

which gives the usual conditions for balance, viz.

$$L = C_4R_2R_3 \quad \text{and} \quad R_2C_4 = R_1C_3.$$

These conditions are independent of the frequency but until the bridge is actually balanced, one component of the current through the detector depends upon ω; the other does not. In fact, in all bridges in which the final balance is independent of frequency, one of the above-mentioned quadrature currents will be inversely proportional to the angular frequency, while the other is not.

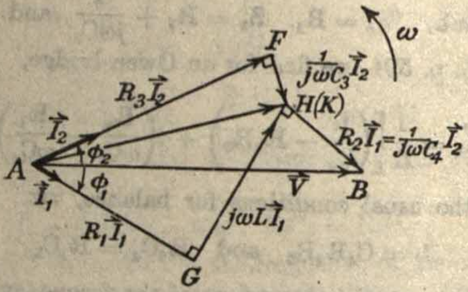
Some further remarks on Owen's bridge.—In Fig. 17·07 (*a*), the vector $\vec{V}$ is represented by AB, i.e. it is the vector representing the voltage across the bridge. Let $\vec{I}_1$ and $\vec{I}_2$ be the current vectors for the arms AHB and AKB respectively, when the bridge is balanced; $\vec{I}_1$ will lag behind $\vec{V}$ while $\vec{I}_2$ will lead so that $\vec{I}_1$ and $\vec{I}_2$ may be drawn. If $\overrightarrow{AF} = R_3\vec{I}_2$ and $\overrightarrow{FH}$ is normal to $\overrightarrow{AF}$ and of length $\frac{1}{\omega C_3}\vec{I}_2$, then $\overrightarrow{AH}$ gives the potential drop across AH; it is equal to that across AK, viz. $\overrightarrow{AG} + \overrightarrow{GK}$, where $\overrightarrow{AG} = R_1\vec{I}_1$ and $\overrightarrow{GK}(= \overrightarrow{GH}) = j\omega L\vec{I}_1$. Now $\overrightarrow{HB}$ is $R_2\vec{I}_1$ and since it is also $\frac{1}{j\omega C_4}\vec{I}_2$, it must be normal to $\vec{I}_2$; thus FHB is a straight line, and the currents $\vec{I}_1$ and $\vec{I}_2$ are therefore in quadrature. This fact having been discovered, the correct vector diagram is as shown in Fig. 17·07 (*b*).

The fact that the vector diagram for an Owen bridge has enabled us to discover that the two currents under discussion are in quadrature when the bridge is balanced is so surprising that it is wor while for us to establish it analytically. We have

$$\tan\phi_1 = \frac{\omega L}{R_1 + R_2}, \quad \text{and} \quad \tan\phi_2 = \frac{1}{\omega CR_3},$$

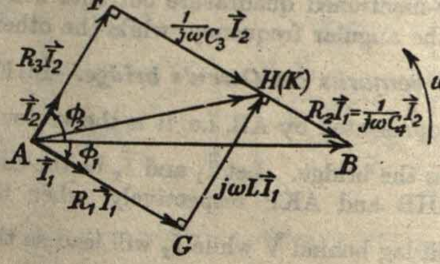
where

$$\frac{1}{C} = \frac{1}{C_3} + \frac{1}{C_4}.$$



'Rough' attempt to draw vector diagram

(a)



$[\phi_1 + \phi_2 = \tfrac{1}{2}\pi]$

(b)

FIG. 17·07.—A vector diagram for Owen's bridge.

Thus

$$\begin{aligned}\tan\phi_1 \tan\phi_2 &= \frac{L}{CR_3(R_1 + R_2)}\\ &= C_4R_2R_3 \cdot \left(\frac{C_3 + C_4}{C_3C_4}\right) \cdot \frac{1}{R_3(R_1 + R_2)}\\ &= 1, \qquad [\because\ R_1C_3 = R_2C_4]\end{aligned}$$

which establishes the proposition.

Maxwell's L/C bridge.—One of the earliest forms of bridge for the measurement of self inductance is due to MAXWELL; with its aid a self inductance is measured in terms of a capacitance and two standard resistances. Originally (1873) the method was a ballistic one but in 1891 WIEN made use of the bridge with alternating current. The circuit arrangement is shown in Fig. 17·08 and needs no particular comment except to say that for accurate work the resistances used must have specially low residual inductances. This latter remark applies to any a.c. bridge for the measurement of inductance.

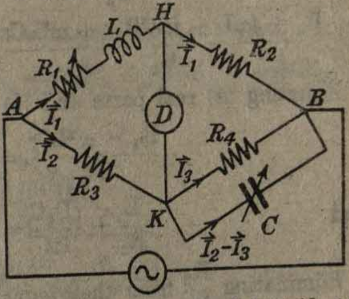


FIG. 17·08.—Maxwell's L/C bridge.

When the bridge is balanced, we have

$$\frac{R_1 + j\omega L}{R_3} = \frac{R_2}{\left[\dfrac{R_4}{1 + j\omega CR_4}\right]} = \frac{R_2}{R_4}[1 + j\omega CR_4].$$

Equating real and imaginary parts in turn, we have

$$\frac{R_1}{R_3} = \frac{R_2}{R_4}, \quad \text{and} \quad L = CR_2R_3.$$

If the two variable quantities are resistances, Maxwell's bridge does not meet the requirement that the two conditions for balance shall be mutually independent. For many years there was a tendency, on this account, not to use this bridge. With the advent, however, of good quality mica condensers arranged in decade and variable in steps of 0·001 μF. from zero to 1·11 μF. and a 0·001 μF. air-condenser which could be arranged in parallel with the decade condensers, the capacitance C became continuously variable and Maxwell's bridge became a practical method for the measurement of self inductance.

To examine the effect of using an imperfect capacitor across R_4 let us assume that such a defect may be represented by a resistance r in series with C. Since the impedance operator for the arm KB is then given by

$$\frac{1}{Z_4} = \frac{1}{R_4} + \frac{j\omega C}{1 + j\omega Cr},$$

the conditions for balance are given by

$$\frac{R_1 + j\omega L}{R_3} = R_2\left(\frac{1}{Z_4}\right) = R_2\left[\frac{1}{R_4} + \frac{j\omega C}{1 + j\omega Cr}\right].$$

$$\therefore \quad \frac{(R_1 + j\omega L)(1 + j\omega Cr)}{R_3} = \frac{R_2}{R_4}(1 + j\omega Cr) + j\omega R_2 C.$$

$$\therefore \quad \frac{R_1 + j\omega L + j\omega CR_1 r - \omega^2 LCr}{R_3} = \frac{R_2}{R_4} + j\omega Cr\frac{R_2}{R_4} + j\omega R_2 C.$$

Equating (*a*) real parts and (*b*) imaginary parts, we have

$$\frac{R_1 - \omega^2 LCr}{R_3} = \frac{R_2}{R_4},$$

and

$$\frac{L}{R_3} + \frac{rR_1C}{R_3} = rC\frac{R_2}{R_4} + R_2C.$$

Eliminating $\frac{R_2}{R_4}$ from these equations we find

$$L = \frac{CR_2R_3}{1 + \omega^2C^2r^2}.$$

This expression shows that the correction term involves the power factor of the capacitor. For a high-class mica condenser the power factor is less than 10^{-4}, so that the correction is less than 1 in 10^8; this may be taken to be zero.

Anderson's bridge for the measurement of self inductance.—This bridge, which was first described in 1891, is not a true Wheatstone network but is a modification of the L/C bridge due to Maxwell. One of its advantages is that it enables the double balance to be obtained by adjusting resistances only, the standard condenser being invariable. Anderson's bridge, shown in Fig. 17·09 (*a*), is formed by placing the condenser C and a non-inductive resistance r in series, the combination being in parallel with R_4; the other components and the detector are placed as shown. When the bridge is balanced let the current vectors in the different parts of the bridge be as indicated. Then by Kirchhoff's laws, we have

$$(R_1 + j\omega L)\vec{I}_1 - r\vec{I}_3 - R_3\vec{I}_2 = 0, \qquad \text{[for loop (i)]}$$

$$R_2\vec{I}_1 - \frac{1}{j\omega C}\vec{I}_3 = 0, \qquad \text{[for loop (ii)]}$$

$$\left.\begin{aligned} r\vec{I}_3 + \frac{1}{j\omega C}\vec{I}_3 - R_4(\vec{I}_2 - \vec{I}_3) &= 0, \\ \text{or} \quad \left(r + \frac{1}{j\omega C} + R_4\right)\vec{I}_3 &= R_4\vec{I}_2, \end{aligned}\right\} \qquad \text{[for loop (iii)]}$$

Eliminating $\vec{I}_3$ from the second and last equations we get

$$\left(r + \frac{1}{j\omega C} + R_4\right)(R_2 j\omega C)\vec{I}_1 = R_4\vec{I}_2,$$

i.e.

$$(j\omega CrR_2 + R_2 + j\omega CR_2R_4)\vec{I}_1 = R_4\vec{I}_2.$$

Eliminating $\vec{I}_3$ from the first two equations we have

$$[(R_1 + j\omega L) - j\omega C r R_2]\vec{I}_1 = R_3\vec{I}_2,$$

Eliminating $\vec{I}_1$ and $\vec{I}_2$ from the equations thus obtained, we get

$$j\omega C r R_2 R_3 + R_2 R_3 + j\omega C R_2 R_3 R_4 = R_1 R_4 + j\omega L R_4 - j\omega r C R_2 R_4.$$

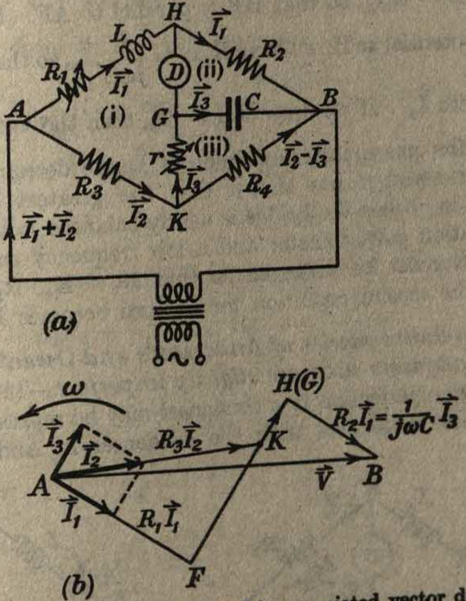


FIG. 17·09.—Anderson's L/C bridge and its associated vector diagram.

Equating (*a*) real parts and (*b*) imaginary parts we find

$$R_1 R_4 = R_2 R_3,$$

as the first condition of balance, and

$$L R_4 - r R_2 R_4 C = C r R_2 R_3 + C R_2 R_3 R_4,$$

which gives

$$L = C R_2\left[r\left(1 + \frac{R_3}{R_4}\right) + R_3\right],$$

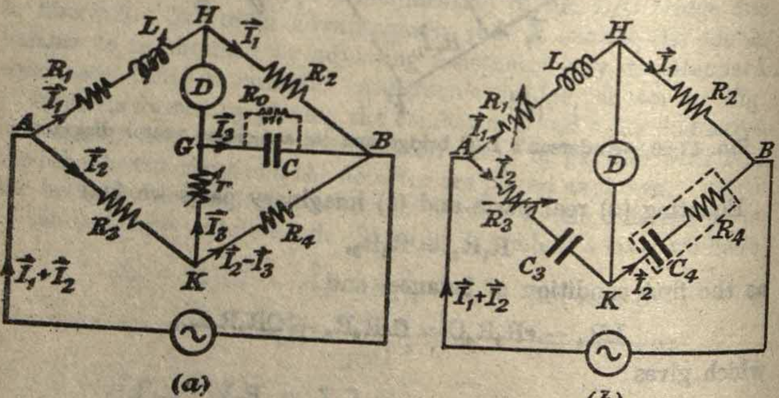
as the second condition. These conditions can be satisfied independently of each other, viz. by adjusting R_1 and r.

Originally (1891), Anderson's bridge was used with interrupted currents and a ballistic galvanometer; in 1898 ROWLAND used the bridge with alternating current, and in 1905 ROSA and GLOVER of the Bureau of Standards replaced the telephone detector by a vibration galvanometer so that with these modifications the bridge became a precision one for the measurement of self (and mutual, cf. pp. 234 and 612) inductance.

Fig. 17·09 (*b*) shows the vector diagram for a balanced Anderson bridge. In it AB is $\vec{V}$, the potential vector for the applied voltage. If $\vec{I}_1$ represents the current in AHB, then AF is $R_1\vec{I}_1$, FH is $j\omega L\vec{I}_1$ and $HB = R_2\vec{I}_1$, so that HB is parallel to AF. Since G is at the same potential as H, we have $R_2\vec{I}_1 = \frac{1}{j\omega C}\cdot\vec{I}_3$, so that $\vec{I}_3$ is in quadrature with $\vec{I}_1$. If we draw $KG = r\vec{I}_3$, then the vector AK is $R_3\vec{I}_2$; hence the phase and magnitude of $\vec{I}_2$ are determined.

Butterworth has shown that the greatest sensitivity to a change in r when the bridge is nearly balanced is attained by using a vibration galvanometer and a low frequency supply; also, the resistors must be arranged so that $R_1 = R_2$, $R_3 = R_4 = 0{\cdot}5R_1$. Then the second condition for balance becomes $L = 2CR_1^2$.

The relative merits of Anderson's and Owen's bridges when the condensers used are slightly imperfect.—It will be assumed that the imperfection in a condenser may be represented by a high resistance in parallel with the condenser in Anderson's bridge;



(*a*) (*b*)

Fig. 17·10.—Anderson's and Owen's bridges with imperfect condensers.

this is tantamount to saying that there is a small leak in the condenser. Let R_0 be the 'fictitious' resistance in parallel with the condenser C. We wish to see how this defect affects the balance of the bridge. The bridge is shown in Fig. 17·10 (*a*), and when the bridge is balanced let the current vectors be as indicated.

Then, in the usual way, we have

$$(R_1 + j\omega L)\vec{I}_1 - r\vec{I}_3 - R_3\vec{I}_2 = \vec{0}, \quad . \quad . \quad \text{(i)}$$

$$R_2\vec{I}_1 - \left(\frac{R_0}{1 + j\omega CR_0}\right)\vec{I}_3 = 0, \quad . \quad . \quad \text{(ii)}$$

and

$$\left[r + \frac{R_0}{1 + j\omega CR_0}\right]\vec{I}_3 - R_4(\vec{I}_2 - \vec{I}_3) = 0 \quad . \quad . \quad \text{(iii)}$$

Eliminating $\vec{I}_2$ from (i) and (iii) we have

$$\left[r + R_4 + \frac{R_0}{1 + j\omega CR_0}\right]\vec{I}_3 - R_4\left[\frac{(R_1 + j\omega L)\vec{I}_1 - r\vec{I}_3}{R_3}\right] = 0 \quad . \quad \text{(iv)}$$

Eliminating $\vec{I}_1$ and $\vec{I}_3$ from (ii) and (iv) we find

$$R_2\left(r + R_4 + R_4\frac{r}{R_3}\right)(1 + j\omega CR_0) + R_0R_2 = \frac{R_0R_4}{R_3}(R_1 + j\omega L).$$

Equating (*a*) real parts and (*b*) imaginary parts, we get

(*a*)
$$R_2r + R_2R_4 + \frac{R_2R_4r}{R_3} + R_0R_2 = \frac{R_0R_1R_4}{R_3},$$

which reduces to

$$R_1R_4 = R_2R_3 + \frac{R_2}{R_0}\{r(R_4 + R_3) + R_3R_4\},$$

and (*b*)
$$L = \frac{R_3C}{R_4}\left[R_2\left(r + R_4 + R_4\frac{r}{R_3}\right)\right]$$

$$= CR_2\left[r\left(1 + \frac{R_3}{R_4}\right) + R_3\right].$$

Thus the imperfection has no effect * on the measured value of the self inductance and only a small effect on the measured value of R_1.

When Owen's bridge is used with imperfect condensers any defect in C_3 is really included in R_3 if the defect is represented by a small resistance in series with the condenser; it is therefore only necessary to consider the effect of an imperfection in the condenser C_4. It will be represented by a small series resistance R_4, as shown in Fig. 17·10 (*b*). The conditions for balance are given by

$$\frac{R_1 + j\omega L}{R_2} = \frac{R_3 + \dfrac{1}{j\omega C_3}}{R_4 + \dfrac{1}{j\omega C_4}}.$$

$$\therefore\ R_1R_4 + j\omega LR_4 + \frac{L}{C_4} + \frac{R_1}{j\omega C_4} = R_2R_3 + \frac{R_2}{j\omega C_3}.$$

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* Only if C_2 is taken as C—cf. p. 562.

Equating (*a*) real parts and (*b*) imaginary parts, the conditions for balance are

$$R_1R_4 + \frac{L}{C_4} = R_2R_3,$$

and

$$\omega LR_4 - \frac{R_1}{\omega C_4} + \frac{R_2}{\omega C_3} = 0.$$

[It should be noted that the first of these conditions does not involve C_3.]

Thus both conditions for balance are affected and the second condition involves ω, so that the source must be free from harmonics. In this respect, the Owen bridge is inferior to the Anderson bridge.

If the defect in C_4 is represented by a large resistance R_p in parallel with the condenser, it is left as an exercise to the reader to show that the conditions for balance in an Owen bridge are

$$R_1 = \frac{R_2R_3}{R_p} + \frac{C_4R_2}{C_3},$$

and

$$L = R_2R_3C_4 - \frac{R_2}{\omega^2C_3R_p}.$$

A null-dynamometer method for the measurement of a large self inductance.—This method is due to PEGLER (1935) and involves the use of an electrodynamometer. The fixed coil system, F.C., Fig. 17·11, consists of two parallel circular coils each with about 500 turns of fine insulated copper wire and a moving coil, M.C., of some 200 turns of similar wire; it is suspended by means of a phosphor bronze strip. This serves to carry current to M.C. and also as a unifilar elastic control with a small torsional constant; the current leaves via a loosely coiled spring below the moving coil. The period of swing is about 6 sec. and a damping vane of mica attached between the moving coil and the loosely coiled spring causes its movement to be nearly critically damped. A concave mirror attached to M.C. is used to enhance the sensitivity.

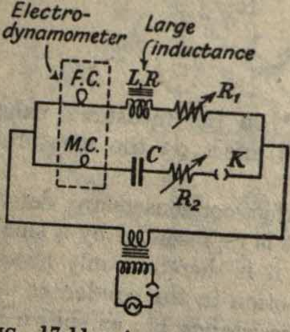


FIG. 17·11.—A null-dynamometer method for measuring the self inductance and effective resistance of an iron-cored choke.

The coil on test, e.g. an iron-cored choke, is denoted by L,R and is arranged in series with the fixed coils of the electrodynamometer and an adjustable resistor R_1 with a range 0–10,000 Ω. A capacitor C, 1 μF, a resistor R_2, similar to R_1, and a key K are placed in series with the moving coil M.C.; this branch of the circuit is in parallel with that containing the choke. This arrangement is chosen since, when the system is fed from an a.c. supply, the current in the capacitor branch is small and therefore should be passed through the moving coil of the electrodynamometer.

The electrodynamometer is set up with the plane of its moving coil approximately normal to that of the fixed coils. The terminals of the moving coil are short-circuited and alternating current is passed through the fixed coils. The suspension is adjusted until there is no change in the position of the moving coil when the current through the fixed coils is switched on or off; the mutual inductance between the coils is then zero. The connexion across the moving coil is removed and the instrument is ready for use.

Now the phase of the current in the inductive branch lags behind that of the applied voltage by $\tan^{-1}\left(\frac{\omega L}{R + R_1}\right)$, while that in the capacitive branch leads by $\tan^{-1}\left(\frac{1}{\omega C R_2}\right)$; in writing down these expressions the small resistances and reactances of the coils in the electrodynamometer have been neglected. When the currents in the two branches are in quadrature the mean torque on the moving coil is zero, cf. p. 527, and it is not deflected from its position of static equilibrium. The necessary condition is

$$\left(\frac{\omega L}{R + R_1}\right) \cdot \frac{1}{\omega C R_2} = 1,$$

since $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, and the denominator of this fraction is zero if $A + B = \frac{1}{2}\pi$. The above condition may be written

$$L = C(R + R_1)R_2.$$

In using this method, R_1 or R_2, or both, are adjusted to such values R_1' and R_2', say, that the deflexion is zero; a second balance is then obtained using different values R_1 and R_2, say R_1'' and R_2''. Then

$$L = C(R + R_1')R_2' = C(R + R_1'')R_2''.$$

From these equations we find

$$L = \frac{C(R_1' - R_2')R_2'R_2''}{R_2'' - R_2'},$$

and

$$R = \frac{R_1'R_2' - R_1''R_2''}{R_2'' - R_2'}.$$

Experiment shows that the value of R may be several times the z.f. value for the resistance of the coil.

The modified Carey Foster method for the determination of a mutual inductance in terms of resistance and capacitance.—[More often this bridge is used to measure a capacitance in terms of M and R.] The original Carey Foster bridge for this purpose was used with direct current and a ballistic galvanometer.

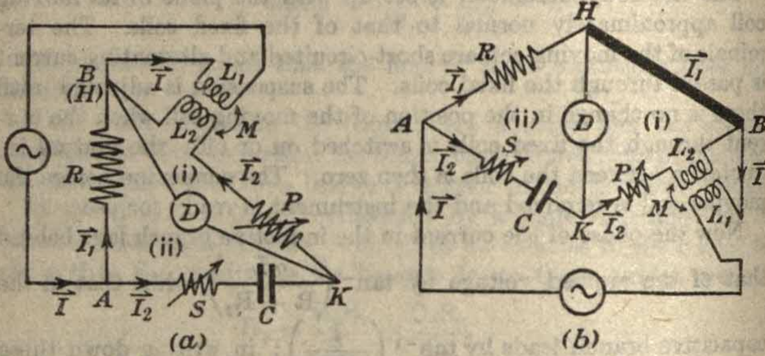


FIG. 17·12.—Campbell's bridge for the measurement of capacitance [or mutual inductance].

When alternating current and a suitable detector are used the simple bridge can only be balanced if one particular condition is satisfied. HEYDWEILLER (1894) and CAMPBELL (1907) modified the bridge by placing an adjustable resistance in series with the condenser. The bridge is generally known as the Campbell bridge. The circuit is shown in Fig. 17·12 (*a*) and the conditions for balance may be obtained as follows. Applying Kirchhoff's laws to the loops (i) and (ii) we have

$$\mathrm{P}\vec{\mathrm{I}}_2 + j\omega\mathrm{L}_2\vec{\mathrm{I}}_2 + j\omega\mathrm{M}\vec{\mathrm{I}} = 0, \quad . \quad . \quad . \quad \text{(i)}$$

and

$$\mathrm{R}\vec{\mathrm{I}}_1 - \left(\mathrm{S} + \frac{1}{j\omega\mathrm{C}}\right)\vec{\mathrm{I}}_2 = 0. \quad . \quad . \quad . \quad \text{(ii)}$$

Since $\vec{\mathrm{I}} = \vec{\mathrm{I}}_1 + \vec{\mathrm{I}}_2$, the first of these equations becomes

$$(\mathrm{P} + j\omega\mathrm{L}_2 + j\omega\mathrm{M})\vec{\mathrm{I}}_2 = -j\omega\mathrm{M}\vec{\mathrm{I}}_1 \quad \text{(iii)}$$

Eliminating $\vec{\mathrm{I}}_1$ and $\vec{\mathrm{I}}_2$ from equations (ii) and (iii), we have

$$\mathrm{R}(\mathrm{P} + j\omega\mathrm{L}_2 + j\omega\mathrm{M}) = -j\omega\mathrm{M}\left(\mathrm{S} + \frac{1}{j\omega\mathrm{C}}\right)$$

$$= -j\omega\mathrm{MS} - \frac{\mathrm{M}}{\mathrm{C}}.$$

Equating in turn the real and imaginary parts, we have

$$\mathrm{RP} = -\frac{\mathrm{M}}{\mathrm{C}}, \text{ and } (\mathrm{L}_2 + \mathrm{M})\mathrm{R} = -\mathrm{MS}.$$

These conditions require that M shall be negative, i.e. $M = -|M|$. Then

$$|M| = CRP, \quad \text{and} \quad L_2 = |M|\left[1 + \frac{S}{R}\right].$$

[Adjust P and S, keeping R constant, to obtain balance, for the two conditions are then mutually independent.]

It must be noted that P is the total resistance of the arm containing L_2, and also a necessary condition for balance is that $L_2 > |M|$. Hence of the two coils forming the mutual inductance that coil whose self inductance exceeds the value of $|M|$ must be placed in the secondary circuit. Of course both L_1 and L_2 may be greater than $|M|$ and in such an instance either coil may be used in the secondary circuit. If the condenser has a low power factor then S is known and consequently L may be determined.

If the condenser is imperfect, S includes 'r', the 'imperfection' of the condenser. If such be the case let $S = S_0 + r$, where S_0 is known. Then

$$L_2 = |M|\left[1 + \frac{S_0 + r}{R}\right],$$

so that

$$S_0 = R\left(\frac{L_2}{|M|} - 1\right) - r.$$

Thus by plotting S_0 against R we get values for $\frac{L_2}{|M|} - 1$ and r. When r is known the power factor for the condenser, which is ωCr, may be calculated.

Since, in general, P includes the resistance of the coil L_2, for P we ought to write $P_0 + r'$, where P_0 is known. Then

$$P_0 = \frac{1}{R}\frac{|M|}{C} - r',$$

which suggests that P_0 should be plotted against $\frac{1}{R}$ in order to find $|M|$ if C is known, or to find C if $|M|$ is a known mutual inductance. The same graph gives r', if this is required.

If Figs. 17·12 (*a*) and (*b*) are compared it will be noticed that in Fig. 17·12 (*b*) the Campbell bridge has been made to resemble a four-branch network by adding a conductor of zero resistance between the points B and H; this latter form of diagram is preferable for some purposes, cf. p. 619. The conditions for balance are obtained by using the facts that

$$R\vec{I}_1 = \left(S + \frac{1}{j\omega C}\right)\vec{I}_2, \quad \text{and} \quad (P + j\omega L_2)\vec{I}_2 + j\omega M(\vec{I}_1 + \vec{I}_2) = 0.$$

At this stage it is interesting to examine the effect on the balance conditions if the arm HB in Fig. 17·12 (*b*) has a small but finite resistance ΔR. Using Kirchhoff's laws we have

$$\mathrm{P}\vec{\mathrm{I}}_2 + j\omega\mathrm{L}\vec{\mathrm{I}}_2 + j\omega\mathrm{M}(\vec{\mathrm{I}}_1 + \vec{\mathrm{I}}_2) - \Delta\mathrm{R}.\vec{\mathrm{I}}_1 = 0,$$

and

$$\mathrm{R}\vec{\mathrm{I}}_1 - \left(\mathrm{S} + \frac{1}{j\omega\mathrm{C}}\right)\vec{\mathrm{I}}_2 = 0.$$

After some reduction these equations give

$$(\Delta\mathrm{R} - j\omega\mathrm{M})\left[\mathrm{S} + \frac{1}{j\omega\mathrm{C}}\right] = \mathrm{R}(\mathrm{P} + j\omega\mathrm{L}_2 + j\omega\mathrm{M}).$$

$$\therefore\ \mathrm{S}.\Delta\mathrm{R} - \frac{\mathrm{M}}{\mathrm{C}} = \mathrm{RP},$$

and

$$\mathrm{L}_2 = -\mathrm{M}\left[1 + \frac{\mathrm{S}}{\mathrm{R}}\right] - \frac{\Delta\mathrm{R}}{\omega^2\mathrm{CR}}.$$

These equations show that the balance for the bridge is independent of frequency only if ΔR = 0; this condition must be satisfied if the bridge is to be satisfactory.

The use of a self inductance bridge for determining a mutual inductance.—If two coils with self inductances L_1 and L_2 are connected in series so that M is their mutual inductance, then the self inductance of the combination is $\mathrm{L}_1 + \mathrm{L}_2 \pm 2\,|\,\mathrm{M}\,|$, the sign before M depending upon the manner in which the coils are connected together, cf. p. 234. If we write

$$\mathrm{L}' = \mathrm{L}_1 + \mathrm{L}_2 + 2\,|\,\mathrm{M}\,|,$$

and

$$\mathrm{L}'' = \mathrm{L}_1 + \mathrm{L}_2 - 2\,|\,\mathrm{M}\,|,$$

it follows that

$$|\,\mathrm{M}\,| = \tfrac{1}{4}(\mathrm{L}' - \mathrm{L}'').$$

Hence if L′ and L″ are measured by any self inductance bridge, a numerical value for the mutual inductance of the coils may be found. Of course the two coils must remain fixed with respect to each other so that their mutual inductance remains constant.

An inductometer method for comparing two mutual inductances.—The precise determination of a mutual inductance in terms of a standard which is smaller than itself is a somewhat difficult operation since a balance of self inductances is also required. Steps must also be taken to compensate for the impurities present in each mutual inductance. These arise from the presence of self capacitance in the coils of each mutual inductance, of mutual capacitance between them, and also in part from the dissipation of energy by eddy currents in the metallic components and dielectric losses in the insulators on which the coils are wound. To represent these departures from an ideal mutual inductance for which the operator is $j\omega M$, it is usual to replace this by $(\sigma + j\omega M)$, where σ is a small positive term of the nature of a resistance. To satisfy these requirements, HARTSHORN (1925) devised the following method; the circuit is shown in Fig. 17·13. Fundamentally it is a modification of Maxwell's bridge for the comparison of two self inductances, cf. p. 598. If R_1 is the total resistance in the arm AH and L_1 its total self inductance, the balance conditions when K_1 is closed and K_2 open, are

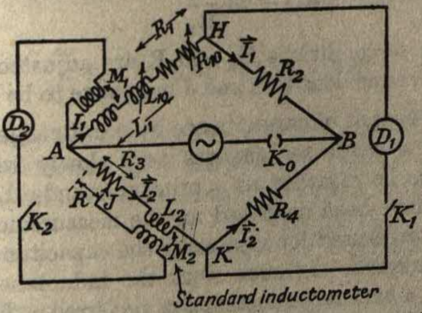


FIG. 17·13.—Hartshorn's inductometer method for comparing two mutual inductances (any ratio).

$$\frac{L_1}{L_2} = \frac{R_1}{R_3} = \frac{R_2}{R_4} \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

A definite value for $\dfrac{R_2}{R_4}$ is chosen and then the above conditions are satisfied by adjusting L_{10} and R_{10}, the variable parts of R_1 and L_1, respectively. When this balance has been effected the points H and K are at the same potential, so that

$$R_2\vec{I}_1 = R_4\vec{I}_2,$$

and in view of the condition that $R_1R_4 = R_2R_3$, we have

$$R_1\vec{I}_1 = R_3\vec{I}_2 \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

The key K_1 is now opened and K_2 closed, while a lead from the secondary coil of M_2, the inductance being determined, is taken to a point J so that a variable portion R of R_3 is now in the circuit containing D_2. The current through the detector D_2 is zero if

$$(\sigma_1 + j\omega M_1)\vec{I}_1 - (\sigma_2 + j\omega M_2)\vec{I}_2 - R\vec{I}_2 = 0,$$

i.e.

$$\frac{\sigma_1 + j\omega M_1}{(\sigma_2 + R) + j\omega M_2} = \frac{\vec{I}_2}{\vec{I}_1} = \frac{R_1}{R_3}, \quad . \quad . \quad . \qquad \text{(iii)}$$

since, at balance, $\vec{I}_1$ and $\vec{I}_2$ are unchanged.

Hence the conditions for balance are

$$\sigma_1 = (\sigma_2 + R).\frac{R_1}{R_3},$$

and
$$M_2 = M_1\frac{R_3}{R_1}.$$

To satisfy these conditions M_2 and R are adjusted alternately; in addition, the connexions to A and J may have to be interchanged.

The measurement of capacitance by a.c. bridges.—Since the capacitance of a condenser changes very slowly with time and cannot therefore be regarded as a primary standard, some of the bridges which have been described for the measurement of inductance are more often used for measuring the capacitance and power factor of a condenser. In such work the inductance must be a standard one; a mutual inductance is preferred. This is a very desirable procedure since inductances can be calculated from geometrical considerations and, practically, they are more durable than condensers. The Carey Foster bridge, as modified by Campbell, is particularly suitable for this purpose. Another bridge for the measurement of capacitance and power factor will now be described.

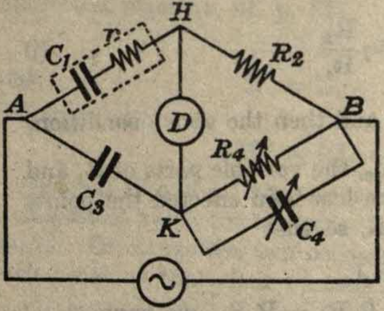


FIG. 17·14.—The Schering bridge.

The Schering bridge.—For the measurement of the capacitance and power factor of a condenser, when a reliable condenser is available, the bridge designed in 1920 by SCHERING * is one of the best and is very largely used. At the N.P.L. it has been modified so that extremely small condensers may be tested. In its simplest form the bridge is shown in Fig. 17·14. The condenser under test, (C_1, r), is placed in the arm AH of the bridge; C_3 is a standard condenser and C_4 a variable air-condenser. The bridge is balanced when

$$\frac{r + \frac{1}{j\omega C_1}}{\frac{1}{j\omega C_3}} = \frac{R_2}{\frac{R_4}{1 + j\omega C_4 R_4}},$$

or
$$j\omega r C_3 + \frac{C_3}{C_1} = \frac{R_2}{R_4}[1 + j\omega C_4 R_4].$$

---

* In a paper by Semm, *Zeits. f. Instk.*, 40, 124, 1920.

[It should be noted that this equation is most easily obtained by inverting the term below the thick line on each side of the first equation and transforming it to a place above the line; the usual method of cross-multiplying becomes very involved in this instance.]

Equating in turn the real and the imaginary parts, the balance conditions are

$$\frac{C_3}{C_1}=\frac{R_2}{R_4}, \quad \text{and} \quad rC_3 = C_4R_2.$$

These can be satisfied independently of each other. The power factor of the condenser C_1 is $\cos\phi$, where $\tan\phi = (\omega C_1 r)^{-1}$. For all condensers with a small power factor $\phi \to \frac{1}{2}\pi$, i.e.

$$\cos\phi \doteqdot \omega C_1 r = \omega . C_3\frac{R_4}{R_2}.\frac{C_4R_2}{C_3} = \omega C_4R_4.$$

The measurement of the frequency of an a.c. supply.—When the frequency of the source approximates to that of a standard tuning fork, the frequency is readily determined by connecting the source through a high resistance to a telephone, setting the fork in vibration, and counting the beats.

When this method is not applicable, the frequency of a source of low frequency current may be found by applying a portion of the voltage available from the source to a suitable a.c. bridge and adjusting the necessary components so that the current through the detector is zero. Hitherto, in connexion with the a.c. bridges described for the measurement of inductance or capacitance, the aim has been to make the balance conditions independent of frequency so that the freedom of the source from harmonics is not an essential requirement. Now, however, a bridge is required in which the balance can only be fulfilled for one particular frequency. A major difficulty at once arises unless the source is free from harmonics, for if they are present the balance point is blurred when a telephone is used as a detector; a vibration galvanometer may be used to overcome this difficulty, provided it is tuned afresh to every new frequency that is to be measured and an appropriate filter used to eliminate harmonics.

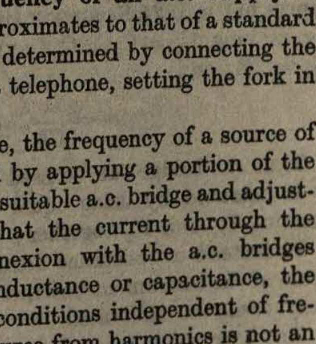


FIG. 17·15.—Robinson's [illegible] bridge.

Robinson's frequency bridge.—In Fig. 17·15, which shows the essential details of a bridge originally due to WIEN, C_1 and C_3 are two high-class capacitors while R_1, R_2, R_3 and R_4 are non-inductive resistors. Since Z_1, the impedance operator for the arm AH of the bridge, is given by

$$\frac{1}{Z_1} = \frac{1}{R_1} + j\omega C_1, \quad \text{or} \quad Z_1 = \frac{R_1}{1 + j\omega R_1 C_1},$$

the conditions for balance are determined by

$$\left(\frac{R_1}{1 + j\omega R_1 C_1}\right) \div \left(R_3 + \frac{1}{j\omega C_3}\right) = \frac{R_2}{R_4}.$$

Equating in turn the real and imaginary parts, we find, after some reduction,

$$\frac{R_4}{R_2} = \frac{R_3}{R_1} + \frac{C_1}{C_3}, \quad \text{and} \quad \omega^2 = \frac{1}{C_1 C_3 R_1 R_3}.$$

Originally the bridge was used for the comparison of two capacitances but in 1924 ROBINSON simplified these conditions for balance by making $C_1 = C_3 = C$ (say), and $R_1 = R_3 = R$ (say). Then $R_4 = 2R_2$ and

$$\omega = \frac{1}{CR}, \quad \text{or} \quad f = \frac{1}{2\pi CR}.$$

Thus the bridge can be used for the measurement of frequency; it is balanced by using equal capacitors, making $R_4 = 2R_2$, and varying R_1 and R_3 together; they must be kept equal.

If, instead of resistance boxes, conductance boxes are used, the scale of the instrument becomes linear; if C remains fixed the scale may easily be calibrated.

Any harmonics present in the input voltage will appear in the detector arm at balance, since the conditions for balance involve ω. Even weak harmonics become prominent as balance is approached, but an experienced observer, with a sensitive ear, finds it quite possible to balance out the fundamental even in the presence of harmonics.

Many other frequency bridges are available but Robinson's bridge has the great merit that its arms are made up entirely of resistors and capacitors, i.e. there is no inductive coupling between the bridge and other neighbouring sources, or even between the components of the bridge.

The Wagner earthing device.—If accurate results are to be obtained with the a.c. bridges described, difficulties, due to 'earth capacitances' which arise, especially at high frequencies, have to be overcome. An ingenious device due to WAGNER is used for this.

Let us suppose that Z_1, Z_2, Z_3 and Z_4, Fig. 17·16, comprise the four impedances forming an a.c. bridge and that one of them is on test. The Wagner earthing device is an auxiliary circuit connected across the a.c. supply. It consists of two impedances Z_5

to earth potential without having either point actually connected to earth. The stray capacitances C_h and C_k have no effect on the balance conditions and the relation $\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4}$ is satisfied. Moreover, the detector is at earth potential and can be used without introducing capacitance currents to earth.

Maxwell's L/C bridge for the measurement of self inductance (with a Wagner earth).—As already mentioned on p. 603, this bridge is a very popular one for the measurement of inductance; such popularity is due mainly to the fact that easily variable capacitors are now available. The bridge with a Wagner earth and an amplifier in the detector circuit is shown in Fig. 17·17. AHBK$_1$ represents the original bridge, the conditions for balance being

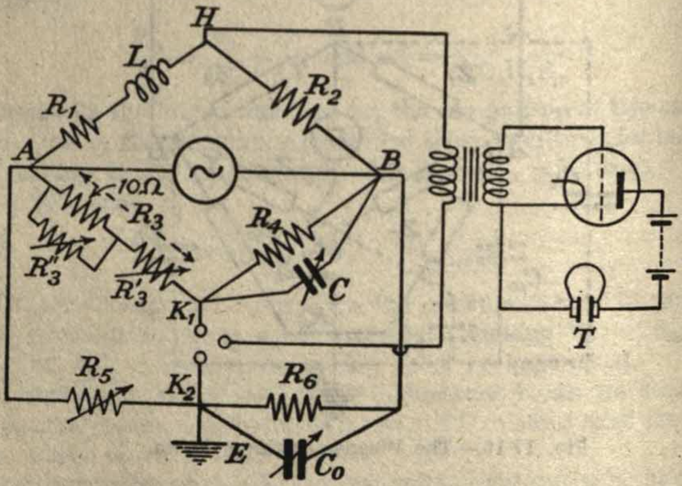


FIG. 17·17.—Maxwell's L/C bridge with a Wagner earth.

$$R_1R_4 = R_2K_3,$$

and

$$L = CR_2R_3.$$

The Wagner earth consists of the two arms AE and EB, with components similar to those in AK$_1$ and K$_1$B; the point K$_2$ is earthed. In order to satisfy the first condition for balance, R$_3$ consists of a variable resistor R$_3$′ in series with a 10 Ω coil shunted by a variable resistor R$_3$″ with a maximum value of 500 Ω. A valve amplifier, in conjunction with a pair of earphones, T, is used to detect when the bridge is balanced.

The Campbell (Carey Foster) a.c. bridge with a Wagner earth.—The principal bridge, shown by heavy lines, is arranged as in Fig. 17·18. The Wagner earth, shown by hatched lines, is placed across the source of a.c.; it consists of a resistance R″ in series with an inductance coil, (R′, L′), the point H_1 being earthed. An additional detector D_1 is shown, but in practice only one detector is used, suitable switches enabling it to be placed in the desired positions. The auxiliary bridge consists of R″, R′ and L′, together with the two arms R and L_1. The principal bridge is first balanced approximately and then the auxiliary bridge. These operations are repeated and the bridges rebalanced at each stage until finally the current through the detector is zero whichever bridge is used. The points H and K on the principal bridge are then at earth potential without actual connexion to earth and so the effects of stray capacitances to earth are eliminated. The conditions for balance are

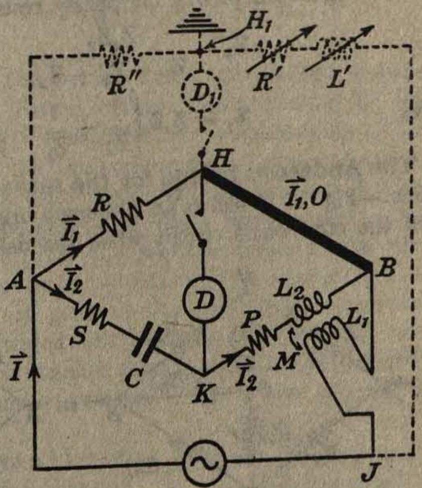
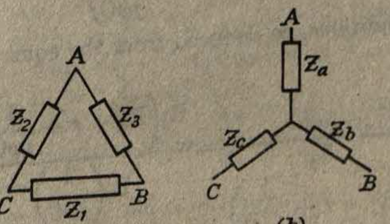


FIG. 17·18.—Campbell's bridge with a Wagner earth.

$$\frac{|M|}{C} = PR, \quad \text{and} \quad L_2 = |M|\left[1 + \frac{S}{R}\right],$$

as already established.

Equivalent delta-star transformations in a.c. networks.—The methods already described for obtaining the conditions of balance in a.c. bridges are usually sufficient for all practical purposes, but when the number of loops involved becomes large, a delta-star transformation may be used to reduce the necessary



(a) (b)

FIG. 17·19.—Equivalent delta-star transformation in a.c. networks.

algebraic work; a star-delta transformation may similarly be used. If the impedance operators for the delta and star networks concerned are as shown in Fig. 17·19 (*a*) and (*b*) respectively, then, proceeding as on p. 212 for purely resistive elements, we find

$$\mathrm{Z}_a = \frac{\mathrm{Z}_2\mathrm{Z}_3}{\mathrm{Z}_1 + \mathrm{Z}_2 + \mathrm{Z}_3},$$

and

$$\mathrm{Z}_1 = \mathrm{Z}_b\mathrm{Z}_c\left[\frac{1}{\mathrm{Z}_a} + \frac{1}{\mathrm{Z}_b} + \frac{1}{\mathrm{Z}_c}\right], \text{ etc.}$$

The Anderson bridge for the measurement of self inductance.—Fig. 17·20 shows an Anderson bridge (for details, cf. p. 604) and the equivalent circuit when the delta assembly between the points B, G and K is replaced by a star with components for which the impedance operators are Z_b, Z_g and Z_k. When the bridge is balanced there is no current in the impedance associated with Z_g; its value need not therefore be written down. Using the above formulae, we find

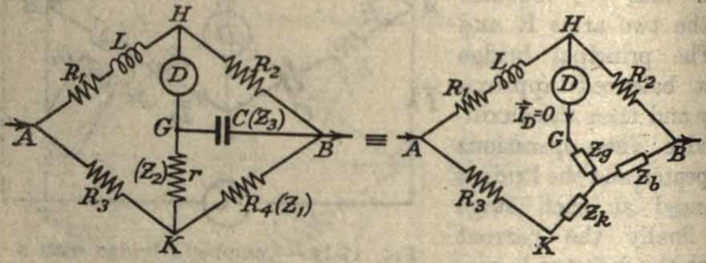


FIG. 17·20.—Anderson's L/C bridge.

$$\mathrm{Z}_b = \mathrm{R}_4\left(\frac{1}{j\omega\mathrm{C}}\right) \div \left(\mathrm{R}_4 + r + \frac{1}{j\omega\mathrm{C}}\right),$$

and

$$\mathrm{Z}_k = \mathrm{R}_4 r \div \left(\mathrm{R}_4 + r + \frac{1}{j\omega\mathrm{C}}\right).$$

The balance conditions, as deduced from the equivalent circuit, are

$$\frac{\mathrm{R}_1 + j\omega\mathrm{L}}{\mathrm{R}_3 + \dfrac{\mathrm{R}_4 r}{\left(\mathrm{R}_4 + r + \dfrac{1}{j\omega\mathrm{C}}\right)}} = \frac{\mathrm{R}_2\left(\mathrm{R}_4 + r + \dfrac{1}{j\omega\mathrm{C}}\right)}{\mathrm{R}_4\left(\dfrac{1}{j\omega\mathrm{C}}\right)}.$$

$$\therefore \quad \frac{\mathrm{R}_1 + j\omega\mathrm{L}}{\mathrm{R}_3\mathrm{R}_4 + r\mathrm{R}_3 + \dfrac{\mathrm{R}_3}{j\omega\mathrm{C}} + \mathrm{R}_4 r} = \frac{j\omega\mathrm{C}\mathrm{R}_2}{\mathrm{R}_4}.$$

Cross multiplying and equating in turn the real and imaginary parts, we obtain, as on p. 605,

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}, \quad \text{and} \quad L = CR_2\left[r\left(1 + \frac{R_3}{R_4}\right) + R_3\right].$$

The determination of the permittivity (or dielectric constant) of an insulating liquid.—Most measurements for the determination of permittivity consist of two measurements of the capacitance of a condenser, the first measurement being made when the dielectric medium of the condenser is air, and the second when the dielectric medium is the substance under investigation. The ratio of these capacitances gives the required value of the permittivity, κ. The capacitance to be measured is usually small—less than 0·001 μF.—and, in general, the capacitance when the dielectric medium is air [which will be referred to as the 'air-capacitance'] is considerably smaller than that when the dielectric medium is the substance under investigation [it is here referred to as the 'liquid capacitance'].

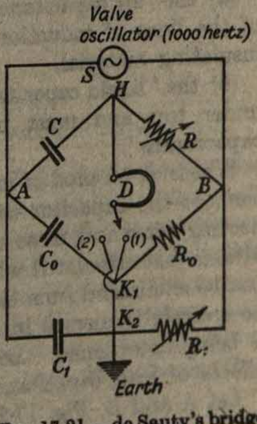


FIG. 17·21.—de Sauty's bridge with a Wagner earth, for the determination of the permittivity of an insulating liquid.

Since such capacitors have a high reactance, any bridge in which they are used must have a Wagner earth. A simple circuit for this purpose is a de Sauty bridge with a Wagner earth arranged as indicated in Fig. 17·21. C is a wireless receiving capacitor of the better type and has a nominal maximum capacitance of $5 \times 10^{-4} \mu$F or 500 pF. It is carried below a polystyrene plate supported on the upper edge of a cylindrical glass jar of ample size. This plate has two terminals connected respectively to the fixed and movable vanes of the capacitor which is placed in the arm AH of the bridge. C_0 and C_1 are two similar capacitors placed as shown, while R, R_0 and R_1 are three non-inductive resistance boxes. The bridge is fed from a valve oscillator with a frequency of 1000 cycle.sec.$^{-1}$; the detector D is a pair of earphones.

With air between the plates of the capacitor C, alternate balances for the bridge proper and then for the Wagner earth circuit are found by varying in turn R and R_1. The vessel containing C is then filled with the liquid (nitrobenzene, $C_6H_5NO_2$, benzene, C_6H_6, or transformer oil) and by slightly tilting the vessel backwards and forwards all air bubbles may be removed from the system. The bridge is again balanced by alternately adjusting R and R_1; let R′ be the value of R under these conditions. Then

$$C_0R_0 = CR = (\kappa)CR',$$

or

$$(\kappa) = \frac{R}{R'},$$

where (κ) is the relative permittivity of the liquid. If the Wagner earth is not used, errors as high as five per cent occur.

A precision method for the determination of the permittivity of a liquid.—This method is due to HARTSHORN and OLIVER* who used it to determine the permittivity of benzene and its variation with temperature. These investigators pointed out that the value of the permittivity is given accurately by the ratio, defined in the preceding paragraph, only if certain conditions are satisfied in making the measurements, and these conditions cannot be fulfilled unless a condenser of special design is used. The essential conditions are :—

(*a*) the 'air capacitance' must be located entirely in air, i.e. none of the lines of induction representing it must pass through solid insulating material,

(*b*) the 'liquid capacitance' must be located entirely in the liquid under test and must be geometrically identical with the 'air capacitance'.

Whatever method of measurement is used, it will be necessary to connect the capacitor to the measuring circuit by a pair of connecting wires and these will possess 'lead capacitance'. The conditions just stipulated will only be satisfied if the lead capacitance can be eliminated from the measurements, since the leads can never be entirely immersed in the liquid when the 'liquid capacitance' is being determined. Let us therefore consider in detail how the effects of lead capacitance may be eliminated.

Let A and B, Fig. 17·22 (*a*), be the two plates of the condenser and let these be connected to the measuring circuit by means of the two leads L_1 and L_2. Suppose it is desired to measure C_{AB}, the capacitance between A and B. It is important to recognize that the system comprises several other capacitances, viz. the capacitance c_2 between L_1 and L_2 and the 'cross-capacitances' c_3 and c_4 between the two plate systems and the opposite leads. The total capacitance is

$$C_{AB} + c_2 + c_3 + c_4.$$

When the lead L_1 is disconnected from A, as in Fig. 17·22 (*b*), then we have $(C_{AB} + c_3)$ and c_5 in series together with c_2 and c_4

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* *Proc. Roy.Soc., A,* **123**, 664, 1929.

in parallel with the series combination. Now

$$\frac{1}{C_{AB}+c_3}+\frac{1}{c_5}=\frac{c_5+C_{AB}+c_3}{(C_{AB}+c_3)c_5}.$$

$$\therefore \text{ Capacitance of system} = \frac{(C_{AB}+c_3)c_5}{c_5+C_{AB}+c_3}+c_2+c_4.$$

If the plates A and B are removed we are left with a single lead capacitance c_2', which is slightly different from c_2. The important point to emphasize is that it is not permissible to determine the capacitance C_{AB} by taking the difference of the readings when the condenser is firstly connected to the leads and secondly disconnected, either at one or both plate systems. Almost all previous observers have assumed that satisfactory measurements could be made in this way and many of the existing discrepancies between the results of different observers are probably due to this. Hartshorn and Oliver used a special method whereby the stray capacitances are almost completely eliminated and the single capacitance C_{AB} is determined.

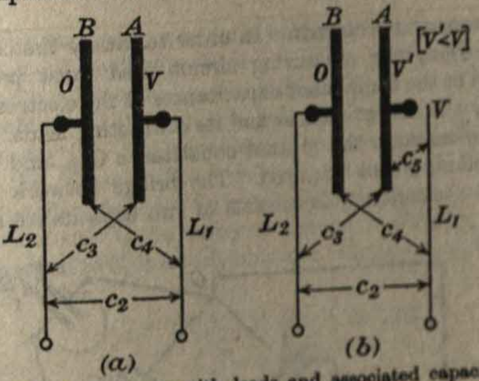

FIG. 17·22.—A condenser with leads and associated capacitances.

The test-condenser and its housing.—It is clear that some form of solid insulating material must be used to separate the two plate systems of the test condenser and a certain capacitance must be associated with this material. In order to satisfy the conditions (*a*) and (*b*) this capacitance must be eliminated from the measurements. For this purpose the test-condenser is given the form shown in Fig. 17·23.

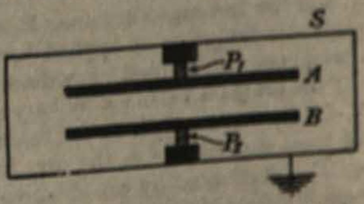

FIG. 17·23.—A condenser with ideal mutual capacitance C_{AB}.

P_1 and P_2 are small insulators which separate the plates A and B from the electrostatic shield S. The whole system may be regarded as consisting of three capacitances, the mutual capacitance C_{AB} of the plates A and B, and the capacitances C_{AS} and C_{BS} between the plate systems and the screen. Now although there is a solid insulator in the electrostatic field represented by each of the capacitances C_{AS}, C_{BS}, the mutual capacitance C_{AB} is located entirely in air, since all the lines of induction which pass through P_1 from A must terminate on the screen ; they cannot possibly reach the plate B.

The measuring circuit. In order to satisfy the conditions for high precision, the measuring circuit used must permit of the separation of the component capacitances of the electrostatic system formed by the test condenser and its connecting leads. It must be possible to measure the mutual capacitance C_{AB}, and to eliminate the capacitances not required. The bridge network is shown in Fig. 17·24 ; essentially it consists of two capacitance arms C_1 and C_2, and the two resistance arms R_3 and R_4. Alternating voltage of telephonic frequency is applied between the points G and E, and a telephone, T, is used as a detector.

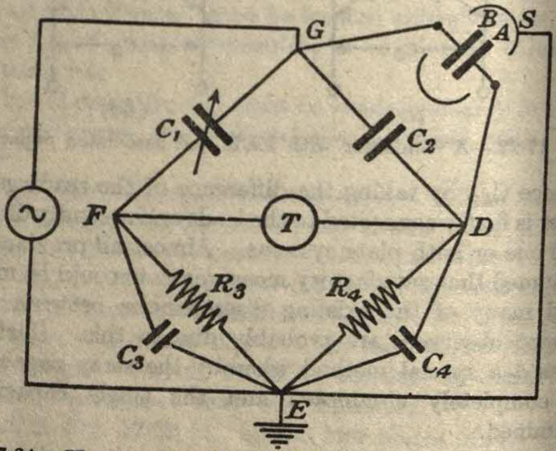

FIG. 17·24.—Hartshorn and Oliver's bridge for determining the permittivity of benzene.

If the bridge is balanced by varying one or more of the quantities C_1, C_2, R_3, R_4, a condition for balance is $C_1R_3 = C_2R_4$, where C_2 is the total capacitance in the arm DG. If the resistance arms R_3 and R_4 are shunted by capacitances C_3 and C_4, and one or both of these is adjusted so as to preserve the balance of the bridge, the same equation still holds. To establish this fact, the conditions for balance are given by

$$\left(\frac{1}{j\omega C_1}\right) \div \left(\frac{R_3}{1 + j\omega C_3 R_3}\right) = \left(\frac{1}{j\omega C_2}\right) \div \left(\frac{R_4}{1 + j\omega C_4 R_4}\right),$$

i.e.
$$\frac{1 + j\omega C_3 R_3}{C_1 R_3} = \frac{1 + j\omega C_4 R_4}{C_2 R_4}.$$

Equating in turn the real and imaginary parts, we have

$$C_1 R_3 = C_2 R_4, \quad \text{and} \quad \frac{C_3}{C_1} = \frac{C_4}{C_2}.$$

The bridge may be conveniently used in the following way for the measurement of small capacitances. With a fixed condenser C_2 in DG and no condenser in parallel with it, C_1, R_3 and R_4 are given suitable values and the bridge is balanced by varying one of these quantities, say C_1. The capacitance to be measured is then connected to the points G, D, i.e. it is placed in parallel with C_2 and balance is again obtained. From the change ΔC_1 in the value of the standard condenser C_1 necessary to balance the bridge, the capacitance ΔC_2 added to C_2 is readily obtained; for assuming R_3 and R_4 to be fixed in value,

$$(C_1 + \Delta C_1)R_3 = (C_2 + \Delta C_2)R_4,$$

and since
$$C_1 R_3 = C_2 R_4,$$
$$R_4 . \Delta C_2 = R_3 . \Delta C_1$$

In taking the difference of two readings in this way greater accuracy is obtained than is possible when only one reading is taken, since errors common to the two readings cancel out.

The capacitance thus measured is the total capacitance added to the points G and D. We have now to show how the bridge will eliminate component capacitances not required and how this property can be used to eliminate the capacitances associated with the leads. Let the test-condenser and its screen, S, be arranged as shown in Fig. 17·24. Under these circumstances the capacitance C_{BS} is connected across the points G and E and merely shunts the source of alternating current; accordingly it cannot affect the bridge reading. The capacitance C_{AS}, however, is across the points D, E, and is thus in parallel with C_4. This will affect the readings of C_3 or C_4 at balance but the equation $C R_3 = C_2' R_4$, where C_2' is the total capacitance between G and D, still holds. Now

$$C_2' = C_2 + C_{AB} + c,$$

where c represents the lead capacitance. Thus the bridge reading, C_1, is independent of C_{AS} and C_{BS} but is a linear function of C_{AB}. Since the test-condenser is perfectly shielded by the earthed screen S, no additional earth capacitances have to be considered.

In order to consider the elimination of the lead capacitance let the test-condenser and leads be represented by the network shown in Fig. 17·25 (a). For a complete representation the screen S and its associated capacitances should be included but since these are eliminated in the bridge measurements, it is not necessary to consider them further. Now let the connexions be made as follows, the screen S being earthed the whole time and the bridge being balanced after each change of connexion.

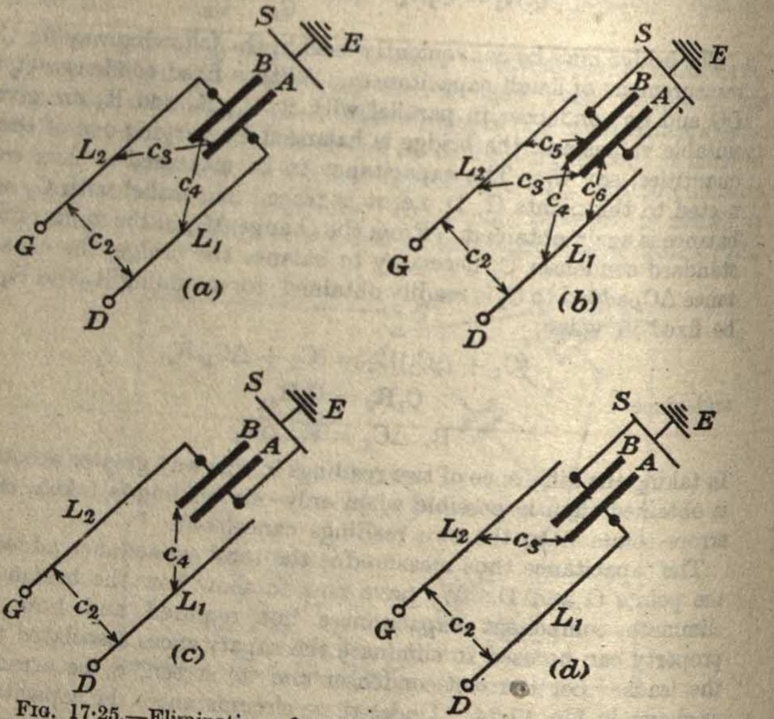

FIG. 17·25.—Elimination of stray capacitances in Hartshorn and Oliver's bridge.

(α) Connect L_1 to A and L_2 to B. Then the total capacitance added to C_2 is

$$C_{AB} + c_2 + c_3 + c_4 = k\alpha \text{ (say)},$$

where k is a constant and α the change in the reading of the standard condenser C_1 necessary to effect a balance.

(β) Break the connexions L_1A, L_2B, and connect A and B to S, as in Fig. 17·25 (b). The capacitances to A and B, viz. c_5 and c_3, c_4 and c_6 have now become capacitances to screen and are therefore eliminated from the measurements as explained earlier. The capacitance now added is $c_2 = k\beta$ (say).

(γ) Make L_2B, disconnecting B from S. The capacitance now added is $c_2 + c_4$, cf. Fig. 17·25 (*c*); this may be called $k\gamma$.

(δ) Disconnect A from the screen; connect A to L_1; disconnect L_2B and connect B to screen. The capacitance now added is $c_2 + c_3$, cf. Fig. 17·25 (*d*); this may be called $k\delta$.

$$\therefore \ C_{AB} \text{ is directly proportional to } (\alpha + \beta) - (\gamma + \delta),$$

and here one must remember that α, β, γ and δ are the changes in the readings of the standard condenser C_1 necessary to effect a balance in each of the four arrangements, the resistances R_3 and R_4 remaining fixed.

Thus, by taking these four readings the various lead capacitances are completely eliminated. The capacitances to screen are also eliminated and the quantity measured is the true mutual capacitance C_{AB}, and this is purely an 'air-capacitance'. If now the test-condenser is immersed in the liquid to be investigated so that it completely fills the screen, and the mutual capacitance of A and B again measured, the ratio of the two capacitances gives the permittivity of the liquid with reference to that of air, no corrections being necessary.

In this research C_1 was a standard air-condenser with a range 200–2900 μF. Its capacitance could be read to 0·1 $\mu\mu$F. at all parts of its scale. Errors due to backlash were avoided in the usual way. The following result was obtained for benzene at 20° C.

$$\kappa_{20} = 2{\cdot}2825 \pm 0{\cdot}0005, \quad [\text{Vacuum} = 1].$$

The permittivities of standard liquids.—Within less than two decades after the investigations by Hartshorn and Oliver on the permittivity of benzene, techniques for the production of hydrocarbons of extremely high purity had been developed by Timmermans, in Brussels, and by Coulson and Herington, at the Chemical Research Laboratory, Teddington. Timmermans had also stressed the need for the determination of the permittivities of at least two more standard liquids. The reason for this is that while liquids of low permittivity may be compared satisfactorily with benzene, for work on liquids with higher permittivities, standard liquids with permittivities of the order 10 to 40 as against 2 for benzene are essential.

Since 1929 the technique of capacitance measurement has been improved considerably. It has already been shown, cf. p. 621, that the experimental determination of a permittivity consists fundamentally in the measurement of the ratio of two capacitances; in 1929 it was considered that this ratio could be determined to 1 part in 10,000 provided the air condenser had a value of 1000 pF., i.e. 10^{-3} μF. Meanwhile, however, it had been found that, with adequate screening, the stray capacitances, inherent in any simple bridge, could be excluded from a measurement with an uncertainty less than 0·01 pF. It had therefore become possible to obtain the same accuracy in a capacitance measurement even though a much smaller condenser was used. This is extremely important for it permits the use of a smaller sample of liquid and it also makes possible the use of a noble metal for the electrodes without making the cost unreasonably high. Both these factors, viz. a small volume of liquid and the use of platinum electrodes, are a great asset when chemical impurities are the chief cause of uncertainty in a physical measurement.

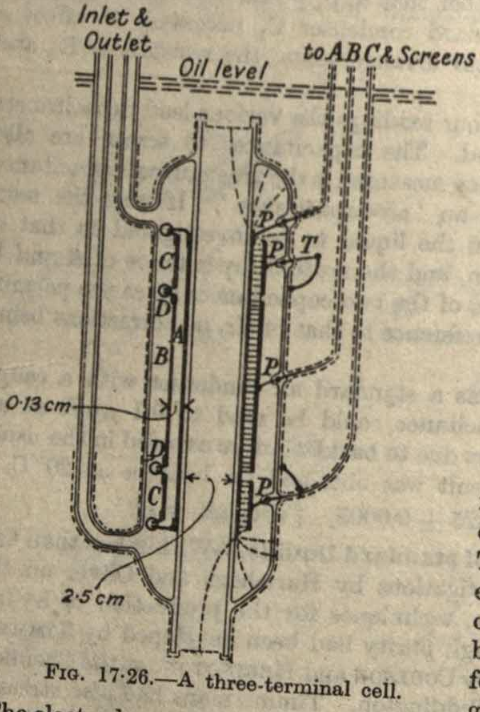

FIG. 17·26.—A three-terminal cell.

Details of the capacitor cell for use with benzene and similar liquids are shown in Fig. 17·26. The electrodes are seamless cylinders of platinum foil 0·03 cm. thick, this being considered the smallest thickness that would provide the necessary rigidity in self-supporting cylinders of the required size. The arrangement consists essentially of two coaxial cylinders, the outer one being divided by circumferential gaps so that guard rings are formed. The electrode assembly is mounted within a double-walled cylindrical shell made of Pyrex glass and carried by the glass tube forming the inner wall of the cell; the inner electrode A fits snugly over this tube. Care is exercised to see that any liquid introduced into the cell makes contact with no metal except platinum. The design of the cell is such that, when it is evacuated, the slight displacement of the cell walls does not affect the geometrical capacitance of the system. When in use the cell is completely immersed in an oil bath whose temperature can be controlled within the requisite limits.

Now, unfortunately, the purest benzene obtainable commercially contains, as shown in 1929, sufficient water to affect its permittivity appreciably, and this water can only be removed by months of treatment with drying agents. In the work carried out by HARTSHORN, PARRY and ESSEN, the water content of the highly purified benzene, supplied in sealed ampoules, was far from negligible and long treatment with drying agents in the sealed enclosure containing the capacitor cell was essential.

The bridge used was similar to that of 1929, the main difference being that a Wagner earth was incorporated. For benzene, C_6H_6, it was found that

$$\kappa_{20} = 2{\cdot}2836 \pm 0{\cdot}0002,$$

and $\kappa_\theta = \kappa_{20} - 0{\cdot}0019(\theta - 20)$, $[20°\,\text{C.} < \theta < 30°\,\text{C.}]$

but the difficulty of removing the water content from benzene is a serious objection to its use as a standard liquid for permittivity measurements. Now cyclohexane, C_6H_{12}, has a permittivity which is unaffected by a drying procedure similar to that used with benzene; thus cyclohexane as a standard liquid has decided advantages over benzene. For cyclohexane

$$\kappa_{20} = 2{\cdot}0250 \pm 0{\cdot}0002,$$

and $\kappa_\theta = \kappa_{20} - 0{\cdot}00155(\theta - 20)$ $[20°\,\text{C.} < \theta < 30°\,\text{C.}]$

When measurements on dichloroethane CH_3CHCl_2, a polar liquid, were attempted new difficulties arose. Its conductivity is so high that at 1 kc.sec.^{-1} the loss tangent, cf. p. 562, is 0·5. Precision measurements were out of the question and so a two-terminal and simpler cell is used. It is standardized with benzene, and then, using a constant volume of liquid and assuming that the capacitance extraneous to this volume is constant, the permittivity of the polar liquid is determined by a comparison method. If C_1 is the measured capacitance of the cell when charged with the fixed volume of liquid, C_V the measured capacitance with the cell evacuated, and C_E the capacitance extraneous to the system, then

$$(\kappa) = \frac{C_1 - C_E}{C_V - C_E}.$$

C_E is determined by using benzene; the same equation is used when determining an unknown permittivity.

For dichloroethane it was found that

$$\kappa_{20} = 10{\cdot}663 \pm 0{\cdot}003,$$

and $\kappa_\theta = \kappa_{20} - 0{\cdot}553(\theta - 20)$, $[20°\,\text{C.} < \theta < 30°\,\text{C.}]$

while for nitrobenzene, $C_6H_5NO_2$,

$$\kappa_{20} = 35{\cdot}72 \pm 0{\cdot}02,$$
$$\kappa_\theta = \kappa_{20} - 0{\cdot}185(\theta - 20). \quad [20°\,\text{C.} < \theta < 30°\,\text{C.}]$$

These investigations showed that the permittivity of each of these two liquids is independent of frequency, and for all practical purposes may be taken as the static permittivity of the pure liquid.

A radio-frequency method for the determination of the permittivity of an insulating liquid.—This method is based on one developed by SUGDEN in 1933–5; the error in his work did not exceed one part in a thousand. Although the apparatus to be described is a simplified form of that designed by Sugden, its components are more modern and it is intended to illustrate the method which he used; the error is less than one per cent.

Consider a tuned circuit consisting of a coil with inductance L connected in series with a variable condenser of capacitance C, and let an alternating e.m.f. of fixed frequency f_0 be induced in the circuit, for example, by coupling the coil inductively with an oscillator as shown in Fig. 17·27 (*a*).

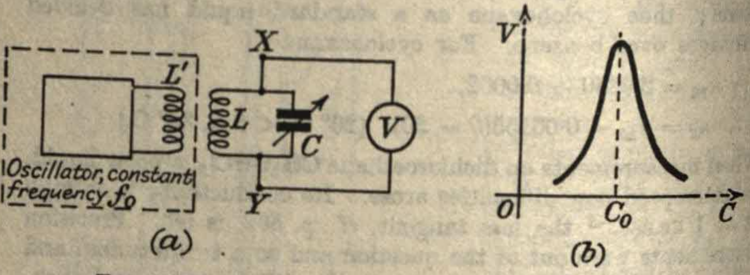

FIG. 17·27.—A tuned circuit and its response curve.

A response curve, cf. Fig. 17·27 (*b*), showing the variation of the r.m.s. value of the potential difference, V, across the coil for different values of C will have a maximum when the resonant frequency of the tuned circuit is equal to f_0, the frequency of the oscillator, i.e. when $f = \dfrac{1}{2\pi\sqrt{LC_0}}$, where C_0 is the capacitance of the condenser at resonance.

Let the condenser be a variable air-spaced condenser of suitable design and suppose it has been calibrated so that its capacitance (with air as dielectric) for any dial reading θ is known.

Response curves are obtained (i) with air as dielectric, resonance being obtained for a dial setting θ_0 corresponding to a capacitance C_0, and (ii) with the insulating liquid, having a permittivity κ, as the medium between the plates, resonance now being obtained for a dial setting θ_1 corresponding to a capacitance C_1, *if air had been the dielectric.* [N.B. C_1 is less than C_0 since $(\kappa) > 1$.]

Now since resonance is obtained at the same frequency f_0 in both instances, $(\kappa)C_1 = C_0$,

i.e.

$$(\kappa) = \frac{C_0}{C_1},$$

where C_0 and C_1 are obtained from the dial readings θ_0 and θ_1 of the condenser at resonance.

Some practical details. ***(a) The oscillator.***—A tuned-anode radio-frequency feedback oscillator may be used. The circuit is shown in Fig. 17·28 (*a*). Oscillations are controlled by the mutual inductance M between the anode and grid circuits ; if M is increased gradually from a small value, the onset of oscillations is usually shown by a sudden change in the anode current and their frequency may be adjusted by means of the variable condenser C′.

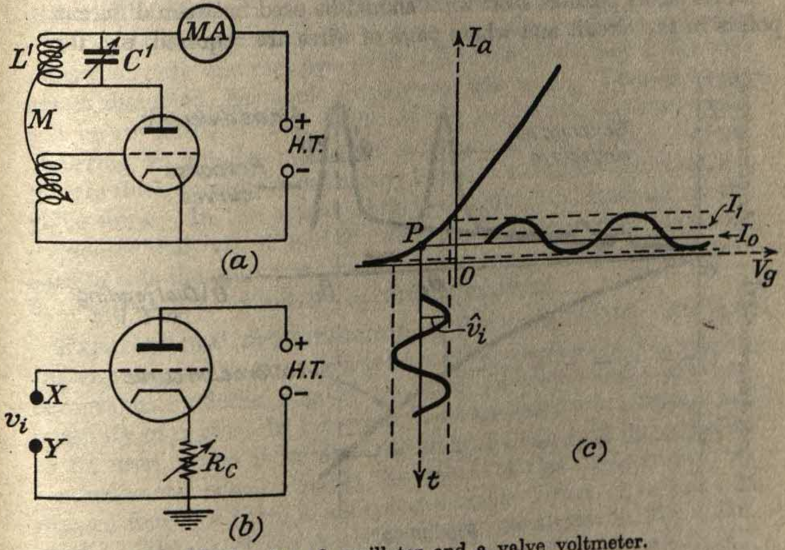

FIG. 17·28.—An r.f. oscillator and a valve voltmeter.

(b) The valve voltmeter.—Any form of high input-impedance voltmeter may be used to measure the whole or a fraction of the voltage across the coil L. A voltmeter based on the anode-detector principle is suitable ; a convenient circuit is indicated in Fig. 17·28 (*b*).

The cathode bias resistor R_C is chosen so that with no alternating input signal between X and Y the quiescent value of the anode current is I_0, and, cf. Fig. 17·28 (*c*), the operating point P lies on the lower bend of the mutual characteristic. When an alternating potential difference is applied between X and Y the anode current wave-form is distorted and the *mean value* of the anode current becomes I_1. The increase in the mean anode current, viz. $(I_1 - I_0)$, which is measured by the milliammeter, is a measure of the amplitude, $\hat{v}_i$, of the input signal.

Determination of the permittivity of benzene.—The condenser C, Fig. 17·27 (*a*), is a calibrated air-spaced condenser surrounded by a metal container which may be filled with liquid. An outer container, filled with water, serves as a constant temperature jacket. The coil L is constructed so that resonance with C is obtained at a suitable frequency and it is mounted so that its distance from the coil L′ (in the anode circuit of the oscillator) may be adjusted to a suitable value. The input to the valve voltmeter is taken from the points X and Y.

Special attention should be paid to the following points :—

(*a*) As far as possible *short* wires should be used between different points in the circuit and where pairs of wires are required, e.g. for filament leads, h.t. leads, wires from X and Y to the voltmeter, etc., they should be well insulated and twisted closely together.

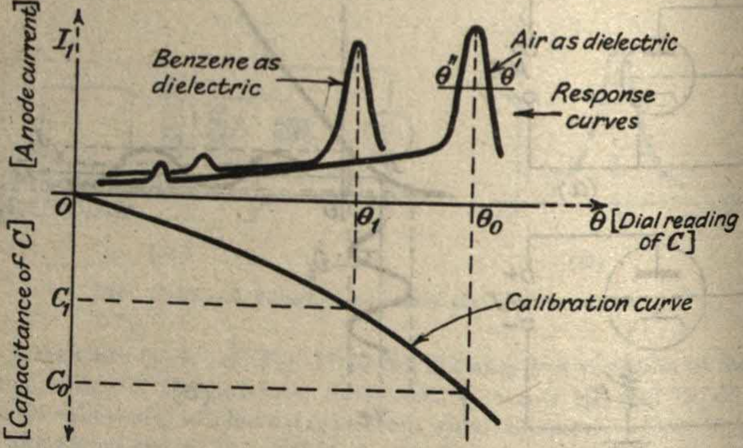

FIG. 17·29.

(*b*) As far as possible all stray coupling between the anode and grid circuits should be avoided (i) by keeping leads short and (ii) by keeping anode circuit components away from grid circuit components.

(*c*) The outer case surrounding the condenser and the movable plates should be 'earthed', e.g. by connecting them to the point Y.

(*d*) The frequency of the oscillator should be adjusted so that the peak of the air-response curve occurs when the dial reading of the condenser is about eighty per cent of its full scale reading; the maximum on the benzene response curve will then be obtained at about half this value.

(*e*) The coupling between the two coils L and L′ should be adjusted so that a smooth response curve is obtained when C is varied in both directions. If the coupling between the two coils is excessive discontinuities in the response curve may be obtained.

(*f*) Response curves should be obtained, i.e. I_1 plotted against θ, (i) with air as dielectric, (ii) with benzene as dielectric, and (iii) with air as dielectric.

Throughout these observations, the frequency of the oscillator must not be changed. As the frequency is likely to be affected not only by the value chosen for C′ but also by the values of M and of the relative spacings of L and L′, all these factors should remain unaltered. Then the readings obtained in (iii) should agree exactly with (i); if they do not, further observations should be made.

(*g*) If the waveform of the output from the oscillator is purely sinusoidal only one response peak will be obtained, but if the output is distorted, harmonic frequencies will occur; their presence will be revealed, cf. Fig. 17·29, by the subsidiary response peaks occurring at smaller condenser settings. Under certain circumstances these subsidiary peaks may be used to obtain an independent value for κ. In the actual work by Sugden two readings on the condenser dial, viz. θ' and θ'', were observed for which the anode current was the same; with sufficient accuracy it was assumed that $\theta_0 = \frac{1}{2}(\theta' + \theta'')$. Similarly θ_1 was determined.

Experimental determination of the permittivity of a gas (or vapour).—The same principles as those discussed in the previous paragraphs underlie Sugden's method for determining the permittivity of a gas. In this work, however, smaller condensers had to be used, when it became necessary to surround them by an earthed metal screen so that the values of their capacitances remained fixed for there is always present the 'stray capacitance to earth'. Details of the receiving circuit are shown in Fig. 17·30.

The resonance coil is tuned by a system of condensers C_3, C_4, C_5 and C_6. C_3 is an inexpensive variable condenser for rough tuning and its capacitance is maintained at a fixed value during a series of measurements. C_6 represents the capacitance of the cell which finally contains the gas or vapour. The small change in the capacitance of C_6 when gas is admitted to the previously exhausted cell or removed from it is measured by the combination C_4–C_5 of two condensers in series. C_4 is about 10 pF., while C_5 is a standard variable air-condenser with a maximum capacitance of 1,200 pF.

It is clear from simple theory that for a given state of response, a small change in capacitance of C_6, will require a large change in C_5 to bring the circuit back to the same state of response. On this principle depends the accurate measurement of the small changes in C_6 as a result of the introduction of gases and vapours. The changes in anode current are measured by a sensitive galvanometer, and it is arranged as a null instrument. The diagram shows one method for making the necessary connexions and it should be noted that if I_a is the current in the detecting circuit, then this will flow through R_1 when $I_g = 0$. Since the drop in potential between H and K must be zero when $I_g = 0$, the necessary value for R_1 is given by $V - RI_a = 0$, where V is the e.m.f. of the cell A.

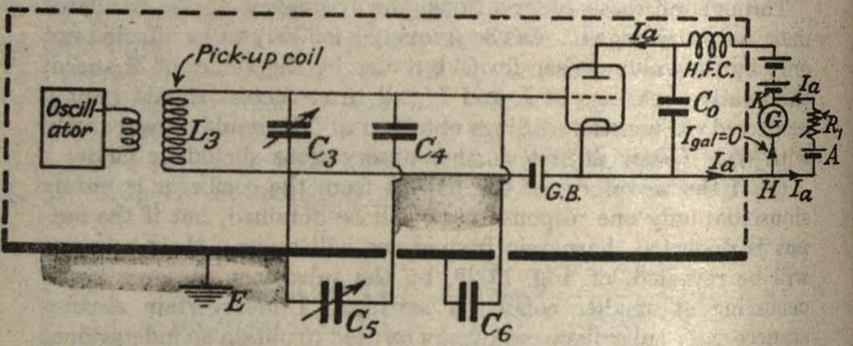

FIG. 17·30.—The receiving circuit in Sugden's work on the permittivity of a gas (or vapour).

C_6 is first evacuated to enable a response curve to be plotted. Gas or vapour is then admitted and C_5 reduced to C_5' until the current flowing through R_1 has the same value as it had before the admission of the substance under examination, i.e. the galvanometer current is again zero. Let ΔC_6 be the change in C_6. The equivalent capacitance of the condensers C_4, C_5 and C_6 before the introduction of gas into C_6 is

$$C_6 + \frac{1}{\dfrac{1}{C_4} + \dfrac{1}{C_5}},$$

and from the nature of the experiments this is equal to the equivalent capacitance of the system after C_6 is filled with gas, viz.

$$C_6 + \Delta C_6 + \frac{1}{\left(\dfrac{1}{C_4} + \dfrac{1}{C_5'}\right)}.$$

$$\therefore \Delta C_6 = \frac{C_4C_5}{C_4 + C_5} - \frac{C_4C_5'}{C_4 + C_5'}$$

$$= \frac{C_4^2(C_5 - C_5')}{(C_4 + C_5)(C_4 + C_5')}.$$

$$\therefore (\kappa) - 1 = \frac{\Delta C_6}{C_6} = \frac{C_4^2}{C_6}.\frac{C_5 - C_5'}{C_5C_5' + C_4(C_5 + C_5') + C_4^2}.$$

[The term C_4^2 in the denominator is very small and may be neglected.]

Among the more important practical details it must be mentioned that the cell consisted of two stainless steel cylinders fixed coaxially with a gap of 0·1 cm. approximately; these were mounted inside a large glass bulb and the system thus formed placed in an electrically heated oil bath. Finally, it must be recorded that the whole potential difference across L_3 was applied to the valve in the valve-voltmeter circuit.

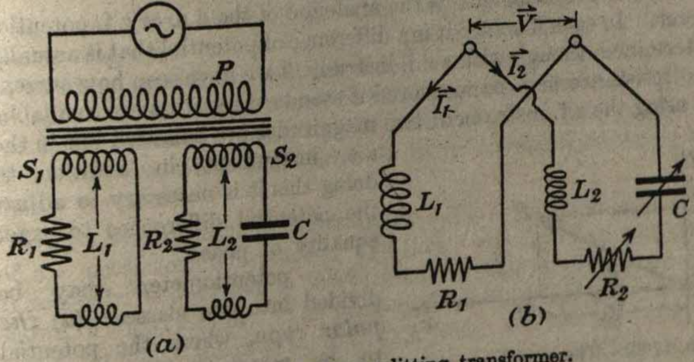

FIG. 17·31.—A phase splitting transformer.

The simple phase splitting condition.—In the near sequel it will become necessary to obtain in two circuits currents which are equal in magnitude but whose phases are in quadrature. To show how this may be done let P, Fig. 17·31 (*a*), be the primary of a transformer with two secondaries S_1 and S_2. Across S_1 let there be placed a resistor and an inductive coil; let R_1 be the total resistance of this closed circuit and L_1 its total self inductance. Across S_2 let there be placed a resistor, an inductive coil and a condenser of capacitance C; let R_2 be the total resistance of all the coils in this circuit and L_2 the total self inductance. It will be assumed that the design of the transformer is such that when it is in operation equal electromotive forces are produced in the secondaries and each may be represented by $\vec{V} = \hat{V}\exp(j\omega t)\vec{v}$. If $\vec{I}_1$ and $\vec{I}_2$ are the vector currents and these are equal in magnitude and, in addition, $\vec{I}_2$ leading and in quadrature with $\vec{I}_1$, then $\vec{I}_2 = j\vec{I}_1$. Accordingly, we have

$$\vec{V} = \vec{I}_1(R_1 + j\omega L_1) = \vec{I}_2\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C}\right)\right],$$

and if $\vec{I}_2 = j\vec{I}_1$, then

$$R_1 + j\omega L_1 = jR_2 - \left(\omega L_2 - \frac{1}{\omega C}\right).$$

Equating in turn the real and imaginary parts we get

$$\left.\begin{aligned} R_1 + \omega L_2 = \frac{1}{\omega C}, \quad \text{or} \quad C &= \frac{1}{\omega(R_1 + \omega L_2)}, \\ \text{and} \qquad R_2 &= \omega L_1. \end{aligned}\right\}$$

These conditions can be attained independently by the variation of R_2 and of either R_1 or C.

The above equations show that some inductance must be present in the in-phase circuit, but none is essential in the quadrature circuit. In such circumstances $C = \frac{1}{\omega R_1}$.

Fig. 17·31 (*b*) shows how the two circuits may be fed from the same source and in such a case the impedance of the source does not enter into the calculations.

A SIMPLE TYPE OF A.C. POTENTIOMETER

The a.c. potentiometer is the analogue of the d.c. or z.f. potentiometer. In each instance it is a difference of potential that is actually determined, though with a z.f. instrument we have seen how current and resistance may be measured if standard resistors are available. In using the z.f. instrument two magnitudes are balanced but in the a.c. instrument, in addition to doing this it is necessary to adjust the potential differences to exact equality of phase.

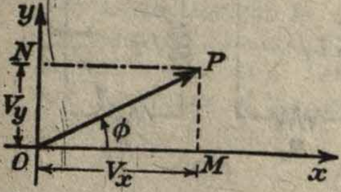

FIG. 17·32.—The coordinate type a.c. potentiometer.

a.c. potentiometers may be divided into two classes: (*a*) ***the polar type,*** where the potential to be measured is reproduced by an independent variation of the magnitude and phase of the balancing potential difference, and (*b*) ***the coordinate type*** so-called (but better described as the rectangular coordinate type), in which the potential to be measured is copied by the vector addition of two potentials, one of which is exactly in quadrature with the other and each of which is capable of independent and continuous adjustment. The present discussion will be limited to the second type of potentiometer.

In general, let $v_x = \hat{V}_x \cos \omega t$ and $v_y = \hat{V}_y \cos (\omega t - \psi)$ be two voltages applied between points A and B, and B and C, respectively. Then the instantaneous potential difference between A and C is given by

$$\begin{aligned} v = v_x + v_y &= (\hat{V}_x + \hat{V}_y \cos \psi) \cos \omega t + \hat{V}_y \sin \psi \sin \omega t \\ &= \hat{V} \cos (\omega t - \phi), \end{aligned}$$

where
$$(\hat{V})^2 = (\hat{V}_x)^2 + (\hat{V}_y)^2 + 2\hat{V}_x\hat{V}_y \cos\phi,$$
and
$$\tan\phi = \frac{\hat{V}_y \sin\psi}{\hat{V}_x + \hat{V}_y \cos\psi}.$$

If v_x and v_y are in quadrature, i.e. $\psi = \frac{1}{2}\pi$, then
$$(\hat{V})^2 = (\hat{V}_x)^2 + (\hat{V}_y)^2, \quad \text{and} \quad \tan\phi = \frac{\hat{V}_y}{\hat{V}_x}.$$

It makes no difference if we use r.m.s. values in these last equations since, in general, $\hat{V} = \sqrt{2}V$. Thus if V_x and V_y are the r.m.s. values of the two components required to balance the voltage on test, cf. Fig. 17·32, then the r.m.s. value of the potential to be measured is $(V_x^2 + V_y^2)^{\frac{1}{2}}$ and the phase angle $\phi = \tan^{-1}\left(\frac{V_y}{V_x}\right)$, with reference to the axis Ox. The diagram is really a rotating vector diagram; it does not imply that the potential differences to be added must be across gaps which lie in straight lines mutually at right angles. In the sequel it will be assumed that all voltages are r.m.s. values; if necessary the peak values are known at once if the supplies, which must have the same circular frequency, are purely sinusoidal.

Gall's simple a.c. potentiometer.—This is a coordinate type of a.c. potentiometer designed by GALL* for teaching purposes. It can only be operated at 50 cycle.sec.$^{-1}$. The circuit diagram is shown in Fig. 17·33 (a). The primary P of the transformer is connected through a rheostat R_0 to the a.c. mains and the resistance R_0 is adjusted until the reading on V is equal to that specified on the instrument. S_1 and S_2 are the secondaries and these are connected to the in-phase slide wire AB and the quadrature slide wire CD, respectively. Each slide wire is 100 cm. long and the values of the various components have been chosen so that when V is set to its fiduciary value the potential drop in each wire is one volt per metre.

The connexions to the slide wires are shown in Fig. 17·33 (b). When the deflexion of the vibration galvanometer, VG, is zero, a condition attained by shifting each sliding contact in turn until the deflexion is a minimum and finally zero, the vector sum of the voltages across AQ_1 and CQ_2 is equal to the voltage on test. In order to allow the phases of the x and y components to be reversed if necessary, reversing switches K_1 and K_2 are provided. These are indicated in Fig. 17·33 (c). The signs to be used with these component voltages are indicated on the switches.

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* *Journ. Sc. Inst.*, 7, 132, 1930.

Application of an a.c. potentiometer to determine the inductance and power factor of an inductive coil.—By means of a potential divider or a transformer a potential difference of one volt (r.m.s.) is applied to a standard resistance R_0 in series with the inductive coil on test. The circuit is shown in Fig. 17·34 (*a*).

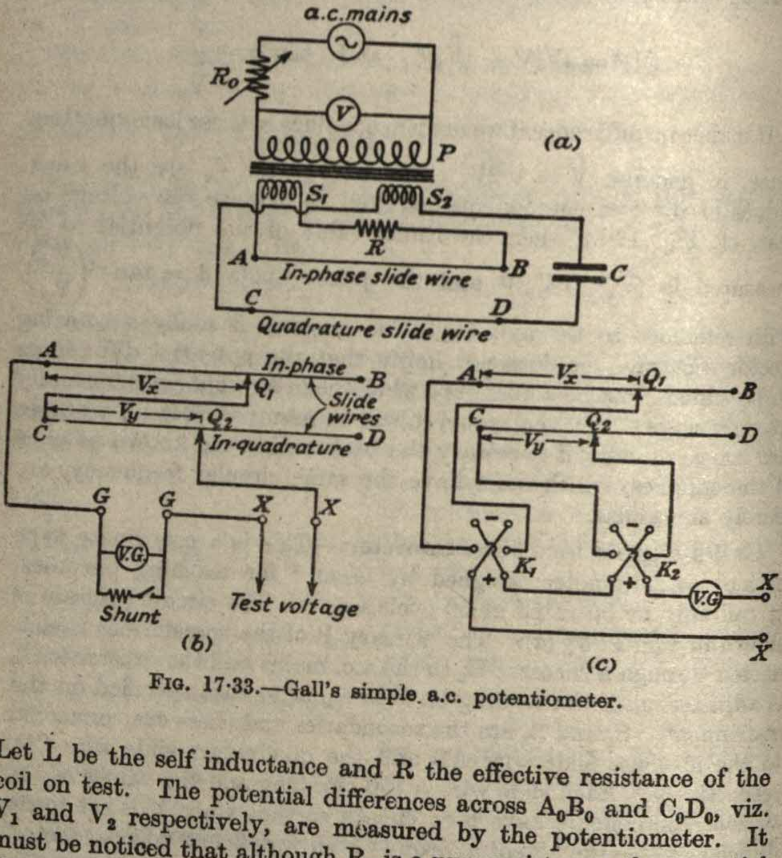


Fig. 17·33.—Gall's simple a.c. potentiometer.

Let L be the self inductance and R the effective resistance of the coil on test. The potential differences across A_0B_0 and C_0D_0, viz. V_1 and V_2 respectively, are measured by the potentiometer. It must be noticed that although R_0 is a pure resistance, the potential drop across it shows a phase difference ϕ_1. This is because the current in A_0B_0 is not necessarily in phase with that in the potentiometer wire AB [cf. Fig. 17·33 (*a*)]. The potential drop across A_0B_0 is therefore represented by a vector OP as in Fig. 17·34 (*b*) and the components measured are OH and HP.

Similarly the potential drop across C_0D_0 is ZI, where Z is the impedance of the coil. This is represented in Fig. 17·34 (*b*) by PQ and the measured components are PK and KQ. Finally it must be observed that the phase of V_2 is ϕ ahead of V_1, cf. Fig. 17·34 (*c*), and $\phi = \phi_2 - \phi_1$.

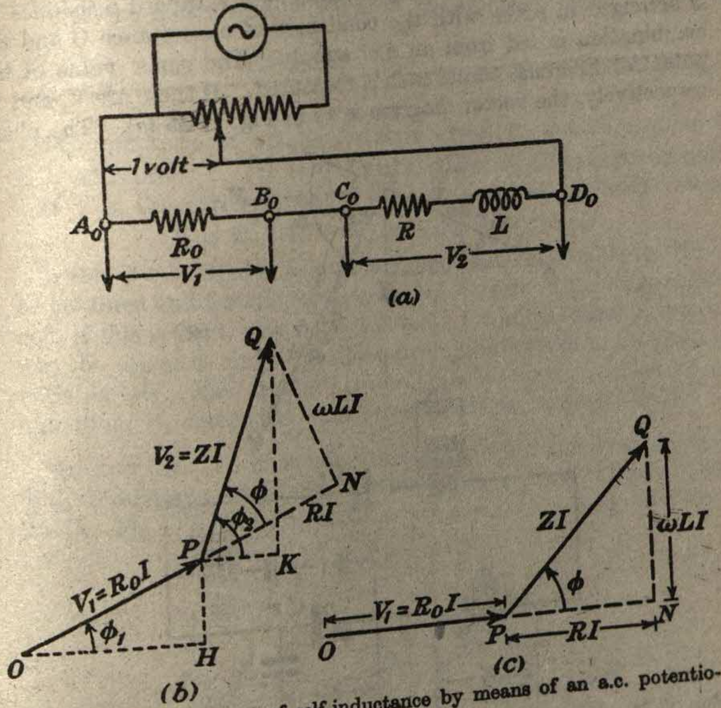


FIG. 17·34.—Measurement of self-inductance by means of an a.c. potentiometer.

Example.—R_0 is a 1 ohm coil. The potentiometer readings when the potential difference across it was measured gave mean values + 19·7 cm. (in.) and + 10·9 cm. (quad.); for the coil the corresponding readings were + 1·4 cm. (in.) and + 19·25 cm. (quad.). Since the potential difference across each wire is 1 volt (r.m.s.), we have

$$V_1 = \sqrt{0 \cdot 197^2 + 0 \cdot 109^2}; \quad V_2 = \sqrt{0 \cdot 014^2 + 0 \cdot 1925^2}$$
$$= 0 \cdot 225 \text{ volt}; \qquad = 0 \cdot 193 \text{ volt.}$$

Also
$$\tan \phi_1 = \frac{109}{197} = 0 \cdot 553, \quad \tan \phi_2 = \frac{1925}{140} = 13 \cdot 76.$$
$$\therefore \phi_1 = 28° \, 57', \qquad \phi_2 = 85° \, 50',$$
$$\therefore \phi = \phi_2 - \phi_1 = 56° \, 53'.$$

Hence
$$Z = R_0 \frac{V_2}{V_1} = 1 \times \frac{193}{225} = 0 \cdot 858 \text{ ohm.}$$

To find L, we have
$$\sin \phi = \frac{\omega L}{Z},$$

i.e.
$$L = \frac{Z \sin \phi}{\omega} = \frac{0 \cdot 858 \times \sin 56° \, 53'}{314} = 2 \cdot 29 \times 10^{-3} \text{ H.}$$

Also
$$R = Z \cos \phi = 0 \cdot 469 \text{ ohm.}$$

The measurement of capacitance.—A standard resistance R_0 is arranged in series with the condenser of capacitance C and the combination is fed from an a.c. supply. The r.m.s. value of the potential difference across each is measured. If these are V_1 and V_2 respectively, the vector diagram is as in Fig. 17·35 (*a*). The phase angle for the condenser is $\phi = \phi_1 + \phi_2$, and this will be very nearly $\frac{1}{2}\pi$ if the condenser has a low power factor.

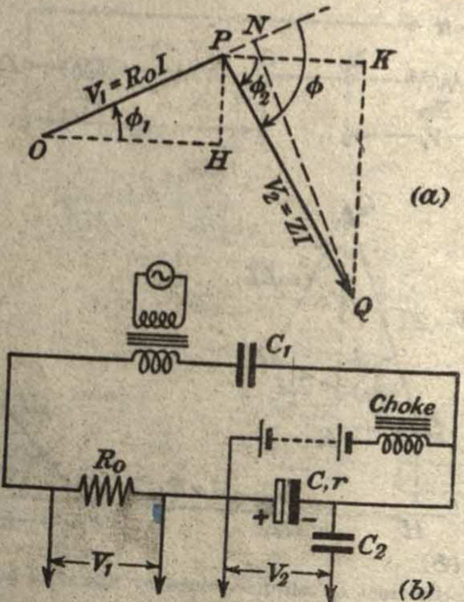


FIG. 17·35.—Measurement of capacitance by means of an a.c. potentiometer.

To test an electrolytic condenser the circuit is as given in Fig. 17·35 (*b*) and if this is compared with Fig. 18·18, cf. p. 680, no further remarks are necessary.

The measurement of an alternating magnetic field.—METHOD (i). Suppose that the alternating magnetic field, cf. p. 530, is given by $h = \hat{H} \cos \omega t$. If it is produced at the centre of a circular coil, of N_1 turns and radius a, by a current $i = \hat{I} \cos \omega t$, then

$$h = \hat{H} \cos \omega t = \frac{2\pi N_1 \hat{I}}{a} \cos \omega t.$$

If a search coil with N_2 turns, each of area A, is placed with its lane normal to the field that is to be measured, the instantaneous ux is

$$\Phi = \mu N_2 A \hat{H} \cos \omega t,$$

where μ is the permeability of the surrounding medium.

Thus the instantaneous value of the induced e.m.f. is given by

$$|e| = \dot{\Phi} = \frac{\mu N_2 A . 2\pi N_1 \omega \hat{I} \cos(\omega t - \tfrac{1}{2}\pi)}{a} = \hat{E}\cos(\omega t - \tfrac{1}{2}\pi) \text{ say.}$$

The r.m.s. value of the induced e.m.f. is therefore given by $E = \mu H N_2 A \omega$. [H is the r.m.s. value of the magnetic field.]

If V is the corresponding value of the r.m.s. induced voltage,

$$V = E \times 10^{-8} = \mu H N_2 A \omega \times 10^{-8} \text{ volt.}$$

If this is measured with an a.c. potentiometer a value for H may be obtained and from it, a value for the current in the large circular coil, if this is used, may be calculated; for all practical purposes μ may be taken as unity for the surrounding medium is almost invariably air. The method really gives the r.m.s. value of the induction, B, associated with the alternating magnetic field.

METHOD (ii). To illustrate a method in which the peak value, $\hat{B}$, of the induction associated with a magnetic field is obtained directly, let C_1, Fig. 17·36, be a coil carrying an alternating current of sinusoidal wave-form, and C_2 a search coil arranged coaxially with C_1. To measure $\hat{B}$, the peak value of the magnetic induction at the centre of C_2, the search coil must have a small mean cross-sectional area A and a large number of turns N. The number of linkages associated with C_2 at time t is given by

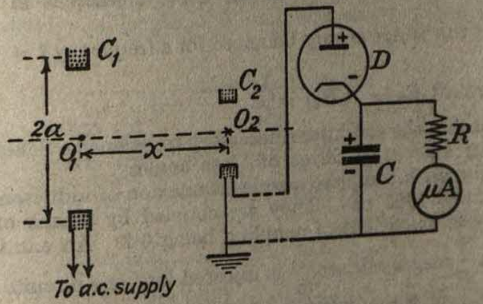


FIG. 17·36.—To measure $\hat{B}$ for an alternating magnetic field.

$$\Psi = NA\hat{B}\cos\omega t,$$

so that the induced e.m.f. is $(NAB\omega \sin \omega t) \times 10^{-8}$ volt. To measure this e.m.f. a diode voltmeter may be used. This consists of a diode valve (or germanium rectifier) connected to a capacitor C. The potential difference across C biasses the anode of the valve D negatively, so that, in the absence of the high resistance R, the capacitor will cease to acquire electricity when the peak value of the applied potential difference equals that across the capacitor. When R is present the charge on the condenser will leak away during the greater part of a cycle but it will be replenished when the applied potential difference is near to its peak value. If the leakage current, as measured by a microammeter, is I, then IR is the potential difference across C. Thus

$$\mathrm{NA}\hat{\mathrm{B}}\omega \times 10^{-8} = \mathrm{IR}.$$

Hence if N, A, ω and R are known, a value for $\hat{\mathrm{B}}$ may be obtained.

EXAMPLES XVII

17·01. R_1, R_2, R_3 and R_4 are four resistances in the arms of a Wheatstone net, which is now modified so that R_4 has a coil of self-inductance L and negligible resistance in series with it, and a condenser of capacitance C may be arranged (*a*) in parallel and (*b*) in series with R_1. Show that in the first arrangement the bridge is balanced if $R_1R_4 = R_2R_3 = \frac{L}{C}$, the balance being independent of any 'overtones' which may be present in the supply, whereas in the second arrangement the bridge is only balanced for a frequency $f = \frac{1}{2\pi}\sqrt{\frac{R_4}{LCR_1}}$ and provided $(R_2R_3 - R_1R_4) = \frac{L}{C}$.

17·02. Obtain the conditions necessary to balance a Schering bridge. Express your answer in terms of 'loss angles'.

Two circuits each comprise a series connexion of inductance 300 μH. and capacitance 1000 pF. They are coupled by means of a mutual inductance, the coefficient of coupling being 0·2. An e.m.f. of amplitude 5 V. at $\frac{\frac{1}{2}}{\pi}$ megacycle.sec.$^{-1}$ is injected into one circuit. Calculate the amplitude of the current in the other circuit. [0·136 A.]

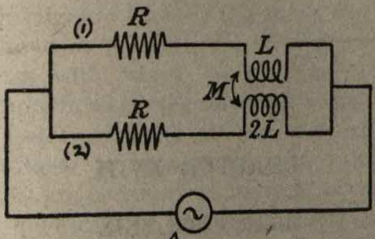
17·03. An Owen bridge was used to measure the self inductance of each of two coils A and B and also their mutual inductance. With the notation of Fig. 17·06 the following values were obtained.

Coil	$C_3 = C_4$ (μF.)	R_2 (ohm.)	R_3 (ohm.)
A	0·33190	200·0	2362
B	0·33190	200·0	2403
A and B in series . .	0·33190	200·0	7095
A and B opposing . .	0·33190	200·0	2240

Show that $L_A = 0·1568$ H., $L_B = 0·1595$ H., $M = 0·0772$ H.

17·04. If in the Owen's bridge shown in Fig. 17·06, p. 600, the condenser C_3 is placed in parallel with R_3, obtain the conditions for balance. Then show that if $C_3 = C_4 = C$ (say), the conditions become $L = C(R_2 - R_1)R_3$ and $R_1 = \omega^2 CLR$. Discuss whether or not any practical use may be made of such a bridge.

17·05. Obtain an expression for the phase difference between the currents in the two branches of the network shown in Fig. Ex. 17·05



$v = \hat{V} \cos \omega t.$

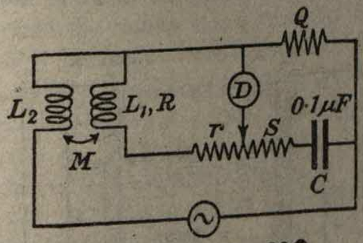
FIG. Ex. 17·05.

when it is fed from an a.c. supply of sinusoidal wave-form.

$$\left[\cot(\phi_2 - \phi_1) = \frac{R^2 + \omega^2(2L - M)(L - M)}{\omega RL}.\right]$$

17·06. Give the theory of the alternating-current network shown in Fig. Ex. 17·06 for the comparison of mutual inductance and capacitance.

One arm of such a network contains, in series with a resistance of 80 ohms, a solenoid of length 100 cm., and mean diameter 4 cm.,



$R = 20\Omega.,\ r = 80\Omega.$

FIG. Ex. 17·06.

wound uniformly with 1000 turns of wire of total resistance 20 ohms. The mutual inductance between the solenoid and a short compact coil of 300 turns wound round it at its centre is compared with a capacitance of 0·1 μF. Calculate values for the additional components needed to balance the bridge. [Assume $\pi^2 = 10$]. [Q = 48 Ω., S = 112 Ω.]

17·07. Two links XY_1Z and XY_2Z are in parallel with the points X and Y connected to an a.c. supply of sinusoidal wave-form and frequency f. The link XY_1Z consists of a resistance R in series with an inductance L; the link XY_2Z is a resistance R in series with a capacitance C. If $L = CR^2$ and I_1 and I_2 the r.m.s values of the currents in the two links, show that

$$\frac{I_2}{I_1} = 2\pi f\sqrt{LC}.$$

If the above ratio is 2·00, calculate a value for f when $L = 3{\cdot}20 \times 10^{-2}$H. and $C = 5{\cdot}00 \times 10^{-2}\mu$F. [7·96 kilocycle.sec.$^{-1}$.]

CHAPTER XVIII

ELECTROLYSIS

When an electric current passes through a metallic conductor there is no obvious change in the composition of the conductor either where the current enters or leaves, or indeed in any part of the conductor through which the current flows. It is otherwise when the conductor is an 'electrolyte', e.g. an aqueous solution of a salt. In this instance the conduction of the current is accompanied by the convection of material substances, parts of the electrolyte, which are frequently deposited upon the metallic 'electrodes', which are the means whereby the current enters and leaves the solution. The deposit upon one electrode is different from that on the other, so that two currents of matter, flowing in opposite directions, accompany the passage of electricity through the solution.

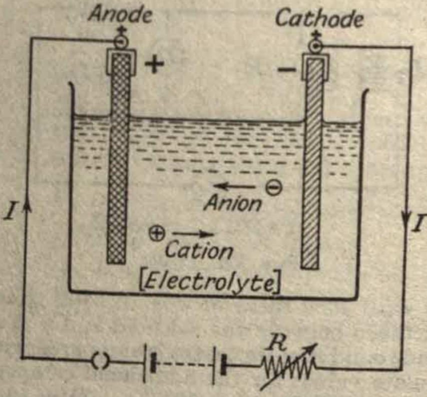


Fig. 18·01.—An electrolytic cell.

The earliest systematic study of electrolysis was made by Faraday. It was through his work that the terms ***anode, cathode, anion*** and ***cation*** came into use. The anode is the metallic conductor at which the current enters the electrolyte, and the cathode is that at which it leaves. The anion is the constituent of the electrolyte which migrates towards the anode and the cation that which migrates towards the cathode—cf. Fig. 18·01.

The electrolytes with which we shall be concerned almost exclusively are those in which the solvent is water. Perfectly pure water is a very feeble conductor of electricity, its conductivity at ordinary temperatures being about 5×10^{-8} ohm.$^{-1}$cm.$^{-1}$, but special precautions are necessary in order to obtain water of this purity.* Small quantities of dissolved substances produce large increases in the conductivity of water. For example, the conductivity of a solution containing 1·5 parts by weight of KCl in 10,000 parts of water is about $2{\cdot}5 \times 10^{-4}$ ohm.$^{-1}$cm.$^{-1}$—an increase of 6000 times. This fact is all the more remarkable when we remember that potassium chloride itself is a very feeble conductor of electricity.

Faraday's laws of electrolysis.—As a result of a long series of researches FARADAY discovered the relations which exist between the quantity of electricity passed through an electrolyte and the mass of the metal, or other substance, liberated at an electrode. Faraday summarized the results of his work in the two following laws which are now known as Faraday's laws of electrolysis.

(*a*) ***The mass of any individual product liberated in electrolysis is directly proportional to the quantity of electricity which has passed through the electrolyte.***

(*b*) ***The masses of the different products liberated during electrolysis by the passage of equal quantities of electricity are proportional to their chemical equivalents, where the chemical equivalent of an element is defined as the mass of that element which, in a chemical reaction, will combine with, or displace, 8 gm. of oxygen.***

The first law of electrolysis may be represented by the expression

$$m \propto \int_0^t \mathrm{I}\, dt,$$

where m is the mass of the substance deposited in time t, and I is the current at any instant. Removing the sign of proportionality, we have

$$m = z\int_0^t \mathrm{I}\, dt,$$

where z is a constant. If m is expressed in grams, and the substance liberated is an element, z is known as the ***electrochemical equivalent*** of that element.

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* For ordinary carefully distilled water, $\sigma = 1{\cdot}5 \times 10^{-6}$ ohm.$^{-1}$cm.$^{-1}$.

If now A^* is the relative atomic mass † of an element of valency ζ, then one gram-equivalent has a mass $\frac{A^*}{\zeta}$ and a mass in grams of any element equal to that of a gram-equivalent is liberated by the passage of a quantity of electricity known as a ***faraday***. This quantity of electricity is usually denoted by the symbol F.

Thus, from the first law of electrolysis

$$\frac{A^*}{\zeta} = zF, \quad \text{or} \quad z = \frac{A^*}{\zeta F}.$$

If we assume the truth of Faraday's laws the value of the electrolytic constant can be determined from experiments upon any electrolyte, but preferably, of course, upon one for which the experimental data can be fixed with precision. If for silver we take $z = 0{\cdot}0011183$ gm.coulomb.$^{-1}$, $F = 96{,}470$ coulomb.gm.-equiv.$^{-1}$. According to BIRGE, $F = 96{,}487{\cdot}7$ coulomb.gm.-equiv.$^{-1}$.

Measurement of quantity of electricity.—Since the mass of any substance liberated at an electrode is given accurately by the expression $m = zQ$, the method of weighing the electrolytic deposit may be utilized in an exact determination of a quantity of electricity, or, if the current is constant, of the strength of that current. An apparatus by means of which this is done is known as a ***voltameter*** or ***coulometer***.

The silver coulometer.—A precision type of coulometer is that designed by F. E. SMITH and its essential features are shown in Fig. 18·02. The silver anode A is in the form of a disc, coated with an electrolytic deposit of silver. This disc is contained in a glass basin D, which is provided with a ground edge. This basin is fused to a glass supporting rod H which, in turn, is held by a suitable clamp. The rod H passes through an aperture in the disc. A silver rod S carries the anode A and a glass tube K fits snugly over S to reduce loss by electrolysis.

G is a glass cylinder with its lower end ground to fit over the rim of the basin D. This cylinder is used to separate the electrolyte into two parts, before and after electrolysis. Just before the current is passed through the coulometer, the cylinder G is raised to the position shown, its lower edge always being covered by the solution.

The cathode C is a platinum bowl containing a freshly prepared 10 per cent solution of re-crystallized silver nitrate—$AgNO_3$. This solution must be free from organic matter and other reducing substances; it should be neutral, or very slightly acid, both before and after the conclusion of an experiment. The dish D is essential if particles which may fall from the anode are to be prevented from reaching the platinum bowl. Earlier workers had covered the anode with filter paper or used a specially prepared porous pot for this purpose. Rosa and Vinal * have shown that when a filter paper is used the increase in mass of the cathode is too great by several parts in 10^5. The use of filter papers is now discarded and the Smith type of coulometer, in which there is no septum, is preferred.

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† A^* is used for the relative atomic mass of an element in order to distinguish it from A, the mass number; this is the integer closest to the relative atomic mass, or the total number of nucleons in a nucleus.

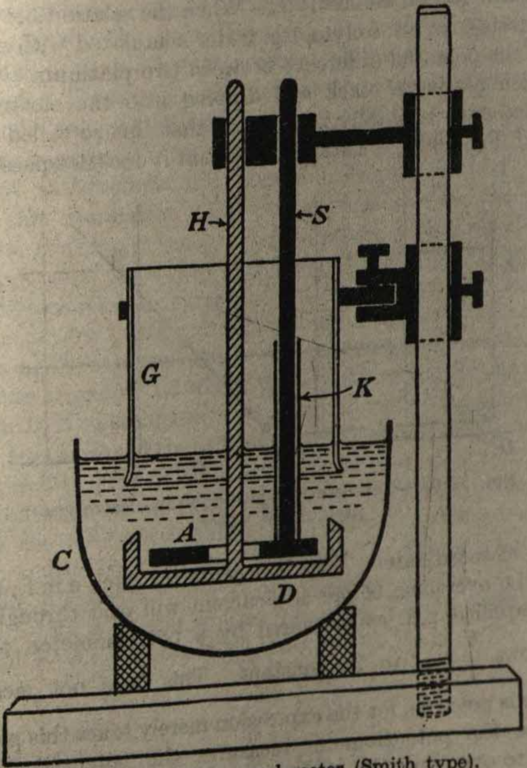

Fig. 18·02.—Silver coulometer (Smith type).

The cathode basin is washed at the end of an experiment, dried at about 150° C. and from the gain in mass the desired information may be obtained. Since the absolute mass of the silver deposit is required, Gauss' method of double weighing must be used and, in general, a correction for buoyancy must be made. Since the surface area of the deposit is large, this tends to take up moisture rapidly and the necessary correction is not known with high precision.

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* *Bur. Stand. Bulletin*, No. 285, 1916.

When it is desired to use this coulometer to measure current the main uncertainty is the time. The duration of the experiment must be 4000 seconds if errors due to making and breaking the circuit are to be negligible. Smith found that for silver, $z = 1{\cdot}11827$ mgm.coulomb.$^{-1}$.

Ohm's law for an electrolyte.—When the relation between the current through an electrolyte, e.g. water acidulated with sulphuric acid, and the potential difference between two platinum electrodes, coated with platinum black and dipping into the electrolyte, is investigated experimentally, it is found that this so-called ***characteristic***, cf. Fig. 18·03, is a straight line, but it does not pass through the origin of coordinates. This is because the back e.m.f., v, in the cell must be overcome before any current will pass through it. If V is the applied p.d. as measured by a potentiometer, then the graph shows $\frac{V - v}{I}$ to be constant. This does not mean that Ohm's law is not true, for the expression merely takes this particular form since the potentiometer measures the potential difference necessary to overcome the back e.m.f. together with that necessary to send the current through the cell. The resistance of the liquid column is $\frac{V - v}{I}$. We can only speak of Ohm's law in reference to electrolytes, i.e. there will only be a linear relationship between the available potential difference $(V - v)$ and the current I, if the current passed through the electrolyte is so small and exists only for such a short time that the concentration of the electrolyte is not altered, for otherwise we should violate one essential condition of Ohm's law, viz. the state of the conductor must remain constant.

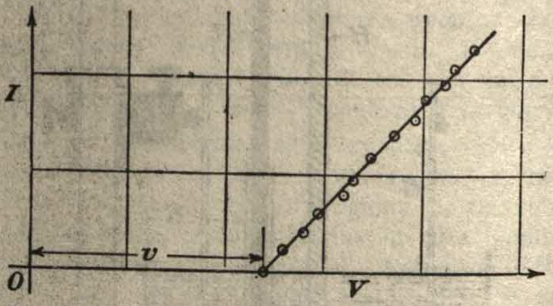

FIG. 18·03.—Characteristic curve for an electrolyte. (Acidulated water.)

With copper electrodes and an aqueous solution of copper sulphate the characteristic obtained in the above manner is a straight line through the origin because the back e.m.f. in this instance is zero.

Experimental determination of the conductivity of an electrolyte.—It has just been shown that Ohm's law is applicable to electrolytes provided that allowance is made for polarization effects. In the early attempts to measure the conductance of an electrolyte many erratic results were obtained and this led to the belief that Ohm's law was not true for electrolytes. The polarization effect, which is the main disturbing factor unless due allowance is made for it, is caused by the gases which are frequently products of electrolysis. These not only increase the resistance between the electrodes of a cell but also set up a counter e.m.f., i.e. one which opposes the passage of the current.

The most satisfactory and almost universal method adopted today for the elimination of polarization effects in conductivity experiments, consists in using a high frequency [1000 cycle.sec.^{-1}] source of potential difference and keeping the current very small. Since the direction of the current is reversed many times per second the polarization due to one pulse of current is completely off-set by the next provided the alternations are symmetrical. This statement is not as concise as one would wish but the justification for the method, which is due to KOHLRAUSCH, is shown by the following approximate analysis. It is assumed that the gas layers give rise to a polarization potential directly proportional to the total charge, q, which exists at a given instant on the positive plate of a fictitious condenser in series with the electrolytic cell, i.e. the instantaneous value of the opposing potential is $\mathrm{P}q$, where P is a constant, and with alternating currents the polarization effects are non-cumulative. If v is the instantaneous value of the applied potential, i is the instantaneous current and R the resistance of the electrolyte,

$$v - \mathrm{P}q = \mathrm{R}i, \quad \text{or} \quad v = \mathrm{R}i + \mathrm{P}q = \mathrm{R}i + \mathrm{P}\int i\,dt.$$

If the applied potential is $v = \hat{\mathrm{V}} \cos \omega t$, we have

$$\hat{\mathrm{V}} \cos \omega t = \mathrm{R}i + \mathrm{P}\int i\,dt.$$

To solve this equation we use the symbolic method, cf. p. 545, and write

$$\vec{\mathrm{V}} = \left(\mathrm{R} + \frac{\mathrm{P}}{j\omega}\right)\vec{\mathrm{I}},$$

where $\vec{\mathrm{V}} = \hat{\mathrm{V}} \exp(j\omega t)\vec{\vartheta}$.

Let $\mathrm{P} = \frac{1}{\mathrm{C}}$, so that $\vec{\mathrm{I}} = \dfrac{\vec{\mathrm{V}}}{\mathrm{R} + \dfrac{1}{j\omega\mathrm{C}}} = \dfrac{\vec{\mathrm{V}}}{\mathrm{R} - j\left(\dfrac{1}{\omega\mathrm{C}}\right)}.$

Hence

$$i = \frac{\hat{V}\,[\exp(j\omega t)\exp j\phi]\ \mathfrak{p}\cdot\mathfrak{p}}{\sqrt{R^2 + \dfrac{1}{\omega^2C^2}}}\left[\text{where } \tan\phi = \frac{1}{\omega CR}\right]$$

$$= \frac{\hat{V}\cos(\omega t + \phi)}{\sqrt{R^2 + \dfrac{1}{\omega^2C^2}}} = \frac{\hat{V}}{\sqrt{R^2 + \dfrac{P^2}{\omega^2}}}\cos(\omega t + \phi).$$

Now the current will be in phase with the applied voltage if $\phi \longrightarrow 0$, i.e. ω must be large. When this is so the term $\dfrac{P^2}{\omega^2} \rightarrow 0$ and the electrolyte behaves as a pure resistance R.

Accordingly, the cell containing the electrolyte is placed in one arm of a Wheatstone bridge and using a supply with a frequency of 1000 cycle.sec.$^{-1}$, $[\omega = 2\pi \times 1000]$, and a suitable detector, such as a telephone receiver, the bridge is balanced. Since the cell behaves as a pure resistance under the conditions stated, the ordinary formula for the comparison of resistances by means of a Wheatstone bridge may be used.

In actual practice a perfect balance is seldom obtained since in addition to the fictitious capacitance C which may be placed in series with a resistance R to represent the effects of polarization in an electrolytic cell, it must be remembered that there is stray capacitance between the electrodes; the effect of this may be represented by placing a condenser of suitable magnitude in parallel with the cell. A more complete analysis is therefore as follows.

ABEF, Fig. 18·04, represents the bridge, the arm AB containing the electrolytic cell whose resistance is to be measured. Such a

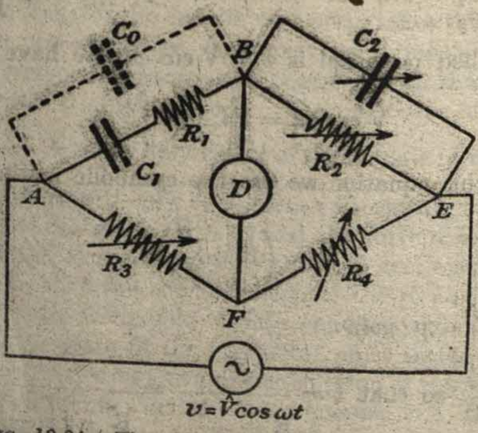

FIG. 18·04.—The theory of a Kohlrausch bridge.

cell may be regarded as equivalent to a resistance R_1 in series with a capacitance C_1 (to represent the polarization effects), the whole being shunted by a capacitance, C_0, to represent any stray capacitance in the cell. To obtain a balance which is almost perfect it is necessary to shunt the arm BE with a variable air-condenser C_2; let this be the value of its capacitance when the bridge is balanced.

If ω is the circular frequency of the applied potential difference, and if Z denotes an impedance operator, we have

$$\frac{1}{Z_1}=\frac{1}{\frac{1}{j\omega C_0}}+\frac{1}{R_1+\frac{1}{j\omega C_1}},$$

so that

$$Z_1=\frac{R_1+\frac{1}{j\omega C_1}}{1+\frac{C_0}{C_1}+j\omega C_0R_1}.$$

Hence, when the bridge is balanced,

$$\frac{\left[\dfrac{R_1+\frac{1}{j\omega C_1}}{1+\frac{C_0}{C_1}+j\omega C_0R_1}\right]}{R_3}=\frac{\dfrac{R_2}{1+j\omega C_2R_2}}{R_4}.$$

Equating in turn the real and imaginary parts, we get

$$R_1R_4+R_2R_4\frac{C_2}{C_1}=R_2R_3\left(1+\frac{C_0}{C_1}\right) \quad . \quad . \quad \text{(i)}$$

and

$$-\frac{1}{\omega^2}=\frac{C_1}{R_4}(C_0R_3-C_2R_4)R_1R_2 \;. \quad . \quad . \quad \text{(ii)}$$

Now it is only when ω is very large that the first term in equation (ii) may be taken as zero; then

$$C_0R_3-C_2R_4=0.$$

When this condition is satisfied, equation (i) gives

$$\frac{R_1}{R_2}=\frac{R_3}{R_4},$$

which is the well-known relationship; but it is only when ω is very large that this equation is valid for use with a Kohlrausch bridge. In precision work it is therefore necessary to investigate experimentally how R_1 apparently depends upon ω and by extrapolating to $\omega \to \infty$ obtain the true value of R_1.

A simple form of Kohlrausch bridge.—As we have seen, a direct current cannot be employed, in general, for the determination of the resistivity of an electrolyte because of the polarization occurring at the electrodes. KOHLRAUSCH used alternating current and designed the following bridge which has been named after him. Fig. 18·05 (*a*) is a diagram showing the principle of the circuit. AB is a uniform wire, S an adjustable known resistance, and K the cell containing electrolyte. Intermittent or alternating current is supplied as indicated. Now Kohlrausch showed that the frequency of the supply must be large, 50 cycle.sec.$^{-1}$ or more, if polarization effects are to be eliminated. A small induction coil was formerly used, but a much more perfect balance is found if the applied voltage is obtained from an oscillator; its wave-form is more nearly sinusoidal. The electrolyte is contained in a glass tube provided with rubber bungs through which the electrodes pass—cf. Fig. 18·05 (*b*). Reference to the bridge shown in Fig. 18·05 (*a*) shows that it is similar to a Wheatstone bridge; since alternating current is used a pair of phones replaces the usual galvanometer. As the jockey moves along the bridge wire the sound in the phones varies. When this sound is a minimum—complete silence is seldom obtained—we have, as shown in the previous paragraph, but with a notation to fit the diagram now used, the ordinary Wheatstone bridge relationship, viz.,

$$\frac{R}{S} = \frac{l_1}{l_2},$$

where R is the resistance of the electrolyte, S a known resistance, and l_1 and l_2 the lengths of the corresponding parts of the bridge wire. [If the sound is a minimum over a range of the bridge wire instead of at some definite point, the extent of this range should be recorded and its centre considered to be the true balance point.]

In practice the above simple circuit is not sensitive since the wire AB almost short circuits the secondary of the induction coil—under such circumstances the rate of supply of energy to the bridge is small. To increase this, AB may be made from very fine wire, but this is not desirable for such wire is easily damaged. The difficulty is overcome by increasing the effective length of AB by placing known resistance coils at its ends as shown in Fig. 18·05 (*d*); generally they are equal, say 10 Ω. To begin the experiment these coils are short circuited and the jockey placed near to the middle of AB. S is adjusted until the bridge is balanced. The short circuiting pieces of wire are then removed and the balance point is accurately determined. This procedure is necessary to ensure that the balance point shall be on AB. Then

$$\frac{S}{R} = \frac{10 + \text{resistance of AD}}{10 + \text{resistance of DB}},$$

so that it is necessary to know the resistance per unit length of the wire AB.

Now, in general, $R = \chi \cdot \frac{l}{A} = \frac{1}{\sigma}\frac{l}{A}$, where the symbols have their usual meanings. In this instance σ cannot be deduced directly because the electrodes do not fit the tube and the lines of current flow, cf. Fig. 18·05 (c), are not linear. To obtain a value for σ, R is determined for various distances, λ, between the electrodes. The relationship is a linear one if λ is not less than about 1·5 diameters of the tube, i.e. $R = \frac{\lambda}{\sigma A} + c$, where c is a constant—cf. Fig. 18·05 (e). The slope of the line gives $\frac{1}{\sigma A}$, so that σ may be obtained.

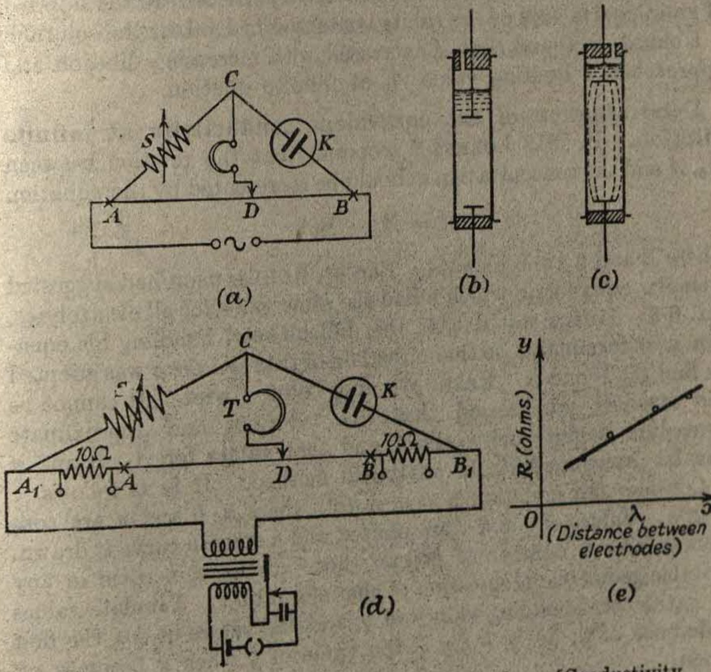

FIG. 18·05.—A simple form of Kohlrausch bridge. [Conductivity of an electrolyte.]

When this bridge is used to determine the conductivity of a copper sulphate solution, copper electrodes may be employed. [In all other cases platinum electrodes, coated with platinum black, should be used.] They must be cleaned with nitric acid and thoroughly washed with distilled water before use. When the first accurate balance point has been obtained the copper electrodes should be again treated in the above manner and then the final observations made. In this way the sharpness of the position for minimum sound in the 'phones is greatly enhanced.

Equivalent conductivity.—The equivalent conductivity, Λ, of an electrolyte is defined as the conductivity divided by the number of gram-equivalents of the solute contained in 1 ml. of the solution.

Kohlrausch found that Λ increased with increasing dilution and approached a limiting value Λ_0 at infinite dilution.

Determination of the equivalent conductivity at infinite dilution.—In 1919 LORENZ [*] proposed that the relation between Λ_0, Λ and the concentration c should be represented by the equation

$$\Lambda = \Lambda_0 - Bc^n,$$

where B and n are constants. Earlier, KOHLRAUSCH had suggested a similar equation in which n had the same value for all electrolytes, viz. 0·5. Lorenz stated that the difficulties of handling his equation were formidable, so that a method of trial and error was adopted to find Λ_0, B and n. FERGUSON and VOGEL [†] wrote 'It cannot be too strongly emphasized that this laborious and approximate procedure is quite unnecessary, and that values for Λ_0, B and n can be determined by an analytical method.' It is as follows.

Consider the equation $y = a - bx^n$, where a, b and n are constants. Values of y and x are plotted and a smooth curve is drawn. Choose now a number of points whose abscissae increase in any suitable geometric progression of common ratio r. Tabulate values of x, the corresponding values of y, and the values of Δy, the first difference of y, i.e. $y + \Delta y$ is the value of y when x becomes rx. Then

$$y = a - bx^n,$$

and

$$y + \Delta y = a - b(rx)^n.$$

$$\therefore\ \Delta y = bx^n(1 - r^n).$$

In practice Δy is found to be negative, say $-\Delta z$. Then, since the logarithm of a negative number is imaginary, we must write

$$\Delta z = bx^n(r^n - 1). \qquad [r > 1].$$

$$\therefore\ \log \Delta z = \log[b(r^n - 1)] + n \log x.$$

Now $\log b(r^n - 1)$ is a constant, so that if the proposed equation is valid a plot of $\log \Delta z$ against $\log x$ should yield a straight line from which the constants b and n may be determined. These being known, from every pair of corresponding values of x and y, the equation

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[*] *Zeit. Anorg. Chem.*, **18, 191, 1919.**

[†] *Phil. Mag.*, **50, 973, 1925.**

$$y + bx^n = a$$

enables one to determine a series of values of a of which the resulting mean value is a reliable one.

Using values obtained by Kohlrausch and Maltby for the equivalent conductivities of different aqueous solutions of potassium chloride, Ferguson and Vogel constructed the curve AA shown in Fig. 18·06. It shows how Λ varies with the concentration.

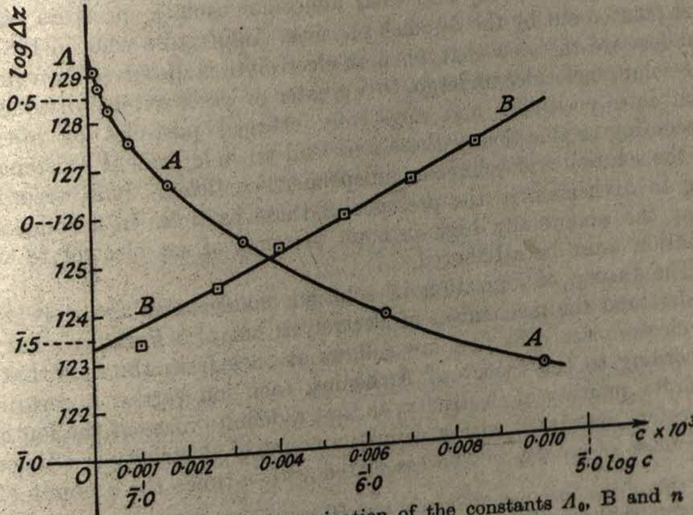

Fig. 18·06.—Graphical determination of the constants Λ_0, B and n in the Lorenz equation $\Lambda = \Lambda_0 - Bc^n$.

The ordinates, Λ, corresponding to a series of abscissae, c, proceeding in a geometrical progression having a common ratio 2 were then read off from the curve AA and tabulated. The first difference $\Delta\Lambda$, $\Delta z = -\Delta\Lambda$, log Δz and log c were tabulated. The following table shows the results of these operations.

$c\left[\dfrac{\text{gm.equiv.}}{\text{ml.}^{-1}}\right]$	Λ	$\Delta\Lambda$	Δz	log Δz	log c
$1\cdot0 \times 10^{-7}$	129·07	− 0·30	0·30	$\bar{1}\cdot4771$	$\bar{7}\cdot0000$
$2\cdot0 \times 10^{-7}$	128·77	− 0·48	0·48	$\bar{1}\cdot6812$	$\bar{7}\cdot3010$
$4\cdot0 \times 10^{-7}$	128·29	− 0·69	0·69	$\bar{1}\cdot8389$	$\bar{7}\cdot6021$
$8\cdot0 \times 10^{-7}$	127·60	− 0·93	0·93	$\bar{1}\cdot9685$	$\bar{7}\cdot9031$
$1\cdot6 \times 10^{-6}$	126·67	− 1·23	1·23	0·0899	$\bar{6}\cdot2041$
$3\cdot2 \times 10^{-6}$	125·44	− 1·67	1·67	0·2227	$\bar{6}\cdot5052$
$6\cdot4 \times 10^{-6}$	123·77				

A plot of $\log \Delta z$ against $\log c$ gives the straight line BB, Fig. 18·06, and this, in turn, gives $n = 0{\cdot}452$, $b = 1390$; thus $\Lambda_0 = 130{\cdot}04$ ohm.$^{-1}$cm.^{2}(gm.-equiv.)$^{-1}$.

The theory of electrolytic dissociation.—Substances which yield aqueous solutions with a van't Hoff factor* equal to unity, i.e. their solutions have a normal osmotic pressure, freezing point, etc., are found to be non-electrolytes whereas salts, acids and alkalis yield solutions with abnormal osmotic pressure, etc., and are found to be electrolytes.

The intimate relationship between the conducting power of solutions for electricity and their abnormal osmotic pressure was first pointed out by the Swedish physicist ARRHENIUS who, in 1887, put forward the view that when an electrolyte is dissolved in water the solute molecules undergo, to a greater or lesser extent, dissociation into positively and negatively charged particles or ***ions.*** According to this theory these ions lead an independent existence in the solution and behave as independent particles. It is, according to Arrhenius, to the presence of these particles in a solution that the abnormally high osmotic pressure of an electrolyte in solution must be attributed.

The theory of ionization in solution enables one not only to understand the mechanism of electrolysis but also Faraday's laws of electrolysis. The first law follows at once from the fact that, according to the theory of Arrhenius, each ion carries a certain definite quantity of electricity, so that a definite mass of the ion is therefore associated with a definite amount of electricity: n-times the mass of an ion will be associated with n-times the amount of electricity and, consequently, the amount of electricity which is transported by the ions through the solution will be directly proportional to the mass of ions discharged at the electrode—in other words the amount of electrolyte decomposed will be proportional to the quantity of electricity which has passed through it.

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* One of the fundamental laws of osmosis is that equimolecular solutions of different substances have the same osmotic pressure and since the depression of the freezing point of a solvent is directly proportional to the osmotic pressure of the solution, it may at once be stated that equimolecular solutions of different substances in water have the same freezing point. Similarly, for the elevation of the boiling point of a solution. These rules, however, cannot be applied directly to solutions of acids, alkalis, and salts.

According to VAN'T HOFF, the osmotic pressure of a solution can be represented not by the simple equation $pV = RT$ (we consider one mole of dissolved substance), but by $pV = iRT$, where i is an empirical factor which varies with the concentration of the dissolved substance. It is known as the ***van't Hoff coefficient.*** In many instances it is found that $i \longrightarrow 1$, but for acids, alkalis and salts i is found to be much greater than unity. This means that the osmotic pressure of solutions of acids, alkalis and bases, determined more especially by freezing-point determinations, is greater than the osmotic pressure of equimolecular solutions of a 'normal' substance such as cane sugar, a 'normal' substance being defined in this connexion as one which in dilute solution obeys the simple law $pV = RT$. The van't Hoff coefficient i therefore represents the ratio of the observed osmotic pressure to the normal osmotic pressure, or $i = \dfrac{p_{\text{obs.}}}{p_{\text{nor.}}}$.

To account for the second law of electrolysis let us consider the deposition of copper and silver by electrolysis ; these are respectively divalent and monovalent elements. Now each gram-atom of an element contains the same number of atoms, N, where N is Avogadro's constant. Thus, if the charge, in coulombs, carried by each silver ion is ε, the charge needed to deposit one gram-atom of silver will be $N\varepsilon$, and since silver is monovalent this is equal to F, i.e. to one faraday. In the case of copper each ion carries a charge 2ε, so that in order to deposit one gram-atom of copper a charge $2N\varepsilon$ will be required, i.e. a charge $N\varepsilon$ will deposit $\frac{1}{2}(A_{Cu})$ gm. of copper, where A_{Cu} is the mass in gm. of copper numerically equal to its atomic weight. In general, for a substance of atomic weight A and valency ζ, the passage of a charge $N\varepsilon$ (or $N\epsilon$) will liberate $\frac{1}{\zeta}A$ grams of substance ; this amount is called a gram-equivalent. Hence a faraday will cause the deposition of one gram-equivalent from solution, i.e. equal quantities of electricity will deposit amounts of different substances proportional to their chemical equivalents.

Some objections to the Arrhenius theory of electrolytic dissociation.—When the theory of electrolytic dissociation was first made known considerable opposition was at once raised against it, mainly on account of the novel, and at that time revolutionary, ideas contained in it. The opponents of the theory maintained that it was impossible for elements such as sodium and potassium, which normally react with water most vigorously, to exist as ions in water and remain quiescent. Also, the production of sodium chloride from its elements was an exothermic reaction of the first magnitude so that the opponents of the dissociation theory maintained that the mere dissolution of this substance in water could not possibly bring about the disintegration of molecules in this way. All such difficulties are at once removed when it is realized that the acquisition of an electric charge by an atom (due to the gain or loss of one or more electrons) is accompanied by very marked changes in its chemical properties. According to modern atomic theory, which attributes the chemical properties of a substance to the outermost shell of electrons surrounding the positively charged nucleus, a sodium ion is more stable than a sodium atom. In this particular instance there are eleven electrons surrounding the nucleus; two in the shell nearest to the nucleus, eight in the next and one in the outermost and incomplete shell—sodium is monovalent. When a sodium atom loses its outermost electron a positive ion is formed and the electronic configuration then simulates that of a neon atom. On the other hand, chlorine requires an additional electron in order for its electronic configuration to consist of complete shells; when this electron is acquired the negatively charged chlorine ion simulates argon. It is for this reason that sodium and chlorine atoms combine with vigour and in the process liberate much thermal energy; the final result is that unstable atoms are converted into stable ions. Crystal analysis, by means of X-rays, confirms the view that it is the ions which are arranged at the corners of a space lattice and not the atoms or molecules.

TRANSPORT NUMBERS AND IONIC MIGRATION

The migration of ions.—The mobility of an ion is defined as its speed per unit fall in potential per unit distance. If the speed is u cm.sec.$^{-1}$, and the space rate of fall of potential E volt.cm.$^{-1}$, then the mobility, μ, is given by

$$\mu = \frac{u \text{ cm.sec.}^{-1}}{\text{E volt.cm.}^{-1}} = \frac{u}{\text{E}} \text{cm.}^2\text{sec.}^{-1}\text{volt.}^{-1}.$$

We shall use μ_+ and μ_- to refer to the cations (+) and anions (−) respectively. Also since the ions travel in contrary directions it will be correct to refer to the speeds of the ions and not to their velocities. In 1853 Hittorf showed that the mobilities of ions could be correlated with the concentration changes occurring in the immediate neighbourhood of the electrodes. The faster moving ion will cause a greater reduction in concentration in the electrode compartment from which it moves away than will the slower ion.

Almost invariably it is found that in any electrolyte the two ions move with different mobilities, yet equivalent quantities of each ion are liberated at the respective electrodes, as required by Faraday's laws of electrolysis.

Kohlrausch's expression for the conductivity of an electrolyte.—Suppose that a solution contains x gm. of a substance whose molecular weight is M, dissolved in y ml. of solution. Then the concentration is $\frac{x}{y\text{M}}$ mole.ml.$^{-1}$. Call this N mole.ml.$^{-1}$. Suppose that of this number a fraction α is ionized where $0 < \alpha < 1$, and that each mole which is ionized splits up into two gm.-ions each of valency ζ. Each gm.-ion will carry a charge numerically equal to ζF, where F is the charge in coulombs carried by a gm.-ion of a monovalent substance, or the charge necessary to deposit one gm.-equivalent of any element. Let $|\mu_+|$ and $|\mu_-|$ be the numerical values of the mobilities of the cation and anion respectively, i.e.

their speeds under a space rate of fall of potential numerically equal to 1 volt.cm.$^{-1}$. If the space rate of fall in potential is $\left|\frac{\partial V}{\partial x}\right|$ volt.cm.$^{-1}$, then the speeds of the ions will be

$$|\mu_+|\cdot\left|\frac{\partial V}{\partial x}\right| \quad \text{and} \quad |\mu_-|\cdot\left|\frac{\partial V}{\partial x}\right|,$$

respectively. Hence the quantity of electricity passing per second across a small area, S, normal to the direction of the flow of electricity is

$$\alpha N\,\zeta FS\Big[|\mu_+|+|\mu_-|\Big]\cdot\left|\frac{\partial V}{\partial x}\right|.$$

This is the current, I, in amperes. Hence

$$I = \alpha N\,\zeta FS\Big[|\mu_+|+|\mu_-|\Big]\cdot\left|\frac{\partial V}{\partial x}\right|,$$

$$= \sigma S\left|\frac{\partial V}{\partial x}\right|,$$

where σ is the conductivity in ohm.$^{-1}$cm.$^{-1}$.

$$\therefore\ \sigma = \alpha N\Big[|\mu_+|+|\mu_-|\Big]\zeta F \text{ ohm.}^{-1}\text{cm.}^{-1}.$$

Experimental determination of the degree of dissociation in an electrolyte.—Since ζN is the number of gram-equivalents per millilitre of solution, we have, for the equivalent conductivity

$$\Lambda = \frac{\sigma}{\zeta N} = \alpha\Big[|\mu_+|+|\mu_-|\Big] F \text{ ohm.}^{-1}\text{cm.}^{-1}\,(\text{gm.equiv.})^{-1}\text{ ml.}$$

If at infinite dilution $\alpha = 1$, then

$$\Lambda_0 = \Big[|\mu_+|+|\mu_-|\Big]F,$$

so that
$$\alpha = \frac{\Lambda}{\Lambda_0}.$$

Thus we have a means of finding experimentally a value for α, the so-called ***degree of ionization*** or ***degree of dissociation.***

It has just been pointed out that the equation $\Lambda = \alpha\Lambda_0$ enables us to find a value for α, but the significance of the value so obtained is obscure unless resort is made to the modern theory of complete dissociation—cf. p. 675. Before that theory came into being the chief use of α was that it enabled electrolytes to be divided into two classes, known respectively as 'strong electrolytes' and 'weak electrolytes'.

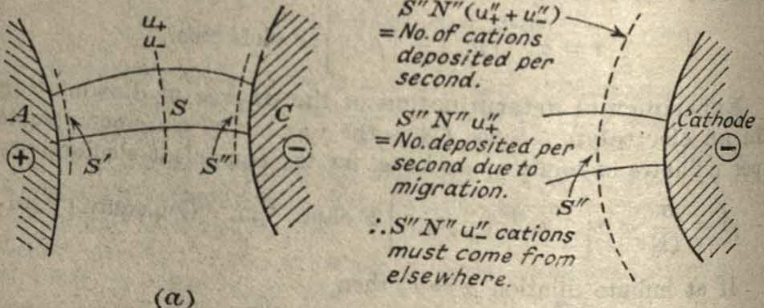
(*a*) In strong electrolytes $\alpha \rightarrow 1$ and α does not vary much with concentration.

(*b*) In weak electrolytes α varies considerably with concentration and at high concentrations $\alpha \to 0$.

All the more common salts and mineral acids belong to the first class, while organic acids and bases, together with some salts of the heavy metals, belong to the second. The classification is, however, only permissible when the solvent is water for the nature of the solvent is found to affect the 'strength' of an electrolyte.

Ionic migration and concentration changes.—Hittorf's investigation of the part played by each ion in the transport of electricity through a solution has been elucidated by S. W. J. Smith * in the following manner.

(*a*) ***Insoluble electrodes :*** Consider the electrolysis of a binary electrolyte, MA, which in solution yields ions of the same valency ζ. Let A and C, Fig. 18·07 (*a*), be parts of the anode and cathode, respectively, the curved lines between them representing a portion of the solution bounded by current stream lines, i.e. lines along which the current is supposed to flow. Then the amount of electricity passing per second across any section normal to the axis of the tube must be constant for a given tube, as otherwise there would be an accumulation of free electricity in different parts of the tube. Let S be any such cross-section of the tube, S′ and S″ being sections very close to the electrodes. Suppose that the potential gradient at S is such that the *speeds* of the cations and anions are u_+ and u_- respectively. Let N be the number of gram-ions of each kind per unit volume near S. Let u_+', u_-' and N′, u_+'', u_-'' and N″ be the corresponding quantities for the sections S′ and S″. If F is the charge carried by a monovalent gram-ion, or gram-equivalent, of



u_+
u_-
A
S
C
+
S′
S″
−
$S''N''(u_+'' + u_-'')$
= No. of cations deposited per second.
$S''N''u_+''$
= No. deposited per second due to migration.
Cathode
−
S″
$\therefore S''N''u_-''$ cations must come from elsewhere.

(*a*) (*b*)

FIG. 18·07.—Ionic migration and concentration changes near the electrodes.

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* *Proc. Phys. Soc.*, 28, 148, 1916.

any metal, then in order that there shall not be any accumulation of free electricity at any part of the tube we must have

$$\begin{aligned} S(u_+ + u_-)N\zeta F = S'(u_+' + u_-')N'\zeta F &= S''(u_+'' + u_-'')N''\zeta F \\ &= \xi \text{ (say).} \end{aligned}$$

If it is further assumed that the ionic speeds are directly proportional to the potential gradients, i.e. the mobilities of the ions are constants. Then

$$\frac{u_+}{u_-} = \left|\frac{\mu_+}{\mu_-}\right| = \frac{u_+'}{u_-'} = \frac{u_+''}{u_-''}$$

and hence $$1 + \frac{u_+}{u_-} = 1 + \frac{u_+'}{u_-'} = 1 + \frac{u_+''}{u_-''} = \beta \text{ say.}$$

$$\therefore\ SN\zeta F u_- = \frac{\xi}{\beta} = \text{const.}$$

i.e. $$SNu_- = S'N'u_-' = S''N''u_-''.$$

Similarly $$SNu_+ = S'N'u_+' = S''N''u^{+''}.$$

Since the current flowing across any section of the tube is $S(u_+ + u_-)N\zeta F$ the number of gm.-cations deposited per second on the cathode is

$$S''N''(u_+'' + u_-'') = SN(u_+ + u_-).$$

Similarly at the anode

$$S'N'(u_+' + u_-') = SN(u_+ + u_-).$$

But the number of gm.-cations brought up to the cathode per second by migration, i.e. by virtue of the current, is $S''N''u_+''$. Hence $S''N''u_-''$ gm. cations must appear in some other way. In addition, at the section S'', cf. Fig. 18·07 (*b*), there must migrate towards the anode $S''N''u_-''$ gm.-anions per second. Thus the net result is that $S''N''u_-''$ gm.-cations and $S''N''u_-''$ gm.-anions must be found. They are taken from the $S''N''u_-''$ gm.-mol. of salt present in the vicinity of the cathode, i.e. there is a loss of solute from the solution in the neighbourhood of the cathode which amounts to $S''N''u_-''$ or SNu_- mole.sec.$^{-1}$.

In a similar manner there is lost from the solution near the anode SNu_+ mole.sec.$^{-1}$. Hence, if the current passes for some time, we have

$$\frac{\text{Loss in concentration of solute round anode}}{\text{Loss in concentration of solute round cathode}} = \left|\frac{u_+}{u_-} = \frac{\mu_+}{\mu_-}\right|.$$

It should be noted that the loss from the electrolyte in the immediate neighbourhood of an electrode is directly proportional to the speed of the ion which is *leaving* it.

The above equation involves the assumption that there is a middle compartment in which the concentration remains unchanged and this implies that the anode and cathode compartments do not 'overlap'. Such overlapping might occur if the electrolysis were prolonged or if the solution were disturbed mechanically.

(*b*) ***Soluble electrodes.*** It must be pointed out that these reductions in concentration of the electrolyte only occur when insoluble electrodes are used, or rather, when the metallic radical of the solute is different from that constituting either of the electrodes, e.g. in the electrolysis of a silver nitrate (or copper sulphate) solution between platinum electrodes. If the anode material and the metallic ion are both of the same metal, e.g. a copper electrode dipping into a copper sulphate solution, or a silver electrode in silver nitrate solution, the result is different. The silver anode loses a mass of silver equal to that deposited upon the cathode, so that while $SN(u_+ + u_-)$ gm.-cations are being deposited on the cathode, the same number leaves the anode and enters the solution as cations. Of these SNu_+ migrate towards the cathode and SNu_- remain within the vicinity of the anode. In addition SNu_- gm.-anions enter the neighbourhood of the anode per second and thus SNu_- moles of silver nitrate accumulate per second in this vicinity. Therefore, there is a gain in concentration of the solute round the anode equal to SNu_- moles per second, and a corresponding loss round the cathode. The total mass of solute in solution remains unchanged.

During the electrolysis of an aqueous solution of silver nitrate between silver electrodes, the mass of silver ions deposited on unit area of the cathode per second will be $N(u_+ + u_-)$ gm.-ions and the mass of metal contained in the salt molecules carried away from the cathode space in the same time is Nu_-. Hence we have

$$\frac{\text{Mass of metallic ion (i.e. the cation) lost from vicinity of cathode}}{\text{Mass of metallic ion deposited on cathode}} = \frac{u_-}{u_+ + u_-}.$$

Transport numbers.—The quantity of electricity carried by the cations per second across a fixed boundary is directly proportional to their speed, u_+ and the quantity carried by the anions in a contrary direction across the same boundary in the same time is directly proportional to u_-. Hence the total current is proportional to $u_+ + u_-$. The fraction of the total current carried by the cations is $\frac{u_+}{u_+ + u_-}$. This fraction is termed the ***transport number*** of the cation and is denoted by t_+. Similarly, for the anion $t_- = \frac{u_-}{u_+ + u_-}$. Hence $t_+ + t_- = 1$.

Finally, it must be noted that

$$t_- = \frac{\text{Mass of metallic ion lost from electrolyte near cathode}}{\text{Total mass of metallic ion deposited on the cathode}}.$$

The experimental determination of transport numbers.—The principle of the method devised by Hittorf is to allow a quantity of electricity to pass through the electrolyte of known concentration and after the passage of the electricity the concentrations at the anode and cathode are determined by chemical methods. Since it is essential that the electrolysis should be stopped before the middle portion of the electrolyte is affected by concentration changes it is customary to remove some of the liquid from the middle portion for analysis. In this way it is demonstrated that the anode and cathode compartments do remain quite distinct. Precautions must always be taken against mixing of the electrolyte which may be caused by mechanical vibrations, convection currents due to temperature variations, or by diffusion. Originally short and wide tubes were used with porous partitions inserted to prevent mixing. It was soon discovered that these partitions were not without effect on the values of the mobilities and hence of the transport numbers of the ions and so special apparatus has been designed in which mixing is avoided without the use of membranes. Moreover, in order that the changes in concentration shall be localized in the vicinity of the electrodes, the current must be kept small and the time the experiment lasts must not be prolonged.

Findlay's apparatus (a modified form of one due to Hittorf) is shown in Fig. 18·08; it will be supposed that a determination of the transport numbers of the silver ion and nitrate ion in an aqueous solution of silver nitrate is contemplated.

A and C are two silver electrodes fixed, by means of corks, in the outer tubes of the apparatus. These electrodes consist of pieces of thick silver rod attached to copper leads, the latter being cemented into glass tubes so that only the silver is in contact with the $\frac{N}{20}$ $AgNO_3$ solution which fills the apparatus. The central portion, M, of the apparatus is attached by means of rubber tubes (with clips) to the rest of the apparatus; this enables M to be removed easily after the completion of the experiment and is a distinctive feature of the apparatus.

The apparatus is connected in series to a 30 V. battery, sliding resistance, milliammeter, etc., and a current of 10 mA. passed through. The current should be measured, as before, by using a Weston cell to balance the potential difference across a selected known resistance (100 Ω); in this way it is possible to observe small variations in the current and to eliminate them. Instead of determining the quantity of electricity which passes in the course of an experiment and calculating the increase in mass of the cathode, this increase may be determined directly.

At the end of the experiment the rubber tubes are closed by means of the clips provided and the anode liquid run off into a weighed flask. The anode vessel and the anode itself are washed with some of the original solution and the washings added to the liquid in the flask. The composition of the solution is determined by a titration method. The solution in M is removed and its composition determined; if it is not the same as that of the original solution the experiment has been allowed to proceed for too long a time and a repetition is therefore necessary. The method of calculation is made clear by the following example.

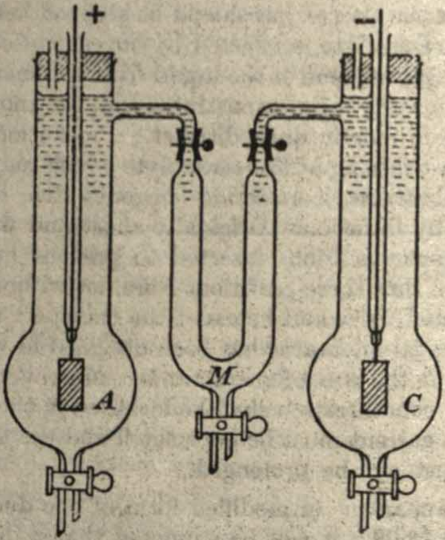

FIG. 18·08.—Findlay's apparatus for determining transport numbers.

Example.—A 0·1004N solution of silver nitrate was electrolysed between pure silver electrodes for several hours. The mass of the cathode increased by 0·1016 gm., and when 121·90 ml. of the solution in the cathode compartment were titrated with 0·100N KCNS the concentration was found to be 0·09429N. The anode compartment contained 109·42 ml. of 0·1029N silver nitrate. Calculate a value for the transport number of a silver ion in this solution.

The mass of silver originally present in the liquid drawn from the anode compartment = (0·10942 × 0·1004 × 108) = 1·1865 gm.

Into this liquid there had passed from the anode 0·1016 gm. of silver. Hence the total mass of silver to be expected in the anode compartment is $(1{\cdot}1865+0{\cdot}1016)=1{\cdot}2881$ gm. The mass of silver actually present is

$$0{\cdot}10942 \times 0{\cdot}1029 \times 108 = 1{\cdot}2160 \text{ gm.}$$

∴ loss of silver from the anode compartment is

$$1{\cdot}2881 - 1{\cdot}2160 = 0{\cdot}0721 \text{ gm.}$$

The loss of silver from the cathode compartment is

$$0{\cdot}12190 \times (0{\cdot}1004 - 0{\cdot}0943) \times 108 = 0{\cdot}0803 \text{ gm.}$$

$$t_+ = \frac{\text{Loss of silver from anode compartment}}{\text{Loss of silver from both compartments}} = \frac{0{\cdot}0721}{0{\cdot}0721 + 0{\cdot}0803}$$
$$= 0{\cdot}473.$$

The moving boundary method.—A second method available for the determination of ionic speeds depends on observations of the actual rate of motion of the ions as they move along under the influence of an electric field. The idea was originated by OLIVER LODGE who endeavoured to follow the movements of certain ions by utilizing their reactions with certain chemical indicators. The principle of the method is as follows. Suppose that an aqueous solution of a salt MA is to be investigated; then two other salts M_1A and MA_1, each in aqueous solution, are required. It will be noticed that each of these two latter salts has one ion in common with the first salt MA. These two solutions are referred to as 'indicator solutions' and they occupy, respectively, the upper and lower portions of the vessel shown in Fig. 18·09, the space between the indicator solutions being filled with the prepared solution of the salt MA. Some care is necessary to fill the apparatus so that the boundaries shall be sharp.

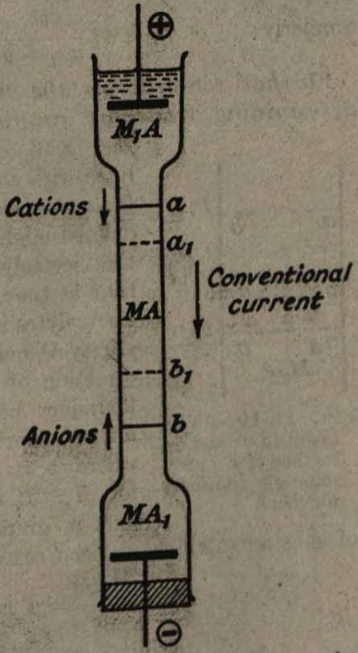

FIG. 18·09.—The principle of the moving-boundary method for determining ionic speeds.

In the diagram *a* and *b* represent the initial boundaries between the three solutions; in order that these boundaries shall remain well defined it is necessary that the speed of M shall be greater than that of M_1 and that the speed of A shall exceed that of A_1. When these conditions apply the boundaries do not become indistinct since M_1 ions do not overtake M ions or A_1 ions overtake A ions. It will be observed that in a given time the boundary a advances to a_1 while the boundary b advances to b_1. Assuming that the potential gradient is constant throughout that portion of the apparatus occupied by MA, the distances aa_1 and bb_1 will be directly proportional to the speeds u_+ and u_- of the ions M and A respectively. Thus

$$\frac{u_+}{u_-} = \frac{aa_1}{bb_1},$$

or
$$t_+ = \frac{u_+}{u_+ + u_-} = \frac{aa_1}{aa_1 + bb_1}.$$

Similarly
$$t_- = \frac{bb_1}{aa_1 + bb_1}.$$

Further remarks on the moving boundary method for determining transport numbers. Consider a vertical tube, Fig. 18·10, containing two solutions of electrolytes with a common negative constituent A. It is essential that there should be a well-defined boundary between the solutions. Let the electrolytes be M_1A and M_2A and suppose that at time $t = 0$ the boundary between the electrolytes is a_0b_0. When a quantity of electricity Q has been passed (the conventional direction of the current is indicated) let the boundary be at ab. Then the mass of M_1 ion transported by the current will be t_+zQ, where z is the electrochemical equivalent for M_1. If c is the concentration of these ions in grams per millilitre, the above mass of substance will have been removed from a volume

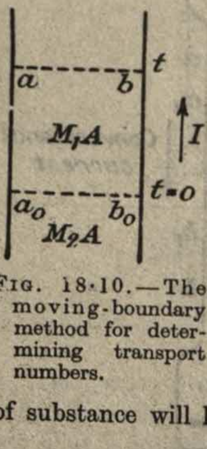

FIG. 18·10.—The moving-boundary method for determining transport numbers.

$$\frac{t_+zQ}{c} = v \text{ (say)}.$$

$$\therefore t_+ = \frac{cv}{zQ} = \frac{cv}{z\int_0^t \mathrm{I}\,dt}.$$

If C is the concentration of M_1 expressed in gram-equivalents per litre, ζ the valency of M_1 and V the volume in litres from which the mass t_+zQ of M_1 ions are removed in time t,

$$\mathrm{C} = \frac{1000c}{\mathrm{M}_0 \div \zeta},$$

where M_0 is the atomic weight of M_1, and $V = \dfrac{v}{1000}$.

Now the charge associated with 1 gm.-equivalent of M_1 is F coulombs, so that

$$zF = \text{mass of 1 gm.-equivalent of } M_1 = \frac{M_0}{\zeta}.$$

$$\therefore t_+ = \frac{C(M_0 \div \zeta)}{1000}\cdot\frac{1000V}{(M_0 \div \zeta)}\cdot\frac{F}{\int_0^t I\,dt},$$

$$= \frac{CVF}{\int_0^t I\,dt}.$$

Experimental determination of the transport number of a hydrogen ion.—The apparatus consists of a graduated glass tube G, Fig. 18·11 (*a*), about 3 mm. in diameter and sealed to a wider portion B.

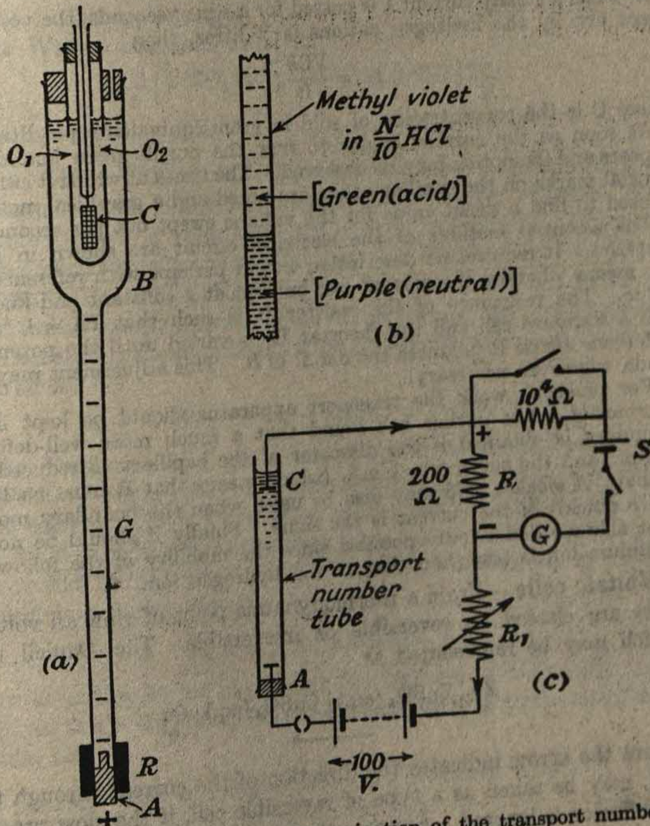

FIG. 18·11.—Experimental determination of the transport number for hydrogen.

A is the anode; this is a rod of cadmium turned to fit G and held in position by rubber tubing R. C is a silver chloride electrode prepared by depositing silver on a platinum gauze fused to a thick platinum wire which, in turn, is sealed into a glass supporting tube. The silver-coated gauze is then made the anode in a potassium chloride solution. The chlorine liberated by electrolysis reacts with the silver to form silver chloride which remains attached to the electrode. This electrode, when mounted in the transport apparatus, is surrounded by a wider glass tube with two apertures, O_1 and O_2. Concentration changes round the cathode cannot then affect the electrolyte within the capillary tube.

The transport apparatus is filled with 0·1N HCl solution which contains a trace of methyl violet; the solution is faintly green. A current of 5 mA. is passed at once (so that the cadmium shall not dissolve appreciably). Chlorine ions are discharged at A and form cadmium chloride, while hydrogen ions at C form HCl. Now cadmium chloride forms a neutral solution in which methyl violet is purple, so that as the hydrogen ions travel upwards the colour changes from green to purple —cf. Fig. 18·11 (*b*).

If, when a steady current I is passed for a time *t* seconds, the volume swept out by the hydrogen cations is V litres, then

$$t_+ = \frac{\text{VCF}}{\text{I}t},$$

where C is the concentration of acid in gram-equivalents per litre.

As soon as the current begins to flow the purple-green boundary appears and its motion may be observed. The times at which it crosses fiducial marks on the capillary tube are noted and a graphical method is used to find a mean value for the volume swept out per second.

The essential features of the electrical circuit are shown in Fig. 18·11 (*c*). It requires no description except perhaps with reference to the means whereby the current can be kept at a constant and known value. The resistance of the resistor R is such that RI ≒ 1 volt. S is a standard cell and the rheostat R_1 is varied until the potential difference across R balances the e.m.f. of S. This adjustment may be made whenever necessary.

For precision work the transport apparatus should be kept in a thermostat. The author has found that a much more well-defined boundary is obtained if the diameter of the capillary is reduced to 1 mm., and the current to 1 mA. (which means that R must be 1000 ohms). A weaker acid may also be used, when the boundary moves more quickly if the current is the same. Finally it should be noted that this method is only possible since the mobility of the following cadmium ion is less than that of the hydrogen ion.

Voltaic cells.—From a thermodynamic point of view all voltaic cells are classed as reversible or irreversible. The Daniell cell which may be represented as

$$\underset{\ominus}{\text{Zn}} \;\Big|\; \text{ZnSO}_4\,(\text{aq.}) \;\Big\vdots\; \text{CuSO}_4\,(\text{aq.}) \;\Big|\; \underset{\oplus}{\text{Cu}}$$
$$\longrightarrow$$

where the arrow indicates the direction of the current through the cell, may be taken as a type of reversible cell, if the slow process of diffusion is ignored. Suppose that a potential difference from an external source is applied to the cell so that it balances exactly the e.m.f. of the cell. If the applied potential difference is reduced by a very small amount, the cell supplies a current in the conventional direction and the ordinary chemical changes occur, viz. zinc passes into solution and copper is deposited on the anode. On the other hand, if the applied potential difference exceeds the e.m.f. of the cell by a small amount, a current flows in the opposite direction and all the chemical changes are reversed, copper passing into solution from the copper plate while at the same time zinc is deposited on the zinc plate.

When a reversible cell is supplying current, the source of energy is the chemical reaction. If the same amount of chemical action were to take place in a calorimeter, without the performance of external work, the calorimetric heat evolved is a measure of the change in internal energy.

The Weston standard cell, viz.

$$\underset{\ominus}{\mathrm{Cd}} \mid \mathrm{CdSO_4\,(aq.)},\quad \mathrm{Hg_2SO_4\,(solid)} \mid \underset{\oplus}{\mathrm{Hg}}$$

is also a reversible cell.

The Gibbs-Helmholtz equation for a reversible cell.—Helmholtz and Kelvin, independently of one another, first suggested that in a voltaic cell the energy evolved in the chemical action associated with the supply of current from the cell, is a direct measure of its e.m.f. According to this hypothesis the whole of the chemical energy appears as electrical energy and when applied to a Daniell cell the hypothesis finds ample support. The reaction involved is

$$\mathrm{Zn + CuSO_4 \rightarrow Cu + ZnSO_4} + 50{,}000 \text{ cal.}$$

i.e. the dissolution of 1 gm.-atom of zinc is accompanied by the deposition of 1 gm.-atom of copper together with the liberation of 50,000 calories. Since 1 gm.-atom of zinc is 2 gm.-equivalents, the quantity of electricity passing through the cell must be two faradays, i.e. 2 × 96,500 coulombs. If E is the e.m.f. of the cell in volts, the electrical energy supplied by it is

$$E \times 2 \times 96{,}500 \text{ joule.}$$

Now according to the hypothesis put forward by Helmholtz and by Kelvin this energy is equal to the energy evolved in the chemical reaction, i.e.

$$E \times 2 \times 96{,}500 = 50{,}000 \times 4{\cdot}184,$$

or

$$E = 1{\cdot}09 \text{ volt.}$$

Now the e.m.f. of a Daniell cell is about 1·09 volt so that the hypothesis appears to be an acceptable one, but when applied to other reversible cells large discrepancies occur. GIBBS and HELMHOLTZ pointed out the error in the argument used by Kelvin, viz. the neglect of any energy exchange between the cell and its surroundings; the Kelvin equation is only valid when this exchange is zero. The equation which Gibbs and Helmholtz obtained may be established as follows.

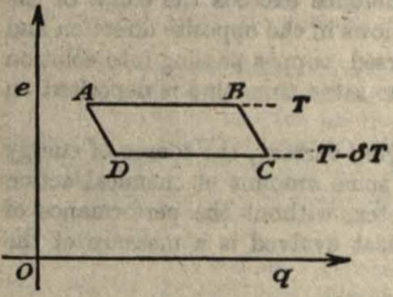

FIG. 18·12.—The Gibbs-Helmholtz equation for a reversible cell.

Imagine a reversible cell (balanced by an equal and opposite e.m.f.) in an enclosure at absolute temperature T; let e be its e.m.f. at this temperature. If the balancing e.m.f. is reduced slightly the cell will supply a small electric current. Let Δq be the finite charge thus supplied in a short time Δt, i.e. before the circuit is broken, and suppose that, in order for the temperature of the cell to remain constant, it takes in heat from its surroundings. The change in state of the cell may be represented on the indicator diagram shown in Fig. 18·12 by the line AB, which is parallel to Oq.

Now the heat taken in from the surroundings

= heat absorbed by the reaction associated with the supply of the charge Δq when the reaction takes place under such conditions that no other form of energy is liberated,

+ the energy gained by the charge Δq as it is raised from the potential of the cathode (zero) to that of the anode (e).

Thus,

$$\text{Heat (in work units) taken in} = \xi\,\Delta q + e\,\Delta q,$$

where ξ is the heat *absorbed* per unit charge supplied by the cell under the conditions stated. [ξ is the work equivalent of the calorimetric heat *absorbed* when an amount of chemical action corresponding to unit transfer of electricity takes place.]

Now let the circuit be remade and a further but very small quantity of electricity be supplied by the cell, the supply of energy from the surroundings being prevented; the temperature of the cell will fall to T − δT and the state of the cell will be represented by the point C on the diagram.

Then let the opposing e.m.f. be raised so that a charge Δq passes in the opposite direction through the cell, when

$$\text{Heat taken in} = -(\xi - \delta\xi)\Delta q - (e - \delta e)\Delta q,$$

where $\xi - \delta\xi$ and $e - \delta e$ are the new values of these variables when the temperature is T − δT. Let the reversible cycle be completed along DA, no exchange of heat with the surroundings occurring.

Since the cycle is reversible, $\oint \frac{dQ}{T} = 0$, where dQ is a small quantity of heat. Thus

$$\frac{\xi + e}{T}.\Delta q = \frac{(\xi - \delta\xi) + (e - \delta e)}{T - \delta T}.\Delta q.$$

Since $[T - \delta T]^{-1} \doteqdot T^{-1}[1 + T^{-1}.\delta T]$, we have

$$\frac{\xi + e}{T} = \frac{1}{T}\left[(\xi - \delta\xi + e - \delta e)\left(1 + \frac{\delta T}{T}\right)\right]$$

$$= \frac{1}{T}\left[\xi - \delta\xi + e - \delta e + \frac{\xi}{T}\delta T + e\frac{\delta T}{T}\right],$$

if second order terms are neglected.

$$\therefore 0 = -\frac{\delta\xi}{T} - \frac{\delta e}{T} + \frac{\xi}{T^2}\delta T + \frac{e}{T^2}\delta T.$$

$$\therefore e = -\xi + T\left[\frac{\partial e}{\partial T} + \frac{\partial \xi}{\partial T}\right].$$

If $\frac{\partial \xi}{\partial T}$ is negligible, we have

$$e = -\xi + T\frac{\partial e}{\partial T}.$$

This is the ***Gibbs-Helmholtz equation*** for a reversible cell. Since it has been tacitly assumed that no gases are evolved, i.e. the volume of the system remains constant, one ought to write

$$e = -\xi + T\left(\frac{\partial e}{\partial T}\right)_v.$$

Finally, it must be noted that we have assumed that the resistance of the cell is so low that the Joule heating effect, which is irreversible, may be neglected; also that diffusion between the electrolytes is negligible, for this is also an irreversible process.

The equation $e = -\xi + T\frac{\partial e}{\partial T}$ is true in any consistent set of units, but it is convenient to measure the e.m.f. in volts, say V, and to consider the heat associated with the production of charge when a gram-atom of the electrode element with the higher valency (ζ) and corresponding quantities of the other elements taking part in the cell reaction are involved. Let λ be the heat, in calories, absorbed when the above change occurs in a calorimeter and without the performance of external work. The charge transferred in the equivalent reaction in the cell will be ζF coulombs, where F is one faraday, so that

$$\xi = \frac{4{\cdot}18\lambda}{\zeta \mathrm{F}},$$

i.e.

$$\mathrm{V} = \frac{-\,4{\cdot}18\lambda}{\zeta \mathrm{F}} + \mathrm{T}\frac{\partial \mathrm{V}}{\partial \mathrm{T}}.$$

[N.B. Almost invariably it is the negative electrode (generally zinc, with $\zeta = 2$) which has the higher valency.] .

It must be noticed that if $\frac{\partial e}{\partial \mathrm{T}}$ or $\frac{\partial \mathrm{V}}{\partial \mathrm{T}} = 0$, which happens to be a condition applicable to a Daniell cell, $e = -\,\xi$, or $\mathrm{V} = \frac{-\,4{\cdot}18\lambda}{\zeta \mathrm{F}}$. The term $\mathrm{T}\frac{\partial \mathrm{V}}{\partial \mathrm{T}}$ must therefore represent the heat exchanged between the cell and its surroundings per unit quantity of electricity supplied under conditions which have been stipulated already.

Experimental investigation of the variation of the e.m.f. of a cell with temperature.—For this purpose the cell may take the form shown in Fig. 18·13 (*a*). The two vertical limbs are each about 1 cm. in diameter and have an overall length of 15–20 cm. They are made of glass and joined together by a thick-walled capillary tube about 0·1 cm. internal diameter; the side opening A permits the capillary to be

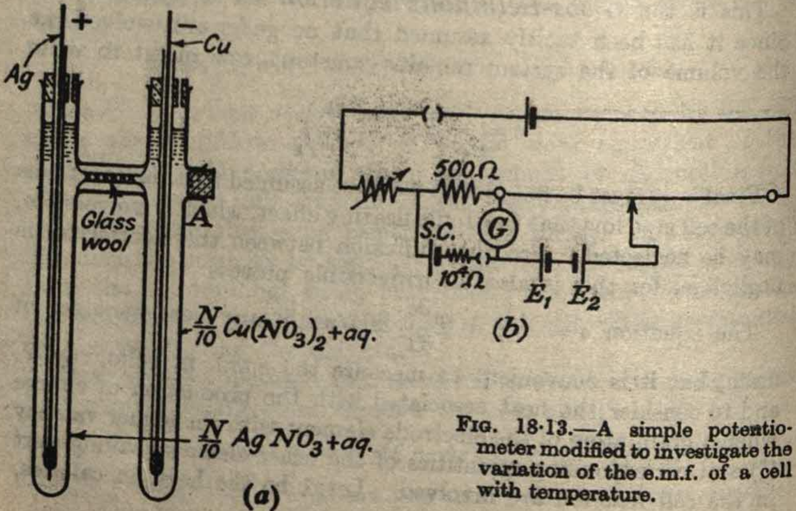
+
Ag
−
Cu
Glass wool
A
$\frac{N}{10}$ Cu(NO$_3$)$_2$+aq.
$\frac{N}{10}$ Ag NO$_3$+aq.
(a)
500Ω
G
S.C.
10⁴Ω
E_1
E_2
(b)

FIG. 18·13.—A simple potentiometer modified to investigate the variation of the e.m.f. of a cell with temperature.

packed lightly with glass wool. One vertical limb is filled with $\frac{N}{10}$ $AgNO_3$ solution and this is permitted to fill the capillary tube. The second tube is filled with $\frac{N}{10}$ $Cu(NO_3)_2$ solution; the glass wool reduces the rate at which diffusion occurs and although the wool will increase the internal resistance of the cell it cannot materially alter its e.m.f. The electrodes are pieces of silver and copper wire, respectively; it is advisable to mount these in glass tubes, held in corks and closed at their lower ends with 'Araldite'. Thus the concentration of each solution, in the regions between which the e.m.f. is measured, will not be much affected by diffusion.

Since the variation of e.m.f. with temperature is small, two such cells are prepared and mounted in a water-bath at 0° C. To measure any difference between their e.m.fs. a modified potentiometer circuit is arranged as shown in Fig. 18·13 (*b*). It is so similar to that used on p. 280 that no detailed description is necessary. Then the temperature of one of the cells is raised by about 5 deg. C.; this temperature should be maintained constant to within 0·2 deg. C. and then the difference between the e.m.fs. of the two cells measured. This procedure is repeated until a temperature of about 50° C. is reached. It will be found that the e.m.f. of this cell is about 0·432 V. at 0° C. and decreases about 1 mV. per degree rise in temperature. A value for the constant in the Gibbs–Helmholtz equation may then be calculated.

Example.—For the reversible cell

$$\overrightarrow{Zn \mid ZnCl_2 + \text{aq.},\quad AgCl \mid Ag}$$

$V = 1\cdot015$ volt at 0° C., and $\frac{\partial V}{\partial T} = -\,4\cdot01 \times 10^{-4}$ volt.deg.$^{-1}$ C.

Now zinc is divalent so that $\zeta = 2$ and we have

$$1\cdot015 = \frac{-\,4\cdot18\lambda}{2 \times 96500} + 273(-\,4\cdot01 \times 10^{-4}).$$

$$\therefore\ \lambda = -\,5\cdot19 \times 10^{4}\ \text{cal.(gm.-atom)}^{-1}.$$

Also, the heat absorbed from the surroundings while the cell supplies a charge equal to one coulomb is

$$\frac{1}{10}T\frac{\partial e}{\partial T}\ \text{ergs} = \frac{1}{41\cdot8 \times 10^{7}}T\frac{\partial V}{\partial T} \times 10^{8}\ \text{cal.} = \frac{1}{4\cdot18}T\frac{\partial V}{\partial T}\ \text{cal.}$$

∴ Heat absorbed when the reaction involves one gram-atom of zinc

$$= -\,\frac{2 \times 96500}{4\cdot18} \times 273 \times 4\cdot01 \times 10^{-4} = -\,5\cdot05 \times 10^{3}\ \text{cal.}$$

The Weston standard cell.—The Weston cell is the present-day standard of electromotive force since wherever it may be set up, provided the conditions for its assembly are adhered to quite strictly, its e.m.f. is always reproducible and, moreover, it is found to remain constant over long periods of time.

The structure of a Weston cell is shown in Fig. 18·14 (*a*). The glass envelope, or container, usually takes the form of a letter H and platinum wires are sealed into the lower extremity of each limb. The mercury and the cadmium amalgam cover the platinum leads;

above the mercury, which forms the positive electrode, there is a paste consisting of mercurous sulphate and cadmium sulphate. This paste and the amalgam are covered with cadmium sulphate crystals and the rest of the apparatus is partly filled with a saturated solution of cadmium sulphate. In preparing the cell, this solution is made saturated at above 40° C. so that when it is placed in the cell and cools to room temperature crystals are deposited on the cadmium sulphate already present. The deposited crystals form a compact coherent layer which prevent the substances beneath them from being displaced. Finally the ends of the two vertical tubes are closed with the aid of a blowpipe flame.

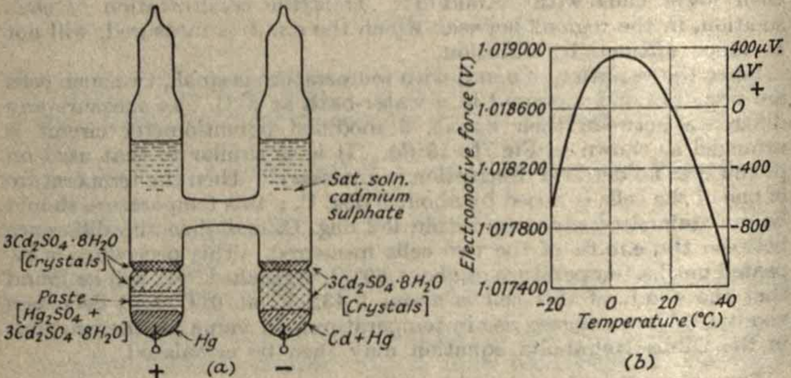

FIG. 18·14.—A Weston standard cell and the variation of its e.m.f. with temperature.

Each limb has a constriction which is most effective in holding the crystals together.

When current is taken from the cell, cadmium dissolves from the amalgam, and mercury is deposited on the anode according to the equation

$$\mathrm{Cd} + \mathrm{Hg_2SO_4} \rightarrow \mathrm{CdSO_4} + 2\mathrm{Hg}.$$

Since, however, an excess of the solids is always present, the state of saturation remains.

The e.m.f. of a Weston cell is 1·018588 volt at 20° C. The e.m.f. at any other temperature, within the range − 20° C. to 40° C., may be obtained by using the formula

$$\Delta V = [-40(\theta - 20) - 0{\cdot}903(\theta - 20)^2 + 0{\cdot}0066(\theta - 20)^3 - 1{\cdot}5 \times 10^{-4}(\theta - 20)^4]\mu V,$$

where $(V + \Delta V)$ is the e.m.f. at a temperature θ° C. [This is an approximation to the N.P.L. formula but it is sufficiently good for academic purposes.] The variation with temperature of the e.m.f. of a Weston cell is shown graphically in Fig. 18·14 (*b*). This diagram shows that the temperature correction will be very small if the cell is at a known temperature close to 0° C. when it is used; better still if the temperature is close to 3° C. when the e.m.f. is a maximum.

If a Weston cell with non-saturated electrolytes is prepared the temperature coefficient is much smaller but the e.m.f. itself is not as constant nor as easily reproducible.

An earlier standard cell was the Clark cell; in this a 10 per cent zinc amalgam replaces the cadmium amalgam and the cell itself is symbolized thus:

$$\mathrm{Hg} \mid \mathrm{Hg_2SO_4}\ (\text{sol.}),\ \vdots\ \mathrm{ZnSO_4}\ (\text{sat. sol.}) \mid \mathrm{Zn}\ (10 \text{ per cent in Hg})$$

The e.m.f. of this cell is given by

$$V = 1{\cdot}4332[1 - 1{\cdot}19 \times 10^{-3}(\theta - 15) - 7 \times 10^{-6}(\theta - 15)^2] \text{ volt,}$$

and it is found to be more constant than that of the Weston cell but unfortunately it has a much higher temperature coefficient.

Modern dissociation theory or the theory of complete ionization.—At the beginning of the present century the view was gradually gaining ground that strong electrolytes are almost completely dissociated at all states of dilution; at the time the view was a very tentative one but soon the results obtained in the field of X-ray crystallography gave very strong support to the new hypothesis. As a result of X-ray analysis it was concluded that in a crystal lattice the unit of structure was not the molecule. For example, in the union of a sodium atom with one of chlorine to form a molecule of sodium chloride it is considered that an electron is transferred from the sodium atom to the chlorine atom. In this way two ions with stable electronic configurations are obtained and these ions are held together by the forces arising from the electrostatic charges which they carry. It is now a universal belief that in a solid salt ionization is complete, but the conductivity of such a substance is necessarily small since powerful forces within the solid prevent any free movement of the charged entities. When, however, the solid is dissolved in water the restraining forces are considerably reduced and the ions are able to move relatively to each other. In very dilute solutions the interionic attractions will be small on account of the relatively large mean distances which separate ions; in concentrated solutions, however, these distances will be much reduced and the attractions correspondingly greater. In such circumstances pairs of ions may be produced or there may be a continual exchange of partners and although ionization is complete the net result will be one of partial dissociation. Thus the theory of complete ionization is at once able to interpret satisfactorily, but in a qualitative manner only, the behaviour of strong electrolytes in aqueous solution.

Arrhenius and the variation of equivalent conductivity with concentration.—The decrease in the equivalent conductivity of an electrolyte as its concentration in aqueous solution increases is a well-known fact. Arrhenius attributed this behaviour of an electrolyte to a decline in the number of ions present per unit mass of the dissolved substance. The equation

$$\Lambda = \alpha[\,|\mu_+| + |\mu_-|\,]\mathrm{F},$$

[cf. p. 659], contains the Faraday constant F and the variables α, μ_+ and μ_-, so that it appears a change in the value of Λ may be due to a variation in one or more of the above variables. Experimentally it is impossible to measure directly the mobility of any one kind of ion, but the ratio of the mobilities of two ions can be obtained by using some form of transport number apparatus. It is found that these ratios vary with the concentration of the solution. Progress towards solving this enigma was slow but when it was realized that the ions in solution were repelled by ions of the same type and attracted by those of the opposite kind, a step in the right direction had been taken. In 1912 MILNER examined theoretically the effect of interionic attractions on mobility but his analysis of the problem did not lead to a complete solution.

The Debye-Hückel theory of solutions.—In 1923 DEBYE and HÜCKEL published an account of their mathematical theory of electrolytes. In this theory it is assumed that the force of attraction or repulsion between ions may be calculated by Coulomb's inverse square law. It is on account of these forces that a perfectly random distribution of ions throughout the solution is impossible. Instead, each positive ion is surrounded by a cloud of ions predominately negative in character, while each negative ion is surrounded by a similar cloud but with positive ions in excess. The particular distribution of ions about a given ion is known as an ***ionic atmosphere.*** If such an ionic atmosphere were always perfectly symmetrical its effect on the motion of an ion in an applied electric field would be nil, for the resultant force due to a symmetrical distribution must be zero. But even if the atmosphere is symmetrical when the applied field is zero it cannot remain so when the field is finite, because the ion will have to build up a new ionic atmosphere in front of it while the atmosphere in the rear gradually gets dispersed. Since a definite time must elapse before that portion of the ionic atmosphere in front of the moving ion assumes its maximum strength and the portion behind is reduced to its minimum strength, the retarding force which the portion of the cloud behind the moving ion exerts on it exceeds the force due to the portion of the cloud in front of it. Thus there arises a resultant retarding force on the ion due to the asymmetrical (but not random) distribution of the ions in the atmosphere surrounding it and the effect is expressed by the ***time of relaxation*** of the ionic atmosphere. This time of relaxation as well as the nature and magnitude of the ionic atmosphere depends upon the concentration of the ions; the radius of the ionic atmosphere decreases as the concentration increases and, likewise, the time of relaxation which is infinite at zero concentration.

Debye and Hückel, in calculating the retarding force on an ion due to the above effect, made the assumption that in all strong electrolytes there was complete dissociation into ions. Then it was postulated that the decrease in the equivalent conductivity with concentration is due mainly to a decrease in the mobilities of the ions and not to a decrease in the extent of the dissociation, which was the hypothesis made by Arrhenius. This decrease in mobility is due, in the first instance, to the time of relaxation associated with the asymmetrical nature of the ionic cloud about a moving ion. In addition to this contribution to the retarding force there is another due to the so-called ***electrophoretic effect***. When an electric field is applied to the electrolyte in solution this field will tend to move the ionic atmosphere, together with its associated solvent molecules, in an opposite direction to that in which the central ion moves. These two retarding effects are additional to that arising from the motion of an ion in a viscous medium. By taking into account these three forces Debye and Hückel showed that

$$\Lambda_0 - \Lambda = kc^{0 \cdot 5},$$

where k is a constant and c the concentration. The equation signifies that the reduction in equivalent conductivity with concentration is directly proportional to the square root of the concentration. The expression is not entirely satisfactory for small discrepancies exist between experimental values for Λ and those calculated from the equation. **Onsager** showed that in calculating the effect of the asymmetrical ionic atmosphere on a moving ion it is necessary to consider the Brownian movement which the ion undergoes and that for dilute solutions of strong electrolytes .

$$\Lambda = \Lambda_0 - \mathrm{A}c^{0 \cdot 5},$$

in which A is a constant whose value can be calculated. The form of the equation remains unchanged; it does not differ from the empirical relationship proposed by Kohlrausch but its value lies in the fact that a value for A can be calculated from theoretical considerations. For a monovalent electrolyte in water at 25° C.

$$\Lambda = \Lambda_0 - (7 \cdot 24\Lambda_0 + 1903)c^{0 \cdot 5},$$

where c is expressed in gram-equivalents per millilitre.

Electrolytic condensers.—Every condenser must have two metallic electrodes and an insulating medium separating them. Electrolytic condensers are no exception to this rule, but they do differ in several distinctive ways from the more usual type of condenser. In the first place, the dielectric has no separate existence but is produced by electrochemical treatment of the surface of one of the electrodes. Secondly, only one electrode is in immediate contact with the dielectric; the other electrode is brought into electrical contact with the opposite surface of the dielectric by filling the space between them with an electrolyte. Fig. 18·15 shows, in diagrammatic form, the essential features of an electrolytic condenser. In the diagram A is one electrode (the positive plate) and this is covered with a layer of dielectric D. C is the other electrode which also serves as a container for the electrolyte E, which fills the space between the outer surface of the dielectric and the second electrode. If this electrolyte were not present and the system AD rested upon C, then contact between C and D would only occur at points so that the capacitance would be reduced considerably. Moreover, at present, all dielectric coatings of the type considered are never ideal, i.e. microscopic cracks are always present in them and these would give rise to 'short circuits' in the condenser. When the electrolyte is present these cracks can be repaired by permitting the 'forming' process of the dielectric layer to be continued after the condenser has been built.

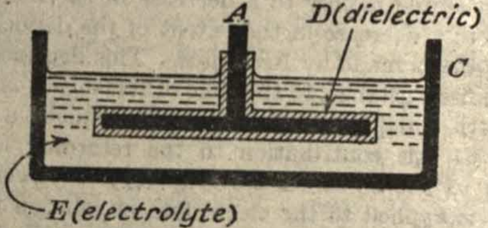

FIG. 18·15.—The principle of an electrolytic condenser.

The 'forming' of the dielectric in an electrolytic condenser.—In Fig. 18·16, B is an electrolytic cell in which the metal which is to constitute the coated electrode of the completed condenser is made the anode A. The cathode, C, is an inactive conductor. When the cell is filled with a suitable electrolyte and a direct current is passed through it, oxygen nearly always appears at the anode and if this is made of a readily oxidizable material an oxide layer is produced over the surface of A. This oxide layer constitutes the dielectric D when A is removed from the bath and used as the positive plate of an electrolytic condenser. The production of this oxide layer is, in general, a separate operation, although, in the smaller types of electrolytic condenser, the forming process may be carried out in the actual condenser itself.

Since the dielectric coating has been formed on the metal while this served as an anode, it is essential that the coated electrode of an electrolytic condenser shall be maintained positive whenever it is in use. Thus the dielectric layer only behaves as an insulator while it is positive with respect to the electrolyte. If it becomes negative the oxide layer tends to be destroyed by the hydrogen which is liberated there and the insulation resistance of the condenser becomes so low that large leakage currents flow and the condenser becomes useless. Electrolytic condensers are said to be polarized; they are represented diagrammatically as in Fig. 18·17.

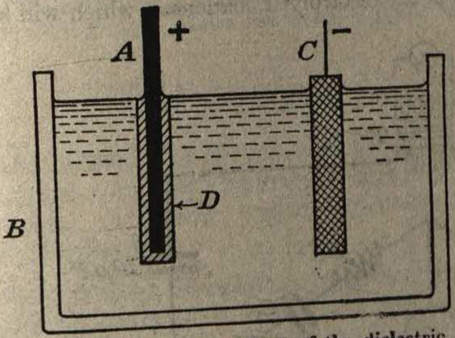

FIG. 18·16.—The 'forming' of the dielectric in an electrolytic condenser.

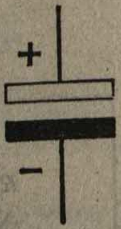

FIG. 18·17.—Schematic representation of an electrolytic condenser.

The dielectric layer is very thin and it is on account of this fact that the capacitance of an electrolytic condenser can have a high value without its bulk becoming excessive.

'Semi-dry' electrolytic condensers.—The electrolytic condensers in present-day use are generally of this type. The positive plate is coated aluminium foil, its area being sufficient to give the required capacitance. The negative plate is also aluminium foil but it is not coated. The two foils are rolled up together with two suitable separators between them to form a compact cylindrical roll. Each separator is wider than the pieces of foil so that the edges of the latter are prevented from coming into direct contact. The aluminium used for the positive plate must be very pure, but commercial aluminium is used for the negative plate. Each separator not only serves as a device for preventing the foils from coming into mechanical contact but also serves to hold the electrolyte between them. Cotton gauze is often used. The electrolyte employed in a semi-dry electrolytic condenser is a viscous mixture of glycerine and either ammonium or sodium borate. The mixture is heated to expel excess moisture and while at a temperature of about 100° C. the cotton gauze is immersed in it; impregnation of the gauze must be complete if the capacitance of the condenser in which it is used is to be high and its power factor low. When properly constructed these condensers are very reliable.

To determine the capacitance and power factor of an electrolytic condenser.—A bridge suitable for this purpose is shown in Fig. 18·18. C_4 is the electrolytic condenser, which will be considered as a condenser of capacitance C_4 in series with a resistance r, while C_2 is a standard condenser with a negligible power factor. Resistors R_1, R_2 and R_3 complete the bridge. T is a transformer while B is a battery to maintain the positive plate of C_4 at the rated voltage. The condenser C_0 permits alternating current to flow to the bridge but isolates the transformer from the battery B.

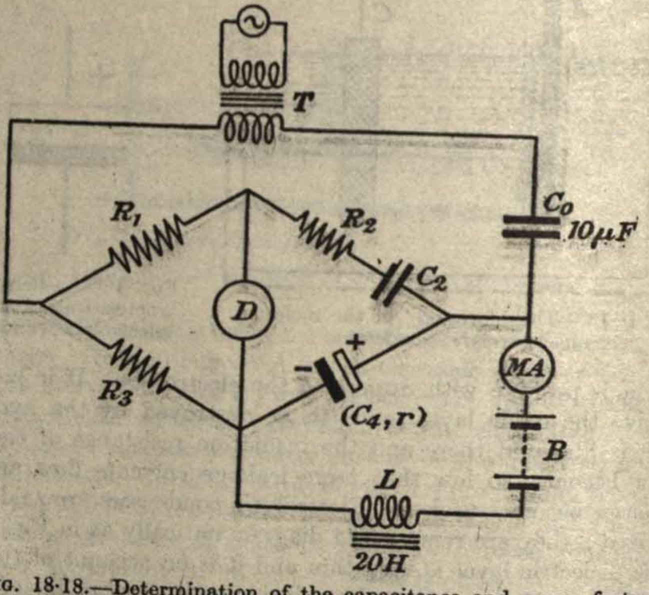

FIG. 18·18.—Determination of the capacitance and power factor of an electrolytic condenser.

In a similar way the iron-cored choke L prevents large alternating currents from flowing through B. The impedance of L must be at least one hundred times the reactance of C_4, i.e. the smaller C_4 is, the less useful is the bridge.

The conditions for balance are given by

$$R_1\left(r + \frac{1}{j\omega C_4}\right) = R_3\left(R_2 + \frac{1}{j\omega C_2}\right),$$

which leads at once to

$$R_1 r = R_3 R_2, \quad \text{and} \quad R_1 C_2 = R_3 C_4.$$

The power factor of the condenser on test is $\cos\phi$, where

$$\tan\phi = \frac{1}{\omega C_4 r} = \frac{1}{\omega C_2 R_2}.$$

Here, since $\tan\phi$ is such that it is not very large compared with unity, we cannot make the usual approximation, but have

$$\cos\phi = \frac{1}{\sqrt{1 + \tan^2\phi}} = \frac{\omega C_2 R_2}{\sqrt{1 + \omega^2 C_2^2 R_2^2}}.$$

The milliammeter MA measures the leakage current through the condenser. Since this current increases with the capacitance of the condenser, the leakage current is generally stated in milliamperes per microfarad.

The faraday, Avogadro's constant and the electronic (or protonic) charge.—If A^* is the atomic weight (or relative atomic mass) of an element, then one gram-atom of this element has a mass A^* gm. and in it N atoms are to be found, where N is Avogadro's constant. If ζ is the valency of the element, one gram-atom is equal to ζ gram-equivalents and one gram-equivalent is deposited in electrolysis by one faraday, i.e. a charge F. Hence one gram-ion, i.e. one gram-atom (or ζ gram-equivalents) which has become ionized, will be deposited by a quantity of electricity equal to ζF. Since the number of ions in a gram-ion is N, it follows at once that the charge on an ion in solution is

$$\frac{\zeta F}{N} = \zeta\,|e|,$$

where e is the electronic charge. Thus

$$F = N\,|e|.$$

By Faraday's first law of electrolysis, however,

$$\frac{A^*}{\zeta} = \text{mass of one gram-equivalent} = zF,$$

where z is the electrochemical equivalent of the element.

$$\therefore\ N\,|e| = F = \frac{A^*}{\zeta z}.$$

These equations are valid in any consistent system of units. Generally, F is given in coulombs. Then 0·1 F × (c), where (c) is the numerical value of the velocity of light in a vacuum, is the value of the charge equal to a faraday, when expressed in e.s.u. If e_s is the electronic charge in e.s.u.,

$$N \mid e_s \mid = \frac{F(c)}{10} = \frac{A(c)}{10\zeta z}.$$

When N is known (cf. Vol. I, p. 649 *et seq.*), this method for determining a value for the electronic charge is capable of great accuracy

EXAMPLES XVIII

18·01. A Daniell cell of e.m.f. E and internal resistance B is arranged in series with an adjustable resistor, an ammeter and an accumulator, the two cells being in opposition. A simple potentiometer, standardized by using a Weston cell, is then used to measure the potential drop, V, across the Daniell cell when the current from the accumulator is I. Show that $V = E + IB$ and hence suggest a method for measuring E and B.

18·02. A liquid resistor consists of two nickel coaxial cylinders, radii r_1 and r_2 $(r_2 > r_1)$; these cylinders are used as electrodes. If χ is the resistivity of the liquid and l the length of each cylinder, show that the resistance between the electrodes is

$$\frac{\chi}{2\pi l} \ln \left(\frac{r_2}{r_1}\right).$$

If $r_1 = 5{\cdot}0$ cm., $r_2 = 5{\cdot}1$ cm., $l = 20$ cm., and $\chi = 2 \times 10^7$ ohm.cm. find the value for the resistance. [$3{\cdot}15 \times 10^3$ ohm.]

18·03. What current would have to pass through an aqueous solution of copper sulphate ($CuSO_4$) in order that 0·05 gm.-equivalent of the salt may be decomposed in one hour ? Assume $F = 96490$ coulomb. (gm.-equiv.)$^{-1}$. [2·68 A.]

18·04. Two points A and B are connected (*a*) to a cell E_1, B_1 and a resistance R_1, (*b*) to a galvanometer, (*c*) to a cell E_2, B_2 and a resistance R_2. The cells are arranged and the resistances adjusted until the galvanometer deflexion is zero. If R_1' and R_2' are other values of these resistances when the deflexion is again zero, show that

$$\frac{E_1}{E_2} = \frac{R_1' - R_1}{R_2' - R_2}.$$

Suggest a method for determining the internal resistance of a Weston cell.

18·05. How may the electrochemical equivalent of silver be determined ? For hydrogen $z = 1{\cdot}04 \times 10^{-5}$ gm.coulomb.$^{-1}$. Assuming Avogadro's constant to be 6×10^{23} mole.$^{-1}$, obtain a value for the charge on a monovalent ion. Express your answer (*a*) in coulombs, (*b*) in e.s.u. [$1{\cdot}603 \times 10^{-19}$ coulomb., $4{\cdot}81 \times 10^{-10}$ e.s.u.]

18·06. An aqueous solution of $BaCl_2$, whose concentration is 0·002 gm.-equiv.litre.$^{-1}$, has a conductivity of $2{\cdot}272 \times 10^{-4}$ ohm.$^{-1}$cm.$^{-1}$. What is its equivalent conductivity ?

[113·6 ohm.$^{-1}$cm.$^{-1}$(gm.-equiv.)$^{-1}$ml.]

18·07. Calculate the conductivity of a decinormal solution of potassium chloride, assuming that it is completely dissociated and that the ionic mobilities for potassium and chlorine ions are $6{\cdot}6 \times 10^{-4}$ cm.^{2}sec.$^{-1}$volt.$^{-1}$, and $5{\cdot}8 \times 10^{-4}$ cm.^{2}sec.$^{-1}$volt.$^{-1}$, respectively.

[0·01196 ohm.$^{-1}$cm.$^{-1}$.]

18·08. A current of 5×10^{-9} ampere, which is just detectable by a certain moving-coil galvanometer, is passed through water acidulated with sulphuric acid. For how long must this current pass in order that the total volume of gas liberated may be 10^{-3} cm.^{3} at s.t.p. ? [Assume that for hydrogen, $\rho = 9 \times 10^{-5}$ gm.cm.$^{-3}$, and $z = 1{\cdot}04 \times 10^{-5}$ gm.-coulomb.$^{-1}$.] [321 hr.]

18·09. Describe and explain how you would determine experimentally the electrochemical equivalent of copper, the current being measured by a potentiometer method.

The resistance of a wire and a column of electrolyte in series is 100 ohms at 0° C. and 50·5 ohms at 20° C. If the resistance-temperature coefficient for the material of the wire is 0·004 deg.$^{-1}$ C. and the conductance-temperature coefficient for the electrolyte is 0·05 deg.$^{-1}$ C., find the resistance of the wire at 0° C. [0·86 ohm.]

18·10. A hydrogen and an iodine coulometer were connected in series. During the passage of a current for a certain period 98·2 ml. of hydrogen were collected at 17·0° C. and 75·1 cm. of mercury pressure. At 17·0° C. the pressure of saturated water vapour may be taken as 1·4 cm. of mercury. The iodine released was titrated with a 0·1N solution of As_2O_3 and 80·4 ml. were required to neutralize it. Estimate the error of the hydrogen coulometer. [0·5 per cent low.]

18·11. At 18° C. a cell containing a solution of silver nitrate whose concentration is 0·0500 gm.-equiv.lit.$^{-1}$ has a resistance of 183·7 ohm. At the same temperature and in the same cell a 0·1N KCl solution has a resistance of 82·2 ohm. If the conductivity of the potassium chloride solution is 0·01116 ohm.$^{-1}$cm.$^{-1}$, calculate a value for the molecular conductivity of the silver nitrate solution. [99·9 ohm.$^{-1}$cm.$^{-1}$mole.$^{-1}$ml.]

18·12. (*a*) Calculate the constant for a cell in which a 0·1N KCl solution gives a resistance of 47·818 ohm. at 25° C. [$\sigma_{25} = 0{\cdot}012886$ ohm.$^{-1}$cm.$^{-1}$.] (*b*) Also for a cell for which a 0·01N KCl solution gives R = 4786·3 ohm. [$\sigma_{25} = 0{\cdot}0014115$ ohm.$^{-1}$cm.$^{-1}$.]

If the water used in making the solutions specified above has a conductivity of 10^{-6} ohm.$^{-1}$cm.$^{-1}$ calculate corrected values for the constants of the two cells.

[(*a*) 0·61619 cm.$^{-1}$, 0·61624 cm.$^{-1}$;
(*b*) 6·7558 cm.$^{-1}$, 6·7607 cm.$^{-1}$.]

18·13. The mobilities of silver ions and nitrate ions in aqueous solution at 18° C. are respectively $5{\cdot}78 \times 10^{-4}$ and $6{\cdot}29 \times 10^{-4}$ cm.^{2}-sec.$^{-1}$volt.$^{-1}$. If the conductivity of a 0·1N $AgNO_3$ solution at 18° C. is $9{\cdot}46 \times 10^{-3}$ ohm.$^{-1}$cm.$^{-1}$, calculate a value for the degree of dissociation of silver nitrate in a decinormal solution. [0·81.]

18·14. In an experiment to determine the conductivity of 0·01N aqueous KCl at 25° C. a right cylindrical cell of length 33·14 cm. and volume 56·92 cm.^{3} and fitted with plane circular electrodes was used. The resistance of the cell was determined by means of a bridge ABCD, where AB, BC are the ratio arms 100 ohms and 1000 ohms respectively and CD the cell. At balance the resistance in AD was found to be 1367·0 ohms. Determine a value for the conductivity of the solution and explain why such a determination cannot be regarded as a precision one. [0·00141 ohm.$^{-1}$cm.$^{-1}$.]

18·15. A conductivity cell is filled with 0·01N KCl solution at 25° C. and placed in a bridge network having equal ratio arms. The resistance required for balance is 2539 ohms. What is the cell constant? The cell is then filled with a 0·02N solution of $Ba(NO_3)_2$ at 25° C. and the resistance required for balance is 1873 ohms. Assuming that the conductivity of a 0·01N solution of KCl is 0·001412 $ohm.^{-1}cm.^{-1}$ at 25° C., obtain values for (*a*) the equivalent conductivity, (*b*) the molar conductivity of the barium nitrate solution.

[3·584 $cm.^{-1}$; (*a*) 95·6 $ohm.^{-1}cm.^{-1}(gm.\text{-}equiv.)^{-1}ml.$; (*b*) 191·2 $ohm.^{-1}cm.^{-1}mole.^{-1}ml.$]

18·16. In an experiment on the electrolysis of silver nitrate 0·603 gm. of silver was deposited. A titration of the solution round the cathode gave 9·420 gm. of silver chloride (AgCl) per litre before, and 9·002 gm. per litre after, the electrolysis. Assuming the relative atomic masses of chlorine and silver to be 35·5 and 108 respectively, calculate a value for each of the transport numbers of the two ions which form the silver nitrate molecule. [$t_- = 0{\cdot}522$, $t_+ = 0{\cdot}478$.]

18·17. In an electrolytic cell a 0·02N solution of copper sulphate was electrolysed between copper electrodes. The anode solution of 50 ml. contained, after electrolysis, 0·221 gm. of copper sulphate ($CuSO_4$). The steady current was such that a potential difference of 1·0186 volt. was maintained across a 50 ohm. coil and the experiment lasted 4 hours. Determine values for the transport numbers for Cu^{++} and SO_4^{--}. [$t_+ = 0{\cdot}418$, $t_- = 0{\cdot}582$.]

18·18. The velocity of a boundary between HCl and KCl is followed in aqueous solution. The current is such that the potential difference between the ends of a 100 ohm. coil is 1·0186 volt. and this current is passed for 30 minutes. The boundary moves 30·1 cm. in a tube whose mean radius of cross-section is 0·40 cm. If the concentration of the HCl is 0·01045N, estimate the transport number of a hydroxonium ion in aqueous solution. [0·83.]

18·19. The transport number for a hydrogen ion was determined with the aid of the apparatus shown in Fig. 18·11. The current was such that the potential drop across a 200 Ω. resistor was 1·018 V. and in 295 seconds the boundary advanced 0·5 cm. If the mean radius of the tube was 0·089 cm. and the concentration of the HCl solution $0{\cdot}108\frac{N}{10}$, obtain a value for the transport number of a hydroxonium ion. [0·84.]

18·20. Establish the Gibbs-Helmholtz relationship for the electromotive force of a reversible cell. The electromotive force of the Weston normal cell at θ° C. is given by the relation:

$$E_\theta = 1{\cdot}01859 - 4{\cdot}06 \times 10^{-5}(\theta - 20) - 9{\cdot}5 \times 10^{-7}(\theta - 20)^2 + 10^{-8}(\theta - 20)^3 \text{ volt.}$$

Assuming mercury to be divalent, find the heat of reaction in the cell at 15° C., if Faraday's constant is 96,500 $coulomb.(gm.\text{-}equivalent)^{-1}$. [$- 4{\cdot}74 \times 10^4$ $cal.coul.^{-1}$.]

18·21. For the cell $\mathrm{Cd}\,\Big|\,\overrightarrow{\mathrm{CdCl_2,\ 10H_2O}\ \vdots\ \mathrm{AgCl}}\,\Big|\,\mathrm{Ag}$, it is found that at 25° C. $V = 0{\cdot}67531$ volt. and at 0° C. $V = 0{\cdot}69153$ volt. Calculate a value for the heat evolved in the chemical reaction corresponding to that in the cell when this supplies two faradays of electricity.
[$-\,4{\cdot}01 \times 10^4$ cal. ($-\,3{\cdot}953 \times 10^4$ cal. by direct experiment).]

18·22. In the cell $\overset{\oplus}{\mathrm{Ag}}\,\Big|\,\mathrm{AgCl,\ ZnCl_2\ aq.}\,\Big|\,\overset{\ominus}{\mathrm{Zn}}$, where the reaction is

$$\mathrm{Zn} + 2\mathrm{AgCl} \longrightarrow \mathrm{ZnCl_2} + 2\mathrm{Ag},$$

the heat absorbed is $-\,5{\cdot}205 \times 10^4$ cal. when one gram-atom of zinc is ionized. Calculate a value for the e.m.f. of the cell at 0° C. if the temperature coefficient is $-\,4{\cdot}0 \times 10^{-4}$ volt.deg.$^{-1}$ C. [1·015 V. is the observed value.] [1·018 volt.]

18·23. The e.m.f. in volts of a reversible cell is given by

$$V = 0{\cdot}931 + (1{\cdot}6 \times 10^{-3})\theta + (7{\cdot}2 \times 10^{-6})\theta^2,$$

where θ is the temperature of the cell on the centigrade scale. Find the heat absorbed from the surroundings when one coulomb is supplied by the cell at a constant temperature of 20° C. [0·133 cal.]

18·24. Test the Onsager equation, $\Lambda = \Lambda_0 - (\alpha\Lambda_0 + \beta)c^{0\cdot5}$, by means of the following equivalent conductivity data for $\mathrm{KNO_3}$ at 25° C. [Shedlovsky, *Journ. Amer. Chem. Soc.*, **54**, 1411, 1932.]

$c \times 10^7$ (gm.-equiv.)ml.$^{-1}$	Λ ohm.$^{-1}$cm.$^{-1}$(gm.-equiv.)$^{-1}$ml.	$c \times 10^7$	Λ
0·69820	144·17	5·8651	142·61
1·7613	143·62	8·6853	141·97
3·8888	142·98	12·119	141·53

[$\Lambda_0 = 144{\cdot}92$; $\alpha\Lambda_0 + \beta = 2951$.]

18·25. Test the Onsager equation for LiCl for which the equivalent conductivity data are as follows:

$c \times 10^7$	Λ	$c \times 10^7$	Λ
0·001	112·5	0·020	104·5
2	111·3	30	102·7
3	110·6	50	100·0
5	109·4	0·100	95·8
0·010	107·2		

[$\Lambda_0 = 115{\cdot}2$; $(\alpha\Lambda_0 + \beta) = 2737$.]

[These last two examples show that Onsager's equation is only valid at low concentrations, i.e. it is really a theoretically limiting law. At higher concentrations, when Λ is plotted against $\sqrt{c}$, the experimental points show positive deviations from the Onsager relation which is linear.]

CHAPTER XIX

UNITS OF ELECTRICAL MEASUREMENT

Any physical quantity which is met in the whole range of physics in its most general sense is always capable of being expressed as the product of a mere number and a unit of the same concrete kind with which it may be compared. Thus

$$[\text{A physical quantity}] = \text{Numeric (N)} \times \text{unit } \{u\} = (\text{N})\{u\}.$$

Any quantity may be expressed in terms of another unit of the same type, provided that the relation between the two units is known. Thus if $\{u_1\}$ and $\{u_2\}$ are two units of the same kind a given quantity may be expressed as $(\text{N}_1)\{u_1\}$ or $(\text{N}_2)\{u_2\}$, where (N_1) and (N_2) are numerics determined by $(\text{N}_2) = (\text{N}_1)\dfrac{\{u_1\}}{\{u_2\}}$.

Units may be chosen arbitrarily, each independent of the other, or each may be based on certain fundamental units, each such unit being in turn based upon some physical property of the universe which is least likely to vary with time or place. Any complete system of units based on certain fundamental units from which all others are derived is known as an ***absolute system*** of units. The term ***absolute*** was first introduced in 1832 by GAUSS in connexion with his determination of H_0.

Absolute electrical measurements are essential for establishing the electrical units in terms of mechanical units; they are often used for examining the constancy of electrical standards and they must be used for the measurement of those quantities for which standards cannot be maintained.

On the other hand it should be noted that comparative measurements are simpler and more precise than absolute measurements; such comparisons are universally employed when a suitable standard is available or when only the ratio of two quantities, of necessity of the same dimensions, is required.

Fundamental principles.—An absolute system of units is one which connects all quantities with a small number of fundamental units, in which they can be expressed and which have been chosen in a manner suited to form a consistent whole and to bring out clearly the interrelations of the various quantities. The absolute system of units on which all electrical measurements are based was originally suggested by Weber (of Göttingen in 1851) in connexion with the magnetic measurements of Gauss; it was brought into general use by the labours of Kelvin and the British Association Committee on Practical Standards of Electrical Measurements established through his advocacy in 1861.

In mechanics it is found that the quantities with which we have to deal can be expressed in terms of the three fundamental units of length, mass, and time. To define electrical and magnetic quantities completely we have to introduce in addition properties of the medium in which the electrical or magnetic action is taking place. These properties turn out to be very intimately connected with the velocity of light through the medium. They are introduced into the equations expressing the relations between electrical quantities in two different forms and in consequence we have, as Weber first pointed out, two different systems of electrical units, known respectively as the electrostatic and electromagnetic units. Each involves the fundamental units of length, mass, and time; in addition, the electrostatic system involves the permittivity of the medium while the electromagnetic system involves its permeability.

The British Association Committee selected as their units of length, mass, and time the centimetre, gramme, and second, which are derived from the French units, the metre, kilogramme, and second.

For the measurement of electrical and magnetic quantities in terms of these units we have recourse to certain facts established by direct experiment. Our belief in the facts depends more on the consequences deduced from them than on the experimental evidence by which they were originally established. The facts in question are expressed in the following statements of the laws to which they lead:

(*a*) ***Coulomb's law of force between electric charges.*** This may be symbolized by the well-known formula

$$F = \alpha\frac{q_1 q_2}{\kappa r^2},$$

where κ is known as the permittivity of the medium—it is a constant for that medium under fixed conditions of temperature and pressure —and α is a pure number. Now κ is really $(\kappa)\kappa_0$, where (κ) is a numeric and κ_0 is the permittivity of a vacuum; and again κ_0 is $(\kappa_0)\{\kappa\}$, where (κ_0) is a numeric and $\{\kappa\}$ is the size of the unit.

(*b*) ***Coulomb's law of the force between magnetic poles.*** This is represented by the equation

$$F = \beta\frac{m_1 m_2}{\mu r^2},$$

where μ is the permeability of the medium and β is a pure number. As above, $\mu = (\mu)\mu_0$ where μ_0 is the permeability of a vacuum; and μ_0 is $(\mu_0)\{\mu\}$.

The two systems of absolute electrical units.—The quantities κ and μ express two different properties of a medium and hence we have two different systems of measurement based respectively on these laws.

(*a*) ***The electrostatic system.*** If we suppose that the two point charges of electricity referred to above are equal, the first law becomes

$$F = \alpha \frac{q^2}{\kappa r^2}, \quad \text{or} \quad q = r\sqrt{\frac{\kappa F}{\alpha}}.$$

Now we can measure r and F but we know nothing about κ or α. To proceed we make the radical assumption that for a vacuum $\kappa = \kappa_0 = 1$ e.s.u. of permittivity and then select the unit of charge so that when two such point charges are at a distance apart of 1 cm. and in a vacuum, $F = 1$ dyne. Then α is unity.

(*b*) ***The electromagnetic system.*** If $m_1 = m_2 = m$ say, we have

$$m = r\sqrt{\frac{\mu F}{\beta}},$$

and by making the fundamental assumption that for a vacuum $\mu_o = 1$ e.m.u. of permeability we may select in a similar way our unit of pole strength.

It is necessary to point out that there will be different systems of electrostatic and electromagnetic units corresponding to the different units of length, mass, and time selected as fundamental, but the only systems which are used here are those based on the c.g.s. system.

From the electrostatic unit of charge we can proceed at once to obtain the electrostatic units of strength of electric field, electrostatic potential, capacitance, current, etc., and in a similar way beginning with the unit magnetic pole, we can derive the magnetic units of strength of field, magnetic potential, intensity of magnetization, etc. If electric phenomena were entirely dissociated from magnetic phenomena the two entirely different sets of units would have no connexion other than that each is based on the c.g.s. system, but Oersted's discovery of the connexion between electric currents and magnetic fields, as expressed by Laplace's law, enables us at once to correlate the two sets of units. Thus we are able to express all electrical quantities in electromagnetic units or in electrostatic units.

DIMENSIONS OF DERIVED ELECTROSTATIC UNITS

Electric charge, q. To determine the dimensions of q we make use of the relation

$$F = \frac{q_1 q_2}{\kappa r^2},$$

where κ is the permittivity of the medium in which two point charges q_1 and q_2 lie at a distance r apart. We have at once

$$[\mathrm{MLT}^{-2}] = \frac{[q^2]}{[\kappa][\mathrm{L}^2]}.$$

$$\therefore\ [q] = [\kappa^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-1}].$$

Since the physical nature of the permittivity of the medium is unknown the symbol $[\kappa]$ is allowed to remain in the dimensional equation; it represents the unknown dimensions of this property of the medium.

Surface density, σ. Since σ is the charge per unit area we have

$$[\sigma] = \frac{[q]}{[\mathrm{L}^2]} = [\kappa^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-\frac{1}{2}}\mathrm{T}^{-1}].$$

***Strength of an electric field,* E.** Since $E = \dfrac{q}{\kappa r^2}$, we have

$$[\mathrm{E}] = [\kappa^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-\frac{1}{2}}\mathrm{T}^{-1}].$$

***Electric induction,* D.** Since $D = \kappa E$, we have

$$[\mathrm{D}] = [\kappa^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-\frac{1}{2}}\mathrm{T}^{-1}].$$

Hence we notice that $[D] = [\sigma]$ which serves as a check since Coulomb's law is $D = 4\pi\sigma$.

***Potential difference,* V.** Since $V = \lim \dfrac{\delta W}{\delta q}$, we have

$$[\mathrm{V}] = \frac{[\mathrm{W}]}{[q]} = \left[\frac{\mathrm{MLT}^{-2} \times \mathrm{L}}{\kappa^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-1}}\right] = [\kappa^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}].$$

***Capacitance,* C.** Since $C = \dfrac{Q}{V}$, we have

$$[\mathrm{C}] = \left[\frac{\kappa^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-1}}{\kappa^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}}\right] = [\kappa\mathrm{L}],$$

so that, for example, when we say that the capacitance of an isolated sphere of radius 2 cm. is 2 cm., we are not quite in order for we have suppressed the dimensions of κ.

It should be noted that in order to find the dimensions of capacitance it is not necessary to find those of q and V for we may use the fact that the energy of a charged condenser is $\frac{1}{2}\frac{Q^2}{C}$, when

$$[ML^2T^{-2}] = \left[\frac{\kappa ML^3T^{-2}}{C}\right]$$

$$\therefore\ [C] = [\kappa L].$$

Similarly the dimensions of potential could be obtained by using the fact that $\frac{1}{2}QV$ is the energy of a charge Q on a conductor at a potential V.

Electric current, i_s. Since current is charge per unit time we have

$$[i_s] = \left[\frac{Q}{T}\right] = [\kappa^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}],$$

Resistance, R_s. Since energy $= i_s{}^2R_st$, we have

$$[R_s] = \left[\frac{ML^2T^{-2}}{\kappa ML^3T^{-4}T}\right] = [\kappa^{-1}L^{-1}T],$$

i.e. the dimensions of resistance on the e.s. system are those of the reciprocal of a velocity if we suppress the dimensions of κ.

Strength of a magnetic field, H_s. Since $H = \frac{2i}{r}$, we have $H_s = \frac{2i_s}{r}$, so that

$$[H_s] = [\kappa^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}].$$

Magnetic pole, m_s. Since MH $\sin\theta =$ couple,

$$[m_sL]\,[H_s] = [MLT^{-2}L].$$

$$\therefore\ [m_s] = [\kappa^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}].$$

Inductance, l_s. Since the energy stored in a magnetic field when a current i_s flows in an inductance l_s, is $\frac{1}{2}l_si_s{}^2$ erg., we have

$$[l_s]\,[\kappa ML^3T^{-4}] = [ML^2T^{-2}].$$

$$\therefore\ [l_s] = [\kappa^{-1}L^{-1}T^2],$$

a fact which could have been obtained by using $e = -l\frac{di}{dt}$. It follows at once that the dimensions of mutual inductance and self inductance are identical.

Specific electronic charge, $\frac{e}{m}$.

$$\left[\frac{e}{m}\right] = \frac{[\kappa^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]}{[M]} = [\kappa^{\frac{1}{2}}M^{-\frac{1}{2}}L^{\frac{3}{2}}T^{-1}].$$

DIMENSIONS OF DERIVED ELECTROMAGNETIC UNITS

Magnetic pole, m. To determine the dimensions of m we make use of the relation $F = \frac{m_1 m_2}{\mu r^2}$, where μ is the permeability of the medium in which two poles m_1 and m_2 lie at a distance r apart, and F is the force on either pole. We have at once

$$[\mathrm{MLT}^{-2}] = \frac{[m^2]}{[\mu][\mathrm{L}^2]}.$$

$$\therefore\ [m] = [\mu^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{3}{2}}\mathrm{T}^{-1}].$$

***Strength of magnetic field,* H.** Since $H = \frac{m}{\mu r^2}$, we have

$$[\mathrm{H}] = [\mu^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-\frac{1}{2}}\mathrm{T}^{-1}].$$

Magnetic potential, Ω. Since $H = -\frac{\partial \Omega}{\partial s}$, or $\delta\Omega = -H\,\delta s$, we have

$$[\Omega] = [\mu^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}].$$

***Magnetic induction,* B.** Since $B = \mu H$, we have

$$[\mathrm{B}] = [\mu^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-\frac{1}{2}}\mathrm{T}^{-1}].$$

***Magnetic moment,* M–M*t*.** Since the couple on a magnet is MH sin θ, we have

$$[\mathrm{M\text{–}M}t] = \frac{[\Gamma]}{[\mathrm{H}]} = [\mu^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{5}{2}}\mathrm{T}^{-1}].$$

***Intensity of magnetization,* J.** This is defined as the magnetic moment per unit volume, so that

$$[\mathrm{J}] = [\mu^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-\frac{1}{2}}\mathrm{T}^{-1}],$$

and it will be noticed that the dimensions of J are identical with those of B and not those of H. It follows therefore that the equation

$$B = \mu_0 H + 4\pi J,$$

where μ_0 is the permeability of empty space, is dimensionally correct.

Strength of a magnetic shell, φ. Since $\phi = Jt$, we have

$$[\phi] = [\mu^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}].$$

Electric current, i. Since in two well-known instances $\delta H = \frac{i\,\delta s \sin\theta}{r^2}$, or $H = \frac{2i}{r}$, we have at once

$$[i] = [\mathrm{HL}] = [\mu^{-\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{\frac{1}{2}}\mathrm{T}^{-1}],$$

or since an electric current is always equivalent to a magnetic shell of strength μi, we have

$$[i] = \left[\frac{\phi}{\mu}\right] = [\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}].$$

Quantity of electricity, q. Since $q = it$, we have

$$[q] = [\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}].$$

Electromotive force and potential difference, v. Since the rate of working is vi, we have

$$[ML^2T^{-3}] = [v]\,[\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}],$$

i.e.

$$[v] = [\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}].$$

Resistance, r. Since i^2r is rate of working, we have

$$[r] = [\mu LT^{-1}].$$

Here it will be noticed that if we neglect the dimensions of μ, the dimensions of a resistance in electromagnetic measure are those of a velocity; it is therefore permissible to speak of a resistance as so many cm.sec.$^{-1}$.

Resistivity, χ. Since the unit of resistivity is the ohm.cm. we have

$$[\chi] = [r]\,[L] = [\mu L^2T^{-1}].$$

Inductance, l. Since $\frac{1}{2}li^2$ is energy, we have

$$[l] = [\mu L],$$

so that an inductance can be expressed in cm. if the dimensions of μ are suppressed.

***Capacitance,* C.** Since $C = \dfrac{q}{v}$, we have

$$[C] = [\mu^{-1}L^{-1}T^2],$$

which may be verified by remembering that $\dfrac{1}{2}\dfrac{q^2}{C}$ measures the energy of a charged condenser.

Magnetic flux (linkages), Φ and Ψ.

$$[\Psi] = [\Phi] = [B \times \text{area}] = [\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}],$$

which could have been used to determine the dimensions of an electromotive force, since

$$e = -\frac{\partial\Psi}{\partial t}.$$

It should be noted that the dimensions of Ψ are those of a pole strength, i.e. $[\Psi] = [m]$.

Specific electronic charge, $\dfrac{e}{m}$.

$$\left[\frac{e}{m}\right] = \left[\frac{\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}}{M}\right] = [\mu^{-\frac{1}{2}}M^{-\frac{1}{2}}L^{\frac{1}{2}}].$$

Thus if $[\mu]$ is suppressed, the dimensions of $\left[\frac{e}{m}\right]$ may be $[\text{cm.}^{\frac{1}{2}}\text{gm.}^{-\frac{1}{2}}]$.

Dimensions of electrical quantities expressed in practical units.—(Strictly speaking the practical system of electromagnetic units.) Since $I = 10i$, where the factor 10 is purely numerical it follows that the dimensions now required are in each instance identical with those of the corresponding electromagnetic units.

Relation between units in the two systems.—In a preceding section the dimensions of a quantity of electricity in both the electrostatic and the electromagnetic systems of units have been obtained. Since the dimensions of any physical quantity must be independent of the particular system of units adopted the dimensions of 'quantity of electricity' may be equated, when we get

$$[\kappa^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}] = [\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}],$$

or

$$[\mu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}] = [LT^{-1}],$$

i.e. the dimensions of $\frac{1}{\sqrt{\mu\kappa}}$ are those of a velocity.

Similarly, using the dimensions of a resistance, we have

$$[\kappa^{-1}L^{-1}T] = [\mu LT^{-1}],$$

which again gives

$$[\mu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}] = [LT^{-1}].$$

Again, if we equate the dimensions of capacitance as expressed by the two systems of units, we have

$$[\kappa L] = [\mu^{-1}L^{-1}T^2],$$

which is consistent with the fact that the dimensions of $\frac{1}{\sqrt{\mu\kappa}}$ are those of a velocity.

Now if we were satisfied to use the dimensions of $\frac{1}{\sqrt{\mu\kappa}}$ to derive for example those of a given electromagnetic quantity when the dimensions of that same quantity on the electrostatic system were known, we should now be in a position to do so. On the other hand, if it were required to know the numerical value of an electrical (or magnetic) quantity expressed in one system when its numerical value in the other system were known then it would be necessary to know all about $\frac{1}{\sqrt{\mu\kappa}}$; its dimensions have been discovered but it remains to find its numerical value. This value can be ascertained for example by measuring experimentally the capacitance of a given air-condenser in both electrostatic and electromagnetic units.

Actually it is only necessary to measure the capacitance in electromagnetic measure for we may choose a condenser whose capacitance in e.s.u. may be calculated from its dimensions.

To correlate the two sets of measurements, we have to remember that when it is stated that the capacitance of an isolated spherical conductor of radius r and lying in a medium of permittivity κ is κr, we really mean that the capacitance is $(\kappa)\kappa_0 r$, where (κ) is a numeric and κ_0 is the permittivity of empty space. Now the size and dimensions of κ_0 are unknown. We may however write

$$\kappa_0 = (\kappa_0)\,\text{gm.}^{\alpha_1}\text{cm.}^{\beta_1}\text{sec.}^{\gamma_1},$$

where (κ_0), α_1, β_1 and γ_1 are numbers. Thus the capacitance of the sphere is

$$(\kappa_0)\,\text{gm.}^{\alpha_1}\text{cm.}^{\beta_1}\text{sec.}^{\gamma_1}\,(r)\,\text{cm.} = (\kappa_0)(r)\,\text{gm.}^{\alpha_1}\text{cm.}^{\beta_1+1}\text{sec.}^{\gamma_1}.$$

Thus if (C_s) is the numerical value of the capacitance in electrostatic units of a certain condenser, we have

$$\text{Capacitance} = (C_s)(\kappa_0)\,\text{gm.}^{\alpha_1}\text{cm.}^{\beta_1+1}\text{sec.}^{\gamma_1},$$

and this is the capacitance in c.g.s. units.

Since on the electromagnetic system of units the dimensions of capacitance are $[\mu^{-1}\text{cm.}^{-1}\text{sec.}^{2}]$, the electromagnetic unit of capacitance will be

$$(\mu_0^{-1})\,\text{gm.}^{\alpha_2}\text{cm.}^{\beta_2}\text{sec.}^{\gamma_2}\text{cm.}^{-1}\text{sec.}^{2},$$

where μ_0, α_2, β_2 and γ_2 are numbers.

Thus if (C_m) is the numerical value of the capacitance of the same condenser when expressed in electromagnetic units, we have

$$\text{Capacitance} = (C_m)(\mu_0^{-1})\,\text{gm.}^{\alpha_2}\text{cm.}^{\beta_2-1}\text{sec.}^{\gamma_2+2},$$

and this is also a measure in c.g.s. units of the capacitance of the condenser under consideration. Equating the two expressions we have obtained for the capacitance in c.g.s. units we have

$$(C_s)(\kappa_0) = (C_m)(\mu_0^{-1}), \quad \text{or} \quad \frac{(C_s)}{(C_m)} = \frac{1}{(\mu_0\kappa_0)},$$

and $\alpha_1 = \alpha_2$, $\beta_1 + 1 = \beta_2 - 1$, $\gamma_1 = \gamma_2 + 2$.

The importance of this argument is that although the α's etc. remain indeterminate it has been shown that

$$\frac{(C_s)}{(C_m)} = \frac{1}{(\mu_0\kappa_0)},$$

and experiments to be described later show that

$$\left(\frac{C_s}{C_m}\right) = (c)^2,$$

where (c) is the numerical value in c.g.s. units of the velocity of light in a vacuum, i.e. $c = (c)$ cm.sec.$^{-1}$.

Thus $$\left(\frac{1}{\sqrt{\mu_0\kappa_0}}\right)=\left(\sqrt{\frac{C_s}{C_m}}\right)=(c)\doteqdot 3\times 10^{10}.$$

Note on Thomson's method for determining the velocity of a charged particle.—Let us suppose a uniform electric field $\vec{E}$ is parallel to the Ox-axis and a uniform magnetic field with induction $\vec{B}$ is parallel to Oy; these fields are said to be 'crossed'. Suppose that the magnitudes of the fields are adjusted so that a charged particle q e.s.u. and mass m moves with constant velocity along Oz. It is desired to determine how this velocity, $\vec{u}=\hat{k}\dot{z}$, where $\hat{k}$ is a unit vector along Oz, may be deduced.

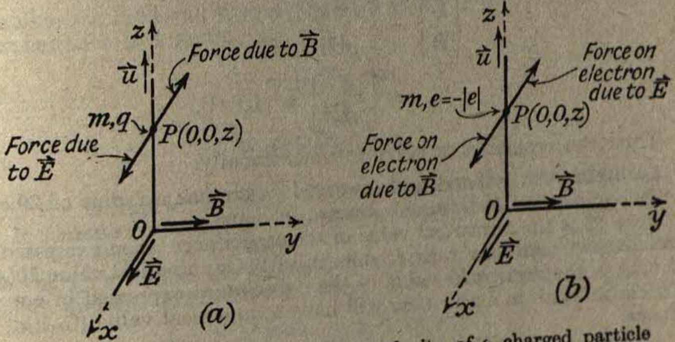

FIG. 19·01.—Determination of the velocity of a charged particle by means of 'crossed' fields.

When the particle is at P, $(0, 0, z)$, Fig. 19·01 (*a*), the force on it due to the electric field is $\hat{\imath}q\mathrm{E}$, where $\hat{\imath}$ is a unit vector parallel to Ox. The force due to the magnetic field is, by Fleming's left-hand rule, $-\hat{\imath}\mathrm{B}\dfrac{q}{(c)}u$, where (c) is the numerical value of the velocity of light in a vacuum; when the resultant force is zero the particle will continue to move along Oz. The necessary condition is

$$q\mathrm{E}=\mathrm{B}\frac{q}{(c)}u,$$

i.e.

$$u=(c)\frac{\mathrm{E}}{\mathrm{B}},$$

and it follows, at once, that we could write

$$u=c\frac{(\mathrm{E}_s)}{(\mathrm{B}_m)},$$

where (E_s) and (B_m) are the numerical values of the electric field strength and magnetic induction when these are expressed in e.s.u. and e.m.u. respectively.

Now as long as the directions of $\vec{E}$, $\vec{B}$ and $\vec{u}$ are unchanged, this equation applies to all charged particles, irrespective of the sign of the charge but, as shown in Fig. 19·01 (*b*), the forces on an electron moving along Oz are reversed. If, however, either $\vec{B}$ or $\vec{E}$ alone is reversed it is impossible to make the resultant force zero.

The real purpose of this note, however, is to enable us to examine the equation $u = (c)\frac{E}{B}$ dimensionally. On the l.h.s. we have

$$[u] = [LT^{-1}],$$

and on the r.h.s.

$$\left[\frac{E}{B}\right] = \left[\frac{\kappa^{-\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}}{\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}}\right]$$

$$= \left[\frac{1}{\mu^{\frac{1}{2}}\kappa^{\frac{1}{2}}}\right] = [LT^{-1}].$$

Thus the equation is correct dimensionally.

Example.—An isolated and insulated sphere has a radius of 20 cm. It is required to find its capacitance in e.m.u.

Since 20 is the numerical value of the capacitance when expressed in e.s.u., the capacitance in c.g.s. units will have a numerical value $20(\kappa_0)$. If (C_m) is the numerical value of the capacitance expressed in e.m.u., the capacitance in c.g.s. units will have a numerical value $(C_m)(\mu_0^{-1})$. Hence

$$20(\kappa_0) = (C_m)(\mu_0^{-1}),$$

or

$$(C_m) = \frac{20}{9 \times 10^{20}} = 2{\cdot}22 \times 10^{-20}.$$

Hence the required capacitance is $2{\cdot}22 \times 10^{-20}$ e.m.u.

Example.—A mutual inductance is 10^7 e.m.u. Express its value in e.s.u.

Let its value be (N_s) e.s.u. Then since the dimensions of inductance in the electromagnetic and electrostatic systems are $[\mu \text{ cm.}]$ and $[\kappa^{-1}\text{cm.}^{-1}\text{sec.}^2]$ respectively, we have

$$10^7(\mu_0) = N_s(\kappa_0^{-1}).$$

$$\therefore (N_s) = 10^7(\mu_0\kappa_0) = \frac{10^7}{9 \times 10^{20}} = 1{\cdot}11 \times 10^{-14}$$

The inductance is $1{\cdot}11 \times 10^{-14}$ e.s.u.

Example.—A charge of electricity is 1 e.s.u. Finds its value in e.m.u.

Let its value be (Q_m) e.m.u. Then since the dimensions of electric charge in the electrostatic and electromagnetic systems are $[\kappa^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]$ and $[\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}]$ respectively,

$$(1)(\kappa_0^{\frac{1}{2}}) = (Q_m)(\mu_0^{-\frac{1}{2}})$$

$$(Q_m) = (\mu_0\kappa_0)^{\frac{1}{2}} = \frac{1}{3 \times 10^{10}}.$$

Hence 1 e.m.u. of electricity is 3×10^{10} e.s.u. of electricity. Since 1 e.m.u. of electricity is 10 coulombs,

$$1 \text{ coulomb} = 3 \times 10^{9} \text{ e.s.u. of electricity.}$$

Alternative method.—If one remembers that 3×10^{10} e.s.u. of electricity are equal to 1 e.m.u. of electricity, the value of one unit in terms of another is readily obtained by making use of expressions for the energy of a system; in dealing with the practical system we may remember that

$$300 \text{ volt.} = 1 \text{ e.s.u. of p.d.}$$

and

$$10 \text{ amp.} = 1 \text{ e.m.u. of current.}$$

Examples.—(*a*) To find how many electrostatic units of capacitance are equal to 1 electromagnetic unit of capacitance.

Now the energy of a charged condenser $= \frac{1}{2}\frac{Q^2}{C} = W$. Let (x) be the number required and suppose that the charge is 1 e.m.u., and that the capacitance is 1 e.m.u.

Then

$$W = \tfrac{1}{2}(1^2)\frac{1}{(1)} \text{ erg.}$$

But

$$W = \tfrac{1}{2}(c^2)\frac{1}{(x)} \text{ erg.}$$

Since the energy is the same we have $(x) = (c^2) = 9 \times 10^{20}$.

(*b*) Inductance: To find how many electromagnetic units of inductance are equivalent to 1 henry.

The energy stored in the magnetic field is $\frac{1}{2}li^2$. Let the current be 1 ampere and the inductande 1 henry. Then the energy is

$$\tfrac{1}{2}.1 \times 1^2 \text{ joule.} = \tfrac{1}{2} \times 10^7 \text{ erg.}$$

But it is also $\frac{1}{2}(x).\left(\frac{1}{10}\right)^2$ erg., where (x) is the number required.

$$\therefore (x) = 10^9.$$

(*c*) To convert a capacitance expressed in electrostatic units of capacitance into one expressed in microfarads.

Consider a condenser of 1 μF. charged to a p.d. of 1 volt. The energy is $\frac{1}{2}CV^2 = \frac{1}{2} \times 10^{-6} \times 1^2$ joule $= \frac{1}{2} \times 10$ erg.

But if 1 μF. $= (x)$ e.s.u. of capacitance, the energy is

$$\tfrac{1}{2}(x)\left(\frac{1}{300}\right)^2 \text{ erg.}$$

$$\therefore (x) = 9 \times 10^5.$$

THE MEASUREMENT OF RESISTANCE IN ABSOLUTE MEASURE

The measurement of the resistance of a standard coil of wire in e.m.u. has been accomplished by a number of experimenters. Early work by Weber, Rayleigh and others is now of historical interest only; two precision methods will be described.

Lorenz's method. Before describing Lorenz's method, as developed at the N.P.L. for determining the value of the ohm in absolute units, the following digression is necessary. Let A, Fig. 19·02, be a circular metal rim of negligible resistance. Suppose that a metal spoke P, of length a, revolves about an axis normal to the plane of the rim and passing through its centre. Let the free end of the spoke be in contact with the rim. Suppose that there is a uniform magnetic field, with induction B, normal to the plane of the rim. Let C_1 and C_2 be two brush contacts connected to a galvanometer G. If the spoke rotates with uniform angular velocity ω, the increase in the number of linkages associated with the closed circuit consisting of the spoke, a portion of the rim, the galvanometer and the leads to it, in time δt is

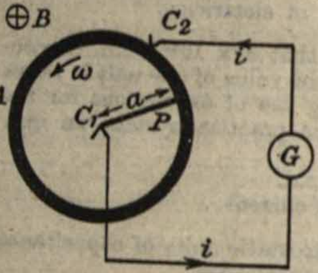

FIG. 19·02.—The equivalence in a magnetic field of a rotating 'spoke' and a rotating disc.

$$\mathrm{B}(\tfrac{1}{2}a^2\omega)\delta t = \delta\Psi \text{ (say).}$$

If the spoke makes n revolutions per second, $\omega = 2\pi n$, and hence

$$\dot{\Psi} = n\mathrm{B}(\pi a^2) = n\mathrm{AB},$$

where A is the area enclosed by the rim. The magnitude of the induced e.m.f. is therefore expressed by

$$\begin{aligned} |e| &= n\mathrm{AB} \text{ e.m.u. of p.d.} \\ &= n\mathrm{AB} \times 10^{-8} \text{ volt.} \end{aligned}$$

In practice there is no difference between the above and a solid circular disc, of area A, revolving at the same angular velocity with its plane normal to the magnetic field.

If the field is not uniform, the term AB must be replaced by an integral of the form $\int \mathrm{B}_n d\mathrm{S}$ or $\int \vec{\mathrm{B}}\cdot\hat{n}\, d\mathrm{S}$ to give the flux of magnetic induction across the disc. If the magnetic field is produced by a current i (e.m.u.) flowing in a suitable arrangement of coils, then the number of linkages is mi, where m is the mutual inductance of the coils and the periphery of the disc.

N.P.L. modification of Lorenz's method for determining a resistance in absolute measure.—This apparatus was devised in 1914 at the N.P.L. by F. E. SMITH.* Earlier arrangements suffered from the fact that there was a considerable thermal e.m.f. in the galvanometer circuit due to heating of the brush contacts. Rayleigh had compensated this but, unfortunately, the compensation can never remain complete since the thermal e.m.f. undergoes slight variations. Moreover, however carefully the apparatus may be mounted, the disc will certainly cut some lines of magnetic induction due to the earth's magnetic field.

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* F. E. Smith, *Phil. Trans., A*, **214**, 27, 1914; and Vigoureux, *Coll. Res. N.P.L.*, **24**, 277, 1938.

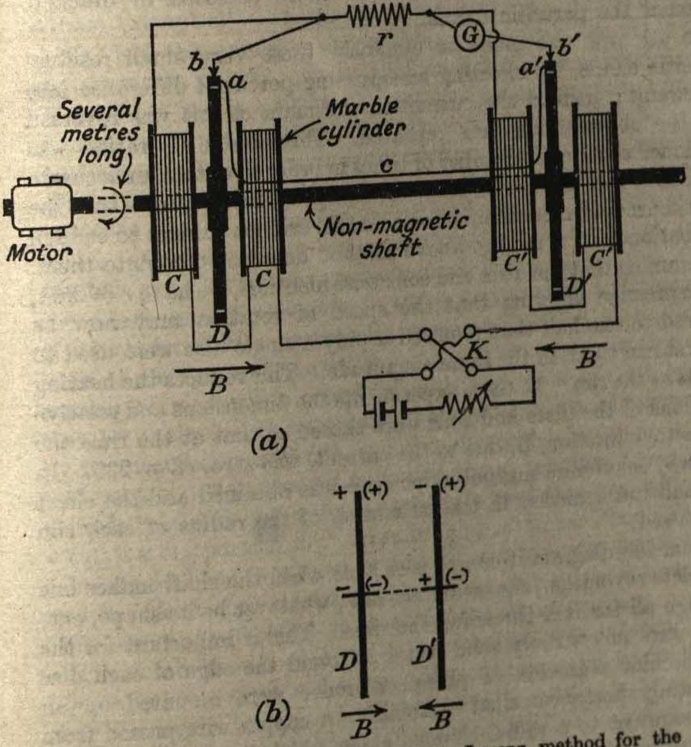


FIG. 19·03.—N.P.L. modification of the Lorenz method for the determination of a resistance in absolute measure.

The N.P.L. apparatus is shown in Fig. 19·03. There was a double system of coils and discs. The discs and driving motor were mounted on a copper-aluminium shaft, since no magnetic material near to the discs could be tolerated. The motor was so far away that its magnetic effect was practically nil and a large fly-wheel placed near to the motor and on the shaft helped to stabilize the rotating system. The discs D, D′ were rotated in the same direction but the magnetic fields were in opposite directions. By arranging the brush contacts, *b*, *b*′, as shown, however, the induced e.m.fs. were added together but the thermal e.m.fs. were now in opposition, as were also any induced e.m.fs. caused by the discs cutting the earth's magnetic field. To verify these statements it is only necessary to consider the potentials at points on the periphery and axle for the two discs as indicated in Fig. 19·03 (*b*); the + and − signs without brackets refer to the e.m.fs. due to the motion in the magnetic field, while those enclosed in brackets represent the parasitic e.m.fs.

No attempt was made to eliminate these very small residual parasitic e.m.fs. by injecting an opposing potential difference into the circuit; instead the current in the main circuit was reversed and the observations were repeated. This double operation was continued a sufficient number of times in order to obtain an accurate value for the resistance on test. Such a procedure assumes that the main sources of these troublesome e.m.fs. are not subject to sudden fluctuations for, if so, the above method does not eliminate them.

Assuming that the discs and coils were identical, we have $r = 2nm$, an expression showing that the speed of rotation may now be reduced to one half that required if only a single disc were used to generate an e.m.f. of the same magnitude. This reduces the heating effects at the rim. In these experiments the dimensions and relative positions of the discs and coils were chosen so that at the rims the magnetic induction, B, due to the current, was zero, cf. p. 235. In this way maximum mutual inductance was obtained and the effect of small uncertainties in the estimation of the radius of each rim minimized.

From the diagram it can be seen that when the shaft makes one complete revolution, the conductor aca', whatever be its shape, cuts through all the flux threading the rims. This is important for the discs were not entirely solid metal. Round the edge of each disc ten circular segments of phosphor-bronze were mounted on an insulating substance called 'stabilit'. A copper wire passed from each segment to a radial channel along which it continued to the shaft. The segments were touched by brushes of phosphor-bronze wire, five on each disc. These brushes resembled violin bows. They were lubricated with paraffin oil. The segments and brushes enabled the e.m.f. obtained to be five times as great as those obtained with a solid wheel; unfortunately the parasitic e.m.fs. were also increased five times, so that, in general, the five brushes on each disc were connected in parallel. In this way small variations in the p.d. across the discs were much reduced.

In Vigoureux's work, new apparatus was constructed for adjusting the coils coaxially with the shaft and for determining the distance between the coils. The speed was 1044 rev.min.$^{-1}$, and a correction was made, as in 1914, for the fact that when the disc revolves its diameter increases by 0·08 mm. On account of the arrangement of coils and discs this correction is almost vanishingly small.

These later experiments showed that the 'international' unit of resistance maintained at the N.P.L. exceeds the ohm by 50 ± 3 parts in 10^5. Smith obtained a value exceeding the above by 2 parts in 10^5. Thus the recent determination failed to detect a drift in the resistance of the working standards at the N.P.L.

The absolute measurement of resistance by the method of Albert Campbell.—The theory of this method is due to Campbell (1925) but the precision experiments were made by Hartshorn and Astbury in 1937. From 1912 until 1937 the results of nearly all precision measurements of electrical resistance were expressed in terms of the international ohm which was defined by the London Conference of 1908. '*The international ohm is the resistance offered to an unvarying electrical current by a column of mercury at the temperature of melting ice, 14·4521 gm. in mass, of a constant cross-sectional area, and of length 106·300 cm.*' Now any measurement in terms of this unit is dependent upon experiments in which a glass tube, the dimensions of which have been accurately measured, is filled with a known mass of mercury and the tube, being at 0° C., the resistance of the column of mercury is compared with that of a resistor which is thereby calibrated in international ohms. Such measurements are by no means easy and although the international ohm is specified to 1 part in 10^5, it has not been possible to get this accuracy with a single tube and even when ten tubes are used the error in the mean result is still 2 parts in 10^5. Results obtained by the various national laboratories have from time to time differed by about 3 parts in 10^5, and it follows that results expressed in terms of the international ohm are always subject to uncertainties of this order.

The international ohm, as defined above, was intended to be 10^9 (c.g.s.) e.m.u. of resistance. Measurements in terms of ohms may be made by any of the so-called absolute electromagnetic methods for determining resistance and in 1912 F. E. Smith showed that measurements made by the method of Lorenz had an error of about 2 parts in 10^5. Shortly afterwards, Grüneisen and Giebe used an alternating current bridge in which the error was again about 2 parts in 10^5. It thus became clear that the ohm could be realized with an accuracy at least of the same order as that obtained for the international ohm. The international system of electrical units is therefore redundant and has been abandoned for an absolute electromagnetic system.

In the modern method, as carried out at the N.P.L., the resistance of the coil, which is to become the standard of resistance, is measured in terms of a mutual inductance and a frequency; in

Fig. 19·04 (*a*), this resistor *r* is arranged in series with the primary winding of the mutual inductor M_1. Under such conditions the potential difference across *r* is proportional to its resistance while the e.m.f. generated in the secondary winding is proportional to the frequency of the a.c. supply and to M_1. If these voltages can be compared then a relation between resistance, frequency and inductance can be established. Since the voltages concerned differ in phase by $\frac{1}{2}\pi$ direct comparison is not possible. To effect a comparison Campbell suggested the use of a second mutual inductor M_2 to rotate the phase of the induced e.m.f. through an additional $\frac{1}{2}\pi$ thus producing an e.m.f. dependent on frequency and inductance and opposite in phase to the p.d. across the resistor *r*, so that the comparison becomes possible by balancing one against the other.

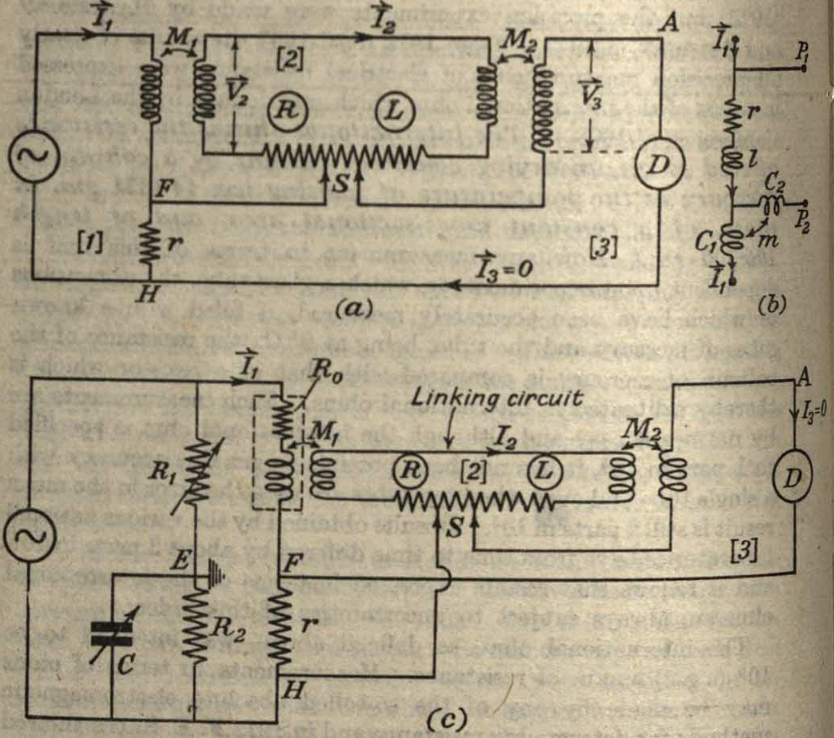


FIG. 19·04.—Campbell's method for determining resistance in absolute measure.

Owing to the self inductance of the secondary windings in the two inductors, their combination cannot produce a phase rotation of exactly π; in order to secure a perfect balance it is necessary to add to the two voltages a third (small) voltage taken from the potential terminals of a resistor S included in the secondary or linking circuit. The essential theory is as follows.

Let R and L denote the total resistance and the total self inductance of the linking circuit; in the diagram these symbols are ringed thus, Ⓡ and Ⓛ, to indicate the facts just stated. Since the e.m.f. introduced into a circuit via the secondary coil of a mutual inductance may be reversed by a reversal of the direction of winding of the coil, or by reversing the connexions to its terminals, M must be considered algebraically. Similarly, the p.d. introduced into a circuit by the potential terminals of resistors such as r and S may be reversed by a reversal of the connexions to their terminals; hence r and S must be considered algebraically. Thus if $\vec{I}_1$ and $\vec{I}_2$ are the current vectors for the primary and linking circuits and if the bridge is balanced so that $\vec{I}_3 = 0$, we have, in the usual way,

$$j\omega M_1\vec{I}_1 = R\vec{I}_2 + j\omega L\vec{I}_2,$$

and

$$j\omega M_2\vec{I}_2 = r(\vec{I}_3 - \vec{I}_1) + S(\vec{I}_3 - \vec{I}_2) = -r\vec{I}_1 - S\vec{I}_2.$$

$$\therefore\ \vec{I}_1[j\omega M_1] = \vec{I}_2[j\omega L + R],$$

and

$$-r\vec{I}_1 = (S + j\omega M_2)\vec{I}_2.$$

Eliminating $\vec{I}_1$ and $\vec{I}_2$ from these equations, we have

$$\frac{j\omega M_1}{-r} = \frac{j\omega L + R}{S + j\omega M_2}.$$

These give, in the usual way,

$$\left.\begin{aligned} \omega^2 M_1 M_2 &= Rr \\ -M_1 S &= Lr \end{aligned}\right\}.$$

The negative sign which appears in one of these equations indicates that the bridge can only be balanced if the connexions are such that the quantities involved have the correct signs. In carrying out an experiment with this bridge, M_1, r and R are fixed, so that the second condition for balance is obtained by altering S only. For a given frequency the first condition is satisfied by adjusting the value of M_2. The two mutual inductances are compared with the Campbell primary standard and the value of the frequency is determined by use of a standard tuning fork, this being checked with the aid of a phonic wheel. $\frac{R}{r}$ is found by means of a d.c. bridge. Thus r can be found; if r is known, the bridge may be used to measure frequencies.

It must be noted that whereas in the Lorenz method the resistance measured is only a small fraction of an ohm, yet in the Campbell method the resistor r can be a coil of nominal resistance one ohm.

The four-terminal resistor r is made of manganin, annealed in nitrogen, and mounted on a keramot holder. This coil, nominally 1 ohm, is wound in the usual bifilar manner but in spite of this has a residual inductance of 0·7 μH. This may be largely compensated for by the following method. Two small coils C_1 and C_2 are arranged, as suggested in Fig. 19·04 (b), so that one is included in a current lead and the other in the corresponding potential lead. If m is the mutual inductance of these coils, the potential difference between P_1 and P_2 is $(r + j\omega l)\vec{I}_1 + j\omega m\vec{I}_1$, and this is only equal to $r\vec{I}_1$, if $(l + m)$ is zero. This is attained by making m negative and separating the small coils until $|m| = l$. The residual inductance is less than 0·02 μH and the departure from absolute stability in the resistance is of the order of a few parts in 10^6 per year.

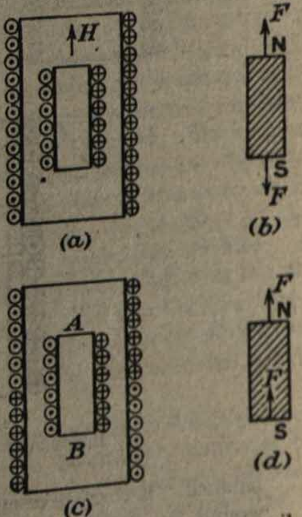
One of the chief sources of error in the circuit so far described arises from the fact that there exists a relatively large stray capacitance between the detector D and earth. This means that some of the induced e.m.f. due to M_2 is used to send current through the stray capacitance. To overcome this the connexions to r in the detector circuit are interchanged (and of course M_1 or M_2 is reversed, but this does not show in a diagram) and the resistors R_1 and R_2 and an adjustable capacitor C added as shown in Fig. 19·04 (c). Since the primary winding of M_1 has resistance R_0 we have, in effect, a Maxwell $\frac{L}{C}$ bridge, cf. p. 602. The point E is earthed and, with the detector placed across EF, the Maxwell bridge is balanced. Then F is at earth potential; the point A is also at earth potential when D is in position in the main part of the network and $\vec{I}_3$ made zero. Thus the effect of stray capacitance between the detector and earth is eliminated. Other stray capacitances still remain, but by actually adding small capacitances from the points concerned to earth it may be shown experimentally that the presence of these residual capacitances does not alter the balance conditions.

The absolute measurement of current.—The value of a current in e.m.u. may be determined from observations on the mutual action between (a) a steady current and a magnet, or (b) between two circuits through which steady currents are passing. The tangent galvanometer may be used for this purpose when H_0, the horizontal component of the earth's magnetic field, is known. The method is not a precise one even when the simple circular coil is replaced by a Helmholtz system; if H_0 were known with the necessary precision the Schuster magnetometer could be used to determine a current in absolute measure.

Greater accuracy is obtained if the current is measured by some form of absolute electrodynamometer, i.e. the current is measured in terms of the force between two or more connected coils carrying the same current. This force is given by $F = i^2 \frac{\partial m}{\partial x}$; cf. p. 302. The Kelvin balance is such an instrument and it has already been shown [cf. p. 319] that

$$i = \frac{1}{2}\sqrt{\frac{m_0 g s}{a}} \div \frac{\partial m}{\partial x}$$

If a silver voltameter is connected in series with a Kelvin current balance the mass of silver deposited by one ampere flowing for one second can be determined. Early determinations gave ($1{\cdot}1180 \times 10^{-3}$) gm.coulomb.$^{-1}$ as the electrochemical equivalent of silver and in 1893 an International Congress adopted as the definition of the international ampere that steady current which will deposit $1{\cdot}1180 \times 10^{-3}$ gm. of silver per second. Later work has shown that the absolute ampere (0·1 e.m.u. of current) deposits silver on the cathode of a silver voltameter at the rate of $1{\cdot}11805 \times 10^{-3}$ gm.sec.$^{-1}$. Since 1940 the absolute ampere has become the legal standard.



(a) (b) (c) (d)

Fig. 19·05.—Forces on a coil within a solenoid.

The forces between coaxial solenoids.—Before entering into a discussion of the N.P.L. form of precision current balance it is necessary to consider the force on a solenoid placed coaxially inside another solenoid when both solenoids carry currents. If the two solenoids are arranged as in Fig. 19·05 (*a*), the magnetic field due to current in the outer solenoid will be directed vertically upwards within the solenoid. If the inner solenoid is replaced by a system of equivalent magnetic shells, free magnetism will only appear at the ends and since the polarity at the ends differs the forces F on the 'poles' will be equal and opposite, i.e. the resultant force, cf. Fig. 19·05 (*b*), is

Suppose now that at the mid-plane of the outer solenoi direction of the current is changed, as indicated in Fig. 19·0

Then the magnetic fields at A and B will be in opposite directions and the equivalent magnet experience at each end a force which is directed upwards, as in Fig. 19·05 (*d*). If this resultant force per unit current can be calculated from the geometry of the system and the resultant force measured when a current is passing through the system, then it should be possible to measure that current in absolute units.

The Ayrton, Jones, Smith current balance.—This electrodynamometer * was designed about 1907 and set up at the N.P.L. It is capable of greater precision than any other form of current balance. It consists essentially of two coils A and B, Fig. 19·06 (*a*), wound on marble cylinders (free from traces of ferromagnetic materials) suspended from the arms of a specially constructed beam balance. The axes of these coils move inside the coaxial fixed cylindrical coils P and Q. The coils are connected in series and in such a way that the forces on the coils A and B are in contrary

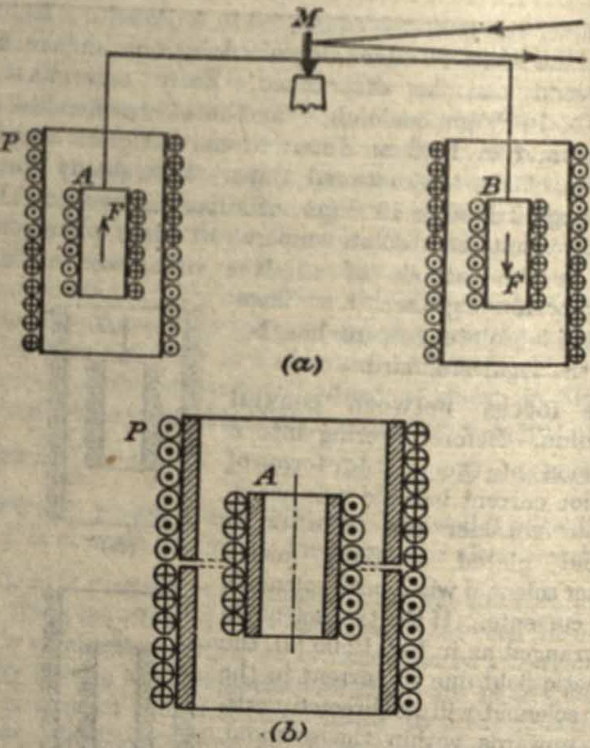


Fig. 19·06.—The N.P.L. current balance.

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* *Phil. Trans.*, 207, 463, 1907.

directions. The force between the fixed and suspended coil was calculated by a formula which is rigorously exact for a helix and a current sheet, and practically so for two helices of fine pitch. The torque due to the current is balanced in much the same way as with a Kelvin balance [cf. p. 319].

Each small coil, cf. Fig. 19·06 (*b*), is about half the length of a large coil. The large coils rest on plates and their axes can be made truly vertical. A number of thin silver wires serve to connect the movable coils to the rest of the system and are arranged to exert no control on the beam of the balance; neither is there any appreciable stray magnetic field permitted for all leads are concentric or twisted together so that the external magnetic field is zero.

Equilibrium is restored by the addition of suitable standard masses to the balance pans and the small resultant of the gravitational and electromagnetic couples is found from observations on the difference between the mean swing and the zero position of the balance and its known sensitivity. In a repetition of this work by VIGOUREUX * increased accuracy was obtained by means of a mirror, M, attached to the beam of the balance; this was used in conjunction with a scale and telescope.

The duplication of the system of coils has great advantages. In all measurements an attempt should be made to keep the errors as small as possible. If, in addition, the observations can be conducted in such a way that the errors tend to neutralize, while the quantity to be measured is doubled, a considerable gain of accuracy ensues.

All this is secured in the instrument just described. The current circulating in the windings warms the marble formers on which the coils are wound and expels moisture; but since it does so to about the same extent on each suspended cylinder, the stability of the balance as a weighing mechanism is not appreciably affected. A similar compensation is secured for the disturbing effect of the air currents produced by the heat generated. The effect of stray non-uniform magnetic fields caused by electric circuits or small magnets is eliminated by reversing the current in the fixed coils and taking two series of readings.

The effects of masses of iron or of large magnets in the neighbourhood of the instrument cannot, however, be entirely eliminated by reversals, because they actually alter the mutual inductance between the coils; no ferromagnetic material can therefore be tolerated in the construction of the dynamometer.

With a duplex system of coils there is, in addition to the force between a small coil and a large coaxial coil, such as A and P, a small axial force due to the action of the large coil Q of the other system on the small coil A. Similarly there is a small axial force on B due to P, but there is, however, no resultant force on coil A due to the current in B, and vice versa. It is not convenient to calculate the small axial force, which is about one per cent of the main force, with all the accuracy required of the instrument, but it is not even necessary to know the force, for it can be reversed by reversing the currents in all the coils of one system, whereas the senses of the direct forces are not changed. This statement is at once verified by considering the forces on the magnetic shells which are equivalent to the A and B coils. The mean of the two weighings gives a measure of the direct force.

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* *Roy. Soc. Phil. Trans.*, 236A, 133, 1936.

A measurement of electric current must of course be referred to concrete standards, and it is usual to choose for the purpose a standard cell and a standard resistance, both of which, when properly constructed and treated with reasonable care, can be relied upon to keep constant to closer than 1 part in 10^5 over a year or more. The current is $\frac{v}{r}$ where v is the e.m.f. of the cell and r the resistance across which the potential difference is v.

(*c*), the ratio of the measure of the electromagnetic unit to the measure of the electrostatic unit of a given electrical quantity.—It has been shown that $\left[\frac{1}{\sqrt{\mu_0\kappa_0}}\right] = [LT^{-1}]$. The separate dimensions of μ and κ are not known. But we have already seen [cf. p. 694] that if (C_s) is the measure in e.s.u. of the capacitance of a given condenser and (C_m) the measure in e.m.u., then

$$\frac{(C_s)}{(C_m)} = \frac{1}{(\mu_0)(\kappa_0)} = (c)^2,$$

where (c) is the measure, in c.g.s. units, of the velocity of light in a vacuum.

In order to determine the value of (c) it is necessary to measure the ratio of the electrostatic and electromagnetic values of at least one electric or magnetic quantity. By measurements of quantity, current, resistance, potential difference, and capacitance, it would appear that there may be five principal methods by which the value of (c) could be obtained. There is, however, no way of measuring directly the value in e.s.u. of a steady current and the number of possible methods is thus reduced to four; the capacitance method is the only one which can be classed as a precision experiment and it alone will be described.

The capacitance method for the determination of (c).—The capacitance of a condenser is defined by the equation $C = \frac{q}{v}$, where the symbols have their usual meanings. This equation may be expressed as

$$[C] = \frac{[\text{current} \times \text{time}]}{[\text{current} \times \text{resistance}]} = \frac{[\text{time}]}{[\text{resistance}]},$$

so that it should be possible to determine the capacitance of a condenser in terms of a resistance and a time. If the resistance is expressed in ohms and the time in seconds, the capacitance will be obtained in terms of the practical unit of capacitance, i.e. the 'sec.ohm.$^{-1}$' or 'farad'. On the other hand if the resistance is expressed in e.m.u., the time again being measured in seconds, the capacitance will be obtained in e.m.u.

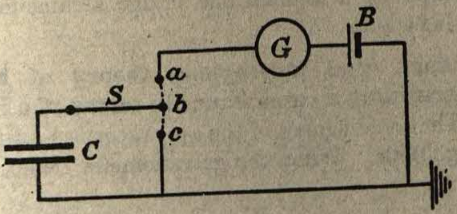

FIG. 19·07.—The steady current due to the rapid charging and discharging of a condenser.

Imagine a circuit as shown in Fig. 19·07; S is a piece of metal strip attached to one terminal of the condenser C. This strip is given a vibratory motion so that its free end *b* alternately makes contact with the studs *a* and *c*. Let V be the potential difference between the terminals of the cell B. If *f* is the number of vibrations made per second by S then the condenser will be charged and discharged *f* times per second, if we suppose that the time of contact between *a* and *b* is sufficiently long for the condenser to be fully charged, i.e. for a potential difference equal to V to be established between its plates, and also if the time of contact between *b* and *c* is sufficiently long for the condenser to be discharged completely, i.e. for no potential difference to exist between its plates. The quantity of electricity flowing per second through the galvanometer G under these circumstances is fVC. [This is the charging current.] If the period of the galvanometer is long compared with f^{-1} sec., i.e. the period of S, the intermittent currents through the galvanometer will produce the same deflexion as a steady current of magnitude fVC.

Let ϕ be the steady deflexion produced. Then

$$f\text{VC} = \text{I} = \kappa\phi,$$

where κ is the reduction factor for the galvanometer.

Now suppose that the condenser and vibrating key are replaced by a resistance R (generally so large that the galvanometer resistance may be neglected) which is adjusted until the deflexion is again ϕ. Then

$$I = \frac{V}{R} = \kappa\phi.$$

From these two equations we have $R = \frac{1}{fC}$, i.e. under the conditions described above the condenser acts as a resistance of magnitude $\frac{1}{fC}$.

The above considerations suggest that the equivalent resistance might be measured by a Wheatstone bridge arrangement. This was done by Maxwell.

The elementary (and incomplete) theory of Maxwell's bridge for determining a capacitance in terms of a resistance and a time.—The arrangement of the apparatus is shown diagrammatically in Fig. 19·08. When the galvanometer deflexion is zero let the current through R_1 and therefore to the condenser be I_1, and the current through R_3 and R_4 be I_2. Then

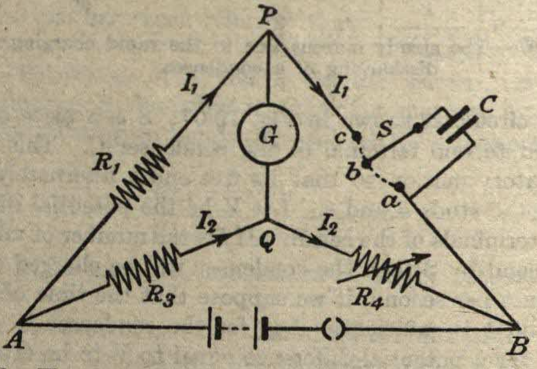

Fig. 19·08.—The essential features of Maxwell's bridge for determining a capacitance in terms of a resistance and a time.

$$I_1R_1 = I_2R_3, \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

since $V_A - V_P = V_A - V_Q$.

Now $V_B - V_Q$ is the effective potential difference, say v, across the condenser when it is charged. It is given by

$$v = I_2R_4 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

But $I_1 = fvC$, since this is the current to the condenser.

From the above equations, we have, as the condition for balance,

$$C = \frac{R_3}{fR_1R_4},$$

which is exactly the equation we should expect if C is regarded as a resistance $\frac{1}{fC}$. The complete theory [cf. Thomson's 'Elements of Electricity'] shows that $\frac{R_3}{R_4}$ should be small.

The bridge is suitable for measuring the capacitance of both large and small capacitors. Thus if $f = 50$ cycle.sec.$^{-1}$, and $C = 2\ \mu$F., then the capacitor may be replaced by a resistor with resistance R_2 where

$$R_2 = \frac{1}{fC} = \frac{10^6}{50 \times 2} = 10^4\ \Omega.$$

If $R_1 = 100\ \Omega$, $R_3 = 10\ \Omega$ and $R_4 = 1000\ \Omega$, the bridge is balanced; such initial values for the components are suitable when capacitances of this order are being measured.

If, however, $C = 0{\cdot}0002\ \mu$F., then with $f = 50$ cycle.sec.$^{-1}$, $R_2 = 10^8\ \Omega$. If then $R_1 = 10^6\ \Omega$, $R_3 = 2000\ \Omega$ and $R_4 = 200{,}000\ \Omega$ the bridge will be balanced; in each instance $\frac{R_3}{R_4} = 0{\cdot}01$ and may easily be reduced to 0·001.

In the first instance we have considered, two accumulators are sufficient to work the bridge but, in the second, at least ten accumulators will be required.

Rosa and Dorsey's * determination of (*c*).—These workers came to the conclusion that the best method for the determination of (*c*) is probably that in which a capacitance is measured in electromagnetic units and in electrostatic units. In this method (*c*) is determined in terms of a resistance, but inasmuch as (*c*) is equal to the square root of the ratio of the numerical values of the capacitances, the uncertainty due to an error in the value of the ohm is halved. All of the methods for determining (*c*) and not involving the ohm are subject to much greater uncertainties.

Rosa and Dorsey used spherical, cylindrical and parallel plate condensers, the second being used both with and without a guard-ring. The spherical condenser was the same one as Rowland had used in 1879. The two spheres were repolished and the two halves of the shell re-ground so that when placed together the joint was water-tight. The condenser together with the silk suspension for the inner sphere and the charging wires W_1 and W_2 are shown in Fig. 19·09 (*a*).

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* *Bull. Bur. Stand.*, **3**, 433, 1907.

At first some difficulty was experienced in determining the capacitance in e.m.u. of the charging wire which passed through a small hole at the pole of the shell to the inner sphere. When this wire is lifted far enough to break the contact with the ball its capacitance changes for two reasons. First, because its length within the shell has changed and, secondly, because its potential is not then equal to that of the inner sphere. This difficulty was finally overcome by using a method of differences.

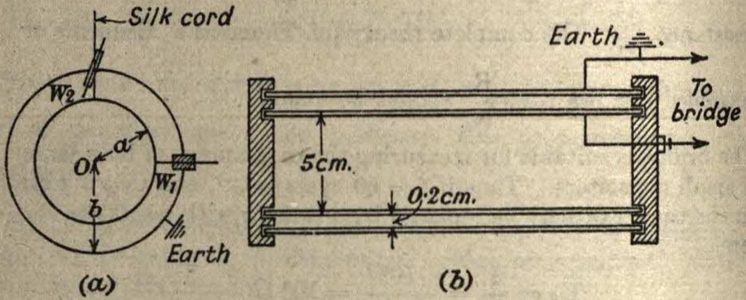

FIG. 19·09.—Spherical and cylindrical condensers used by Rosa and Dorsey.

To examine this method in some detail let Fig. 19·10 (*a*) represent the spherical condenser, capacitance C and the wire W_1 when this alone is used to connect the inner sphere to the vibrating arm S and P; the lead L_1 from W_1 to S and P is at a potential V, say, and likewise the wire which later connects W_2 to S and P. Let α_1 be the capacitance of W_1, while β_1 is the capacitance between the leads L_1 and L_0, where L_0 is the lead from the outer spherical shell to B, which is earthed; similarly, β_2 is the capacitance between L_2 and L_0. If C_1 is the measured capacitance,

$$C_1 = C + \alpha_1 + \beta_1 + \beta_2.$$

Now let W_2 be added to the system, as shown in Fig. 19·10 (*b*). If α_2 is the capacitance of W_2, the measured capacitance, C_2, is given by

$$C_2 = C + \alpha_1 + \alpha_2 + \beta_1 + \beta_2.$$

$$\therefore C_2 - C_1 = \alpha_2.$$

Next W_1 is removed, when the measured capacitance is given by

$$C_3 = C + \alpha_2 + \beta_1 + \beta_2.$$

Finally W_2 is also removed, when

$$C_4 = \beta_1 + \beta_2.$$

$$\therefore C = C_3 - \alpha_2 - \beta_1 - \beta_2 = C_3 - C_2 + C_1 - C_4,$$

and this expression is free from all effects due to the presence of a charging wire. [A similar procedure must be carried out for each cylindrical condenser used.]

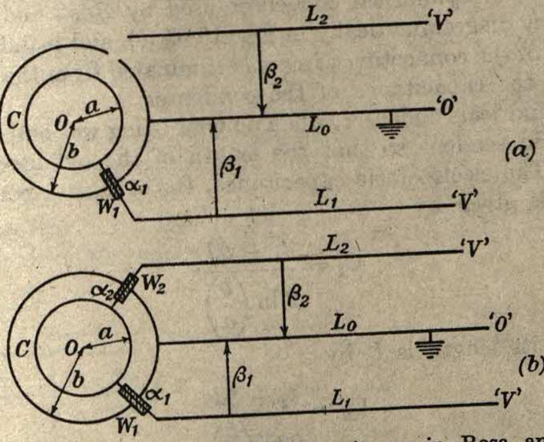

FIG. 19·10.—The elimination of lead-capacitance in Rosa and Dorsey's method for measuring (c).

To adjust the spherical condenser the upper rim of the lower hemisphere was carefully levelled. The ball and upper hemisphere were then put in place and the ball adjusted by the rack and pinion movement which carried the silk suspension until its centre was approximately in the plane of the rim of the lower hemisphere. The upper hemisphere was then raised and propped on blocks, care being taken that the silk cord did not touch the outer shell. The horizontal position of the ball was then tested at four points by means of a brass distance-piece which rested upon the broad rim of the lower hemisphere. The horizontal centring could be attained to within 0·1 mm. The contact between the distance-piece and the inner sphere was determined electrically. The upper hemisphere was then fastened in position and the inner sphere raised or lowered by the rack and pinion movement so that contact, also determined electrically, was made with the top or bottom of the shell. From the scale attached to the rack and pinion the central position of the inner sphere was deduced to within 0·05 mm. The centring was later tested by measuring the capacitance in the adjusted position and then displacing the inner sphere vertically and horizontally by known amounts. The capacitance is a minimum when the spheres are concentric.

The required radii were determined gravimetrically and also by using specially constructed ring-callipers. In one instance $a = 10{\cdot}1180$ cm., $b = 12{\cdot}67159$ cm. The correction for the holes, etc., in the outer shell of the spherical condenser, and for the possible inclination of the axes of the cylinders or want of coincidence, was shown to be negligible with the precautions taken.

The type of cylindrical condenser used by Rosa and Dorsey is shown very diagrammatically in Fig. 19·09 (*b*), and it indicates how the effect of the connecting wires was eliminated from the calculated value for the capacitance of the condenser.

The cylindrical condenser, when no guard-ring was employed, was made up in sections so that the length of the cylinder could be altered. The electrostatic capacitance, C_1, of such a condenser of length l_1 is given by

$$C_1 = \frac{l_1 + \Delta l}{2 \ln \left(\frac{b}{a}\right)},$$

and when its length is l_2 by

$$C_2 = \frac{l_2 + \Delta l}{2 \ln \left(\frac{b}{a}\right)}.$$

Thus the correction term, Δl, can be eliminated, but great care has to be taken to make the end correction identical in the two instances. This is a very important matter and involves many precautions.

Assuming the permittivity of air to be 1·00055, Rosa and Dorsey found

$$(c) = 2{\cdot}9971 \times 10^{10}.$$

This is in terms of the international ohm ; Birge * corrected it to the absolute value and gave

$$(c) = (2{\cdot}9979 \pm 0{\cdot}0001) \times 10^{10},$$

which is in excellent agreement with Essen's value for the velocity of light in a vacuum.

A laboratory determination of $\left(\frac{C_s}{C_m}\right)^{\frac{1}{2}}$ using Maxwell's bridge fitted with a simple electronic switching device.—The two main sources of trouble, associated with the type of rotating commutator usually available for a determination of a capacitance in terms of a resistance and a time, are due to lack of control of the speed of the commutator and to leakage between the 'studs' on the rotating disc. To overcome these defects a simple electronic switch may replace the commutator. The method is due to STEPHENS.

The essential features of the circuit are shown in Fig. 19·11 (*a*).

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* *Phys. Rev. Supplement*, **1**, 10, 1929.

A valve oscillator, the frequency of whose output is determined by means of a double beam cathode ray oscillograph and a standard 1000 cycle.sec.$^{-1}$ source, is connected to the primary of a transformer T. This has two well insulated secondaries, each with a 1 : 1 ratio, and each giving a voltage with a peak value of about 10 V. but in opposite phase. V_1 and V_2 are two valves whose grids are connected to the secondaries of the transformer ; the grids are suitably biased so that each valve is in turn only conducting during a small fraction of a whole cycle. C is the condenser whose capacitance is to be measured and this, together with the switching device, forms one arm of a Maxwell bridge.

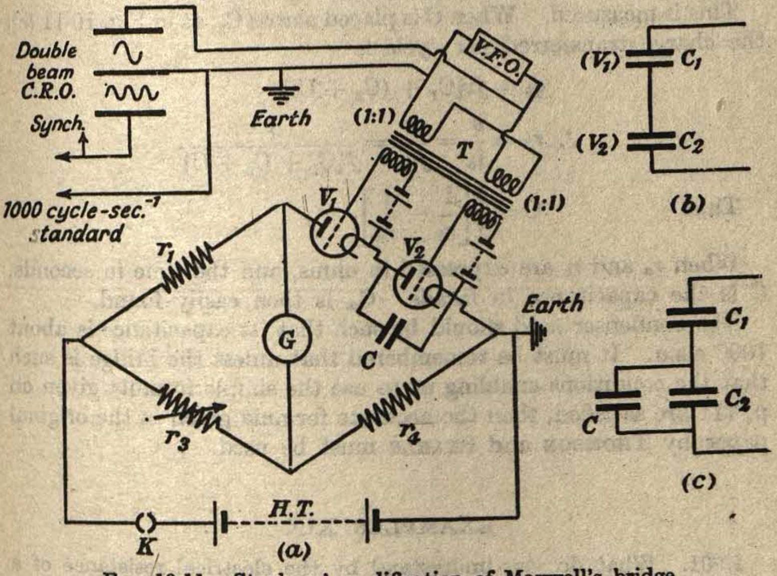


FIG. 19·11.—Stephens' modification of Maxwell's bridge for determining $\left(\frac{C_s}{C_m}\right)^{\frac{1}{2}}$.

As the grid of V_1 goes positive, this valve conducts so that C is charged from the d.c. supply. During the fraction of a half-cycle when this occurs, V_2 is cut off by the standing negative bias plus the negative triggering voltage applied to its grid. In the following half-cycle V_1 is cut out and V_2 conducts so that C is discharged.

Due to interelectrode capacitances and other stray capacitances the charge conveyed per cycle across the arm of the bridge is greater than when the same condenser is used with a rotating commutator at the same frequency. To allow for this let C_1 and C_2 be the total capacitances associated with V_1 and V_2, the condenser C not being present—cf. Fig. 19·11 (b). Then during one complete cycle C_1 is shorted and C_2 charged and then C_2 is shorted and C_1 charged. If v is the d.c. voltage across the bridge, the charge transferred per cycle is

$$q_a = v(C_1 + C_2).$$

Thus, with the usual notation, the current in this arm is given by

$$i_a = fq_a = fv(C_1 + C_2).$$

The equivalent resistance, r_a, is therefore given by

$$r_a = \frac{v}{i_a} = \frac{1}{f(C_1 + C_2)}.$$

This is measured. When C is placed across C_2, as in Fig. 19·11 (c), the charge transferred per cycle is

$$q_b = fv[C_1 + (C_2 + C)].$$

$$\therefore\ r_b = \frac{v}{i_b} = \frac{v}{fq_b} = \frac{1}{f[C_1 + C_2 + C]}.$$

Thus $$C = \frac{1}{f}\left[\frac{1}{r_b} - \frac{1}{r_a}\right].$$

When r_a and r_b are expressed in ohms, and the time in seconds, C is the capacitance in farads. C_m is then easily found.

The condenser used should be such that its capacitance is about 1000 e.s.u. It must be remembered that unless the bridge is such that the conditions enabling us to use the simple formula given on p. 711 are satisfied, then the accurate formula given in the original paper by THOMSON and SEARLE must be used.

EXAMPLES XIX

19·01. What do you understand by the electrical resistance of a wire in absolute measure? Describe and give the theory of an experiment for the absolute determination of a resistance.

19·02. Suppose that an accurate chronometer and c.g.s. standards of length and mass are available, but no other calibrated apparatus or tables of physical constants. Explain what other apparatus is required, and what measurements must be made in order to determine the strength of an electric current. (S)

19·03. Discuss shortly the importance of determining electrical quantities in absolute measure. Describe the experimental method employed to make an absolute determination of *either* a resistance *or* a current. (S)

19·04. Give an account of experimental work leading to the establishment of the standard ohm. (S)

19·05. Find, from first principles, the dimensions of resistance in the electromagnetic system. Describe and explain how the value of a resistance may be determined experimentally in electromagnetic units. (G)

19·06. What is meant by *the dimensions of a physical quantity*? Determine and write down in parallel columns the dimensions of the following quantities in terms of the fundamental units of mass, time and length as derived from both the electrostatic and electromagnetic systems:—quantity of electricity, current, resistance, capacitance, difference of potential and self inductance.

Explain what is meant by the statement that the electrical resistance of a specified conductor is x cm.sec.$^{-1}$.

19·07. If the practical units of potential and current were each increased n-fold, show that there would be no alteration in the size of the units of resistance, capacitance and inductance.

19·08. Check, dimensionally, all the equations which are given on pp. 306, 328, 396, 405 and 527.

CHAPTER XX

ELECTROMAGNETIC THEORY

The divergence of a vector.—Let P, Fig. 20·01, be the point (x, y, z) referred to rectangular axes and suppose that Q is the neighbouring point $(x + \delta x, y + \delta y, z + \delta z)$. Through P and Q let us draw planes parallel to the coordinate planes in order to obtain an element of volume $\delta x\, \delta y\, \delta z$.

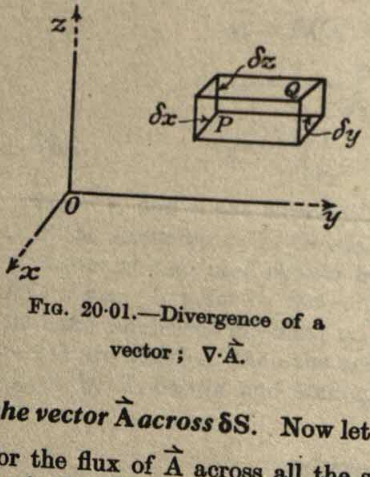


FIG. 20·01.—Divergence of a vector; $\nabla \cdot \vec{A}$.

Let $\vec{A} = (\hat{\imath} A_x + \hat{\jmath} A_y + \hat{k} A_z)$ be a vector function of the independent variables x, y, and z. This function must be finite and continuous at all points; likewise its partial differential coefficients. When δS is an element of a closed surface with an unique outward drawn normal at every point, then $\vec{A} \cdot \hat{n}\, \delta S$, where $\hat{n}$ is a unit vector along the outward drawn normal at δS, is called the ***flux of the vector $\vec{A}$ across δS***. Now let us endeavour to find an expression for the flux of $\vec{A}$ across all the surfaces of the element of volume $\delta x\, \delta y\, \delta z$. Consider the contribution to this flux made by the faces normal to the y-axis. The face nearer to the xOz plane may be called the first y-face; its area is $\delta x\, \delta z$. Since $\hat{\jmath} A_y$ is normal to this face, the flux across it is

$$- A_y\, \delta x\, \delta z.$$

The negative sign occurs since $\hat{n} = - \hat{\jmath}$.

The flux across the second y-face of the element is

$$\left(A_y + \frac{\partial A_y}{\partial y}\, \delta y \right) \delta x\, \delta z,$$

since the new value of A_y is $\left(A_y + \frac{\partial A_y}{\partial y}\, \delta y \right)$. Hence, from both faces, the contribution to the flux is

$$\frac{\partial A_y}{\partial y}\,\delta x\,\delta y\,\delta z.$$

Similarly, $\frac{\partial A_x}{\partial x}\,\delta x\,\delta y\,\delta z$ and $\frac{\partial A_z}{\partial z}\,\delta x\,\delta y\,\delta z$ are the contributions to the flux from the remaining pairs of faces of the element considered. The flux of $\vec{A}$ across the faces of the element is therefore

$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\delta x\,\delta y\,\delta z.$$

i.e. $$\vec{A}\cdot\hat{n}\,\delta S=\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\delta x\,\delta y\,\delta z.$$

Now

$$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{\imath}A_x+\hat{\jmath}A_y+\hat{k}A_z).$$

The quantity $\left(\hat{\imath}\,\frac{\partial}{\partial x}+\hat{\jmath}\,\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)$ is written ∇ and is frequently given the name of ***del***. It is a vector operator and is such that

$$\nabla\cdot\vec{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}.$$

The quantity $\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)$ is known as ***the divergence of the vector*** $\vec{A}$; it is a scalar quantity and sometimes written div $\vec{A}$.

Thus $$\text{div}\,\vec{A}\equiv\nabla\cdot\vec{A}.$$

Gauss' theorem in electrostatics.—If $\vec{D}$ is the electric displacement at a point in a dielectric, Gauss' theorem states that for a closed surface with a unique outward drawn normal at every point,

$$\begin{aligned}\int\vec{D};\hat{n}\,\delta S&=4\pi\Sigma(q)\\&=4\pi Q,\end{aligned}$$

where $Q=\Sigma(q)$ is the total quantity of electricity enclosed within the surface considered.

If we consider a continuous volume distribution of electricity, of density ρ, then for a small rectangular element of volume, Gauss' theorem gives

$$\vec{D}\cdot\hat{n}\,\delta S=4\pi\rho\,\delta x\,\delta y\,\delta z. \quad . \quad . \quad . \quad \text{(i)}$$

But
$$\vec{D}\cdot\hat{n}\,\delta S = \operatorname{div}\vec{D}\,\delta x\,\delta y\,\delta z$$
$$= \nabla\cdot\vec{D}\,\delta x\,\delta y\,\delta z. \quad . \quad . \quad . \quad \text{(ii)}$$

From (i) and (ii) we have
$$\nabla\cdot\vec{D} = 4\pi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

The gradient of a scalar point function.—Let us consider two equipotential surfaces very close together and examine the field in the neighbourhood of a given point A, cf. Fig. 20·02, on the surface V = constant.

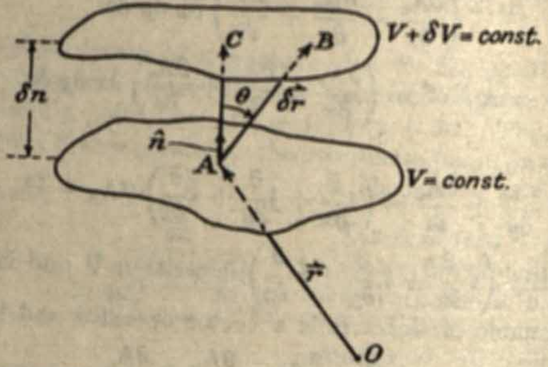


FIG. 20·02.—The gradient of a scalar point function.

Let $\vec{r}$ be the radius vector from a fixed origin O to A; then B, a point on the surface V + δV = constant and not far removed from A, is distant $\vec{r} + \vec{\delta r}$ from O. The least distance at A between the equipotential surfaces is AC, which is drawn in the direction of the unit normal vector $\hat{n}$ at A and of length δn. If δr is the length of AB, the magnitude of the space rate of increase of V at A in the direction AB will be, in the limit, $\frac{\partial V}{\partial r}$. This rate of increase is greatest in the direction of the unit vector $\hat{n}$, i.e. along AC, when it has the value $\frac{\partial V}{\partial n}$; it is such that $\frac{\partial V}{\partial r} = \frac{\partial V}{\partial n}\cos\theta$, where θ is the angle indicated.

Now $\hat{n}\frac{\partial V}{\partial n}$ gives, in magnitude and direction, the greatest space rate of increase of V at A. This vector is called the *gradient* of V at A and is written
$$\operatorname{grad} V = \hat{n}\frac{\partial V}{\partial n}.$$

Thus the gradient of a scalar point function is a vector.

The function ∇V.—Now consider the vector

$$\nabla V = \hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}.$$

To show that this is grad V let us evaluate the scalar products of both grad V and ∇V with $\overrightarrow{\delta r}$. We have

$$\begin{aligned}(\text{grad } V)\cdot\overrightarrow{\delta r} &= \frac{\partial V}{\partial n}\,\hat{n}\cdot\overrightarrow{\delta r} = \frac{\partial V}{\partial n}\,\delta r\cos\theta \\ &= \frac{\partial V}{\partial n}\,\delta n = \delta V \\ &= \frac{\partial V}{\partial x}\,\delta x + \frac{\partial V}{\partial y}\,\delta y + \frac{\partial V}{\partial z}\,\delta z,\end{aligned}$$

in rectangular coordinates. But

$$\begin{aligned}&\left(\frac{\partial V}{\partial x}\,\delta x + \frac{\partial V}{\partial y}\,\delta y + \frac{\partial V}{\partial z}\,\delta z\right) \\ &= \left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)\cdot(\hat{\imath}\,\delta x + \hat{\jmath}\,\delta y + \hat{k}\,\delta z) \\ &= \nabla V\cdot\overrightarrow{\delta r}.\end{aligned}$$

$$\therefore\ \text{grad } V \equiv \nabla V,$$

i.e. the operators grad and ∇ (i.e. del) applied to a scalar point function are identical.

Now $E_x = -\dfrac{\partial V}{\partial x}$, etc.

$$\begin{aligned}\therefore\ \text{grad } V = \nabla V &= -(\hat{\imath}\,E_x + \hat{\jmath}\,E_y + \hat{k}\,E_z) \\ &= -\vec{E}.\end{aligned}$$

Thus the electric field strength is equal in magnitude to $-\dfrac{\partial V}{\partial n}$ and is directed along the normal to the potential surface at the point in question.

Poisson's and Laplace's equations in electrostatics, in isotropic media.—Since $\vec{D} = \kappa\vec{E}$ and $\vec{E} = -\nabla V$, we have

$$\begin{aligned}\nabla\cdot\vec{D} &= \nabla\cdot\kappa\vec{E} \\ &= -\kappa\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)V \\ &= -\kappa\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right),\end{aligned}$$

since when the scalar multiplication is performed it is only those terms such as $(i\cdot i)\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial x^2}$, etc. that remain, since for other terms, $(i\cdot j)$ etc. $= 0$.

The expression $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$ is written $\nabla^2 V$, where ∇^2 denotes the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The equation $\nabla\cdot\overrightarrow{D} = 4\pi\rho$, may therefore be written

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{4\pi\rho}{\kappa},$$

or

$$\nabla^2 V = -\frac{4\pi\rho}{\kappa}.$$

This is ***Poisson's equation*** in electrostatics. If $\rho = 0$, everywhere, we get ***Laplace's equation***, viz.

$$\nabla^2 V = 0.$$

The equation $\nabla\cdot\overrightarrow{B} = 0$.—In a magnetic field, $\overrightarrow{B} = \mu\overrightarrow{H}$, where μ is the permeability of the medium. Since an inverse square law applies to a magnetic field, Gauss' theorem must also be valid, but since free magnetic poles do not exist, the theorem assumes the more restricted form, cf. p. 111,

$$\int \overrightarrow{B}\cdot\hat{n}\, dS = 0.$$

For an element of volume in a magnetic field it therefore follows that

$$\nabla\cdot\overrightarrow{B} = \operatorname{div}\overrightarrow{B} = 0,$$

i.e.

$$\nabla\cdot\overrightarrow{H} = \operatorname{div}\overrightarrow{H} = 0.$$

Some applications of the theorem $\nabla\cdot\overrightarrow{H} = 0$.—(*a*) Suppose that a small magnet of magnetic moment, $\overrightarrow{M}$, lies with its axis along the Ox-axis of a system of rectangular coordinates. Then at a point $(x, 0, 0)$ the field is

$$\begin{aligned}\overrightarrow{H} &= \hat{i}H_x + \hat{j}H_y + \hat{k}H_z = \hat{i}H_x + \hat{j}0 + \hat{k}0\\ &= \hat{i}H_x = \hat{i}\frac{2M}{\mu x^3},\end{aligned}$$

where μ is the permeability of the surrounding medium.

Hence

$$\mathrm{H}_x = \frac{2\mathrm{M}}{\mu x^3}, \quad \text{and} \quad \frac{\partial \mathrm{H}_x}{\partial x} = -\frac{6\mathrm{M}}{\mu x^4},$$

But since $\nabla \cdot \vec{\mathrm{H}} = 0$, i.e. $\dfrac{\partial \mathrm{H}_x}{\partial x} + \dfrac{\partial \mathrm{H}_y}{\partial y} + \dfrac{\partial \mathrm{H}_z}{\partial z} = 0$,

$$\frac{\partial \mathrm{H}_y}{\partial y} + \frac{\partial \mathrm{H}_z}{\partial z} = +\frac{6\mathrm{M}}{\mu x^4},$$

and since the Oy and Oz axes may be taken at random, so long as all three axes are mutually perpendicular,

$$\frac{\partial \mathrm{H}_y}{\partial y} = \frac{\partial \mathrm{H}_z}{\partial z} = \frac{3\mathrm{M}}{\mu x^4}.$$

Thus at the point $(x, a, 0)$ where $a \rightarrow 0$,

$$(\mathrm{H}_y)_{x,a,0} = \left[\left(\frac{\partial \mathrm{H}_y}{\partial y}\right)_{x,0,0}\right] a = \frac{3\mathrm{M}a}{\mu x^4},$$

and there appears to be no easier method of estimating the components of the field at a point near to the axis of a short bar magnet.

(*b*) Now consider the field strength at the centre of a circular coil, radius a, carrying a current i. Let O be the centre of the coil and Ox the axis through O which is normal to the plane of the coil. Then since the field along Ox is symmetrical with respect to the plane of the coil,

$$\left(\frac{\partial \mathrm{H}_x}{\partial x}\right)_{x=0} = 0.$$

Since $\nabla \cdot \vec{\mathrm{H}} = 0$, it follows at once that, at O,

$$\frac{\partial \mathrm{H}_y}{\partial y} = \frac{\partial \mathrm{H}_z}{\partial z} = 0.$$

At the point $(x, 0, 0)$, we have [cf. p. 131]

$$\mathrm{H}_x = \frac{2\pi i a^2}{(a^2 + x^2)^{\frac{3}{2}}},$$

so that

$$\frac{\partial \mathrm{H}_x}{\partial x} = -\frac{6\pi i a^2 x}{(a^2 + x^2)^{\frac{5}{2}}}.$$

Again, since $\nabla \cdot \vec{\mathrm{H}} = 0$,

$$\frac{\partial \mathrm{H}_y}{\partial y} + \frac{\partial \mathrm{H}_z}{\partial z} = \frac{6\pi i a^2 x}{(a^2 + x^2)^{\frac{5}{2}}},$$

and because

$$\frac{\partial \mathrm{H}_y}{\partial y} = \frac{\partial \mathrm{H}_z}{\partial z},$$

on account of the fact that Oy and Oz may, with similar limitations as before, be chosen at random,

$$\frac{\partial \mathrm{H}_y}{\partial y} = \frac{\partial \mathrm{H}_z}{\partial z} = \frac{3\pi i a^2 x}{(a^2 + x^2)^{\frac{5}{2}}}.$$

If $(x, 0, 0)$ is a pseudo neutral point, so that $\dfrac{2\pi i a^2}{(a^2 + x^2)^{\frac{3}{2}}} = |\,\mathrm{H}_0\,|$, where H_0 is the horizontal component of the earth's magnetic field, then

$$\left(\frac{\partial \mathrm{H}_x}{\partial x}\right)_{\text{N.P.}} = -\frac{3x}{(a^2 + x^2)}\,|\,\mathrm{H}_0\,|.$$

$$\therefore \left(\frac{\partial \mathrm{H}_y}{\partial y}\right)_{\text{N.P.}} = \frac{3x}{2(a^2 + x^2)}\,|\,\mathrm{H}_0\,|$$

$$= \left(\frac{\partial \mathrm{H}_z}{\partial z}\right)_{\text{N.P.}},$$

facts which enable us to calculate, in terms of H_0, the field strengths at points close to the pseudo neutral point.

$\nabla \times \mathrm{A}$, or curl $\vec{\mathrm{A}}$.—The vector product of the operator del and a vector $\vec{\mathrm{A}}$ is called the curl of the vector $\vec{\mathrm{A}}$, i.e. curl $\vec{\mathrm{A}} \equiv \nabla \times \vec{\mathrm{A}}$ or $\nabla_{\wedge}\vec{\mathrm{A}}$. Since, in general (cf. Vol. I, p. 7),

$$\vec{\mathrm{A}}_{\wedge}\vec{\mathrm{B}} = \begin{vmatrix} i, & j, & k \\ \mathrm{A}_x, & \mathrm{A}_y, & \mathrm{A}_z \\ \mathrm{B}_x, & \mathrm{B}_y, & \mathrm{B}_z \end{vmatrix},$$

it follows that

$$\text{curl } \vec{\mathrm{A}} = \nabla_{\wedge}\vec{\mathrm{A}} = \begin{vmatrix} i, & j, & k \\ \dfrac{\partial}{\partial x}, & \dfrac{\partial}{\partial y}, & \dfrac{\partial}{\partial z} \\ \mathrm{A}_x, & \mathrm{A}_y, & \mathrm{A}_z \end{vmatrix}.$$

Thus

$$\nabla \times \vec{\mathrm{A}} = i\left(\frac{\partial \mathrm{A}_z}{\partial y} - \frac{\partial \mathrm{A}_y}{\partial z}\right) + j\left(\frac{\partial \mathrm{A}_x}{\partial z} - \frac{\partial \mathrm{A}_z}{\partial x}\right) + k\left(\frac{\partial \mathrm{A}_y}{\partial x} - \frac{\partial \mathrm{A}_x}{\partial y}\right).$$

Example.—Evaluate $\nabla \times \nabla \times \vec{\mathrm{A}}$.

Using the expression for $\nabla \times \vec{\mathrm{A}}$ we have, from the definition for the curl of a vector,

$$\nabla \times \nabla \times \vec{\mathrm{A}} = i\left\{\frac{\partial}{\partial y}\left(\frac{\partial \mathrm{A}_y}{\partial x} - \frac{\partial \mathrm{A}_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \mathrm{A}_x}{\partial z} - \frac{\partial \mathrm{A}_z}{\partial x}\right)\right\}$$

$$+ j\{\ldots\} + k\{\ldots\}$$

$$= \imath\left\{\frac{\partial^2 A_y}{\partial y\,\partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z\,\partial x}\right\} + \jmath\{\ldots\} + k\{\ldots\}$$

$$= \imath\left\{\frac{\partial}{\partial x}\,\nabla\cdot\vec{A} - \nabla^2 A_x\right\} + \jmath\left\{\frac{\partial}{\partial y}\,\nabla\cdot\vec{A} - \nabla^2 A_y\right\} + k\left\{\frac{\partial}{\partial z}\,\nabla\cdot\vec{A} - \nabla^2 A_z\right\}$$

$$= \nabla\nabla\cdot\vec{A} - \nabla^2\vec{A},$$

where $\nabla^2\vec{A} = \imath\nabla^2 A_x + \jmath\nabla^2 A_y + k\nabla^2 A_z$.

Example.—Evaluate $\nabla\cdot\nabla\times\vec{A}$.

$$\nabla\cdot\nabla\times\vec{A} = \left(\imath\frac{\partial}{\partial x} + \jmath\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left\{\imath\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \jmath\cdot\jmath(\ldots) + k\cdot k(\ldots)\right\}$$

$$= \frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_y}{\partial x\,\partial z} + \frac{\partial^2 A_x}{\partial y\,\partial z} - \frac{\partial^2 A_z}{\partial y\,\partial x} + \frac{\partial^2 A_y}{\partial z\,\partial x} - \frac{\partial^2 A_x}{\partial z\,\partial y}$$

$$= 0.$$

Example.—Evaluate $\nabla\cdot\vec{A}\times\vec{B}$.

We have

$$\nabla\cdot\vec{A}\times\vec{B} = \left(\imath\frac{\partial}{\partial x} + \jmath\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left\{\imath(A_yB_z - A_zB_y) + \jmath\cdot\jmath(\ldots) + k\cdot k(\ldots)\right\}$$

$$\begin{aligned} &= A_y\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_y}{\partial x} - A_z\frac{\partial B_y}{\partial x} - B_y\frac{\partial A_z}{\partial x} \\ &+ A_z\frac{\partial B_x}{\partial y} + B_x\frac{\partial A_z}{\partial y} - A_x\frac{\partial B_z}{\partial y} - B_z\frac{\partial A_x}{\partial y} \\ &+ A_x\frac{\partial B_y}{\partial z} + B_y\frac{\partial A_x}{\partial z} - A_y\frac{\partial B_x}{\partial z} - B_x\frac{\partial A_y}{\partial z} \\ &= \vec{B}\cdot\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\vec{B}. \end{aligned}$$

Green's theorem.—This theorem, discovered in 1828, enables a volume integral taken throughout a closed space to be expressed in terms of a surface integral taken over the boundary of that space. All surfaces are assumed to have a unique outward drawn normal at every point. If $\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z$ is a continuous function of the Cartesian coordinates (x, y, z), then Green's theorem states that

$$\int \nabla\cdot\vec{A}\,dv = \int \hat{n}\cdot\vec{A}\,dS,$$

where $\hat{n}$ is the unit vector along the outward drawn normal at a surface element δS.

[It will be noted that only one integral sign is used for surfa and volume integrals, since the differential dS or dv is sufficien indicate the nature of the integral, but to emphasize the natu these integrals we may write

$$\iiint \nabla\cdot\vec{A}\,dv = \iint \hat{n}\cdot\vec{A}dS.]$$

Let S, Fig. 20·03, be the surface of a closed volume; only a section, in the plane of the paper, of this volume is shown. Let a prism whose generators are parallel to Oz and whose cross-sectional area is $\delta x\,\delta y$ cut the surface at $\alpha, \beta, \gamma, \ldots$.

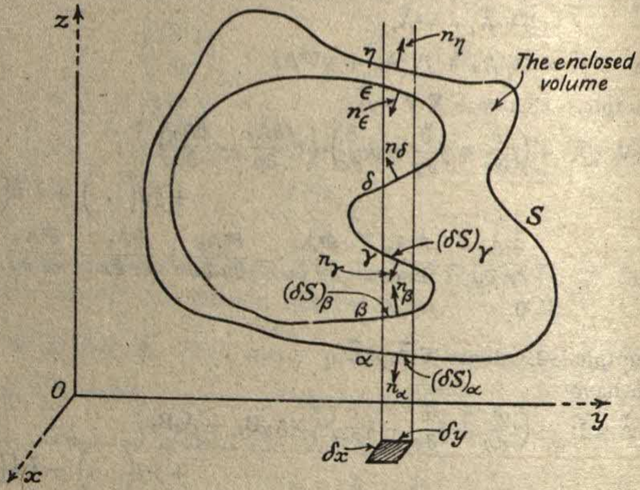


FIG. 20·03.—Green's theorem.

Now $\int \nabla\cdot\vec{A}\,dv \equiv \iiint \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\,dy\,dz.$

Let us therefore consider the contribution to the integral

$$\iiint \frac{\partial A_z}{\partial z}\,dx\,dy\,dz$$

throughout the space enclosed by S arising from the above prismati portion; this contribution is $\left\{\int \frac{\partial A_z}{\partial z}\,dz\right\}\delta x\,\delta y.$

We have

$$\begin{aligned}\int \frac{\partial A_z}{\partial z}\,dz &= \int_\alpha^\beta \frac{\partial A_z}{\partial z}\,dz + \int_\gamma^\delta \frac{\partial A_z}{\partial z}\,dz + \int_\epsilon^\eta \frac{\partial A_z}{\partial z}\,dz + \ldots\\ &= \int_\alpha^\beta dA_z + \int_\gamma^\delta dA_z + \int_\epsilon^\eta dA_z + \ldots\\ &= -(A_z)_\alpha + (A_z)_\beta - (A_z)_\gamma + (A_z)_\delta\\ &\qquad - (A_z)_\epsilon + (A_z)_\eta - \ldots\end{aligned}$$

where $(A_z)_\alpha$ is the value of A_z at α, etc.

Also, since the projection of each area δS_α, δS_β, . . . , on the xOy plane is the element of area $\delta x\, \delta y$, we have

$$\delta x\, \delta y = -\, n_\alpha\, \delta S_\alpha = n_\beta\, \delta S_\beta = -\, n_\gamma\, \delta S_\gamma = \ldots ,$$

where n_α is the value of the third direction cosine for the normal at δS_α, etc. The signs are alternatively negative and positive because as we proceed along the prism the outward drawn normal makes angles which are alternately obtuse and acute with the positive direction of the axis of the prism. Thus

$$\begin{aligned}\delta x\, \delta y \int \frac{\partial A_z}{\partial z}\, dz &= \delta x\, \delta y[-\,(A_z)_\alpha + (A_z)_\beta - (A_z)_\gamma + (A_z)_\delta - \ldots] \\ &= n_\alpha (A_z)_\alpha \delta S_\alpha + n_\beta (A_z)_\beta \delta S_\beta + n_\gamma (A_z)_\gamma \delta S_\gamma + \ldots\end{aligned}$$

Adding together the similar equations which may be written down for all the prisms parallel to Oz, we obtain

$$\iiint \frac{\partial A_z}{\partial z}\, dx\, dy\, dz = \iint n A_z\, dS.$$

Treating A_x and A_y in a similar manner, we get

$$\iiint \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial z} \right) dx\, dy\, dz = \iint (l A_x + m A_y + n A_z)\, dS,$$

where (l, m, n) are the direction cosines of the outward drawn normal at δS.

Hence, in vector notation,

$$\int \nabla \cdot \vec{A}\, dv = \int \hat{n} \cdot \vec{A}\, dS.$$

Solenoidal vectors.—A vector $\vec{A}$ which is such that at every point within a certain region $\nabla \cdot \vec{A} = 0$ is said to be 'solenoidal' within that region. For such a vector Green's theorem shows that

$$\int \vec{A} \cdot \hat{n}\, dS = 0,$$

where dS is an element of a Gaussian surface, etc. In regions free from charge, electric field strength, $\vec{E}$, and electric induction, $\vec{D}$, and also the vectors $\vec{H}$ and $\vec{B}$ are examples of solenoidal vectors.

Stokes' theorem.—This theorem relates to the evaluation of a line integral of a vector along a closed curve. It states that for a vector $\vec{A}$ which is uniform, finite and continuous, and likewise its derivative in any direction, the line integral of $\vec{A}$ round the periphery of a surface is equal to the flux of $\nabla \times \vec{A}$ over the surface, i.e.

$$\oint \vec{A}\cdot\vec{ds} = \int \hat{n}\cdot\nabla \times \vec{A}\, dS.$$

Now

$$\begin{aligned}\vec{A}\cdot\vec{ds} &= (\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z)\cdot(\hat{\imath}\, dx + \hat{\jmath}\, dy + \hat{k}\, dz)\\ &= A_x\, dx + A_y\, dy + A_z\, dz,\end{aligned}$$

and

$$\hat{n} = l\hat{\imath} + m\hat{\jmath} + n\hat{k},$$

where (l, m, n) are the direction cosines of the unit vector $\hat{n}$, the direction of the normal in the second integral being related to the sense in which the first integral is taken round the circuit in the same manner as translation to rotation in a right-handed screw motion—cf. Fig. 20·04 (a).

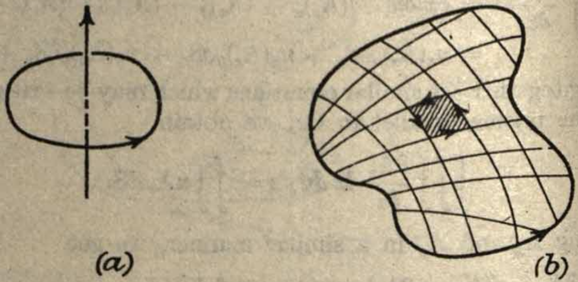


FIG. 20·04.

$$\therefore\ \hat{n}\cdot\nabla \times \vec{A} = (l\hat{\imath} + m\hat{\jmath} + n\hat{k})\cdot\left\{\hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}(\ldots) + \hat{k}(\ldots)\right\}$$

$$= l\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + m\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + n\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$

Hence, in Cartesian coordinates, Stokes' theorem becomes

$$\oint (A_x\, dx + A_y\, dy + A_z\, dz)$$

$$= \int\left\{l\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + m\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + n\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dS\right\}$$

To prove this theorem it will be found that it is only necessary to prove it for an element of surface. The reason for this is that if we consider a finite surface to be divided into small loops by means of a network of lines as shown in Fig. 20·04 (b), then the total sum of all the line integrals taken for each small elementary circuit separately is equivalent to the integral taken round the outer circuit alone, any interior element of a circuit being counted on the whole twice over but with contrary signs in each instance. It is therefore only necessary to establish the theorem for a small element.

Consider the value of $\int A_x\,dx$ taken round the boundary of the element of area δS, shown in Fig. 20·05 (*a*). Let A_x be the value of this quantity at the mean centre O′ of the element. If (x', y', z')

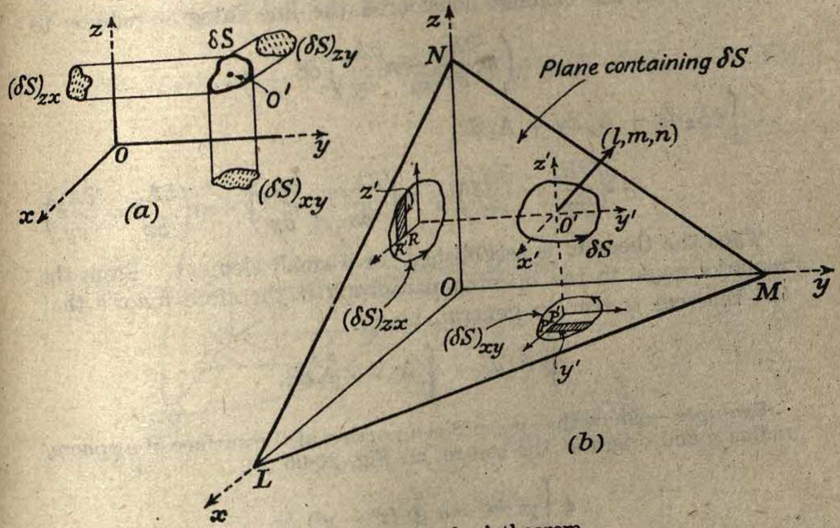


FIG. 20·05.—Stokes' theorem.

are the coordinates of any point on this boundary relative to the above mean centre, then the value of A_x at a point on the boundary is given by

$$A_x + x'\frac{\partial A_x}{\partial x} + y'\frac{\partial A_x}{\partial y} + z'\frac{\partial A_x}{\partial z}.$$

Hence for the element considered we have to evaluate

$$\oint\left(A_x + x'\frac{\partial A_x}{\partial x} + y'\frac{\partial A_x}{\partial y} + z'\frac{\partial A_x}{\partial z}\right)dx',$$

the integration being round the periphery of the element. Since $\oint A_x\,dx' = A_x \oint dx' = 0$ and $\oint x'\,dx' = \left[\tfrac{1}{2}(x')^2\right]_{x'}^{x'} = 0$, the above integral reduces at once to

$$\frac{\partial A_x}{\partial y}\oint y'\,dx' + \frac{\partial A_x}{\partial z}\oint z'\,dx'.$$

Let $(\delta S)_{xy}$ denote the area of the projection of δS on the plane xOy. Then if (l, m, n) are the direction cosines of the normal at O′,

$$(\delta S)_{xy} = n\,\delta S \quad\text{and}\quad (\delta S)_{zx} = m\,\delta S, \qquad \text{[cf. p. 30]}.$$

But, cf. Fig. 20·05 (*b*), $(\delta S)_{xy} = -\oint y'\,dx'$. The appearance of the negative sign is accounted for in the following way. In $(\delta S)_{xy}$ the area of the shaded portion is $y'(PP') = -y'dx'$ since $\delta x' = P'P$.

Similarly $(\delta S)_{zx} = +\oint z'\,dx'$.

Hence for the element considered the line integral reduces to

$$\left(m\frac{\partial A_x}{\partial z} - n\frac{\partial A_x}{\partial y}\right)\delta S.$$

$$\therefore \int (A_x\,dx + A_y\,dy + A_z\,dz)$$

$$= l\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + m\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + n\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$

Thus the theorem is established for a small element. From the remarks made in the opening paragraph it therefore follows that the theorem is true in general, i.e.

$$\oint \vec{A}\cdot\vec{ds} = \int \hat{n}\cdot\nabla\times\vec{A}\,dS.$$

Example.—Show that when S is a portion of the surface of a sphere, radius a and origin at the centre, cf. Fig. 20·06 (*a*),

$$4\int yz\,dS = a\oint (z^2 - y^2)\,dx,$$

where the surface integral extends over the surface S and the line integral is evaluated round the periphery of S.

We have, since the direction cosines of the normal at (x, y, z) are $\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$,

$$\oint \{(z^2 - y^2)\,dx + 0\,dy + 0\,dz\}$$

$$= \int\left[\frac{x}{a}\{0 - 0\} + \frac{y}{a}\left\{\frac{\partial}{\partial z}(z^2 - y^2) - 0\right\} + \frac{z}{a}\left\{0 - \frac{\partial}{\partial y}(z^2 - y^2)\right\}\right]dS$$

$$= \int \frac{4yz}{a}\,dS,$$

which establishes the proposition.

Example.—Show that the volume within a closed surface S is

$$\tfrac{1}{6}\int \nabla(r^2)\cdot\hat{n}\,dS$$

Let δv be the volume of the small cone indicated in Fig. 20·06 (*b*).

Then $\delta v = \frac{1}{3}r \times$ projection of δS on a plane normal to $\vec{r}$

$= \frac{1}{3}r(\delta S\cos\theta) = \frac{1}{3}r\hat{r}\cdot\hat{n}\,\delta S$ $\qquad [\because \cos\theta = \hat{r}\cdot\hat{n}]$

$= \frac{1}{3}\vec{r}\cdot\hat{n}\,\delta S.$

$$\therefore\ v = \tfrac{1}{3}\int (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)\cdot\hat{n}\, dS$$

$$= \tfrac{1}{6}\int \nabla(r^2)\cdot\hat{n}\, dS,$$

since $\nabla(r^2) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z).$

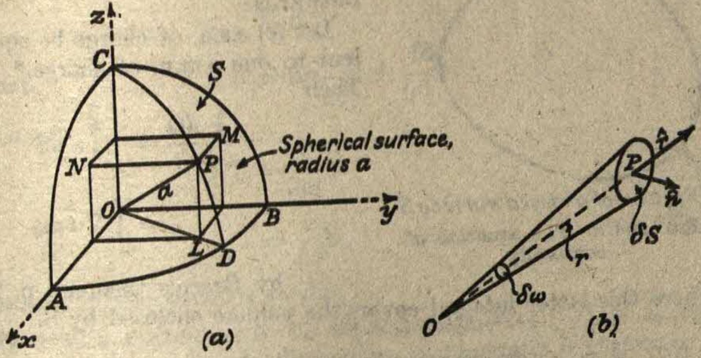


FIG. 20·06.

Maxwell's first law of electromagnetics.—Consider a surface S with a unique outward drawn normal at every point. Let δS be an element of this surface while δs is an element of its periphery. If $\vec{i}$ is the current density in e.m.u., the total current enclosed by the periphery and therefore flowing across the surface S is $\int \vec{i}\cdot\hat{n}\, dS$, where $\hat{n}$ is the unit vector along the outward drawn normal at δS. Then for *steady* currents, as proved on p. 143,

$$\oint \vec{H}\cdot\vec{ds} = 4\pi \times \text{(total current enclosed by the periphery of which } \delta s \text{ is an element)}$$

$$= 4\pi \int \vec{i}\cdot\hat{n}\, dS$$

But by Stokes' theorem,

$$\oint \vec{H}\cdot\vec{ds} = \int \hat{n}\cdot\nabla\times\vec{H}\, dS.$$

$$\therefore\ \nabla\times\vec{H} = 4\pi\vec{i}.$$

But $$\nabla\cdot\nabla\times\vec{H} = 0. \quad \text{[cf. p. 725.]}$$

$$\therefore\ \nabla\cdot\vec{i} = 0.$$

Now consider any closed surface with a unique outward drawn normal at all points and situated within a dielectric medium. Fig. 20·07 is a section in the plane of the paper of such a surface.

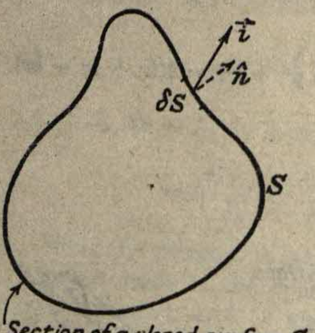


FIG. 20·07.—The equation of continuity.

If Q (e.s.u.) is the total charge inside S at time t, then the rate at which this total charge is decreasing is the total current across S outwards.

Let (c) e.s.u. of charge be equivalent to one e.m.u. of charge.* Then

$$-\frac{1}{(c)}\frac{dQ}{dt} = \int \vec{i}\cdot\hat{n}\, dS$$

$$= \int \nabla\cdot\vec{i}\, dv,$$

by Green's theorem, p. 725, where this latter integral covers the volume enclosed by S.

Now $Q = \int \rho\, dv$, where ρ is the density of the charge.

$$\therefore\ -\frac{1}{(c)}\int \frac{\partial\rho}{\partial t}\, dv = \int \nabla\cdot\vec{i}\, dv,$$

or

$$\nabla\cdot\vec{i} + \frac{1}{(c)}\frac{\partial\rho}{\partial t} = 0.$$

This is the ***equation of continuity*** and it is only when ρ is constant that $\nabla\cdot\vec{i} = 0$.

Since $\nabla\cdot\vec{i} + \frac{1}{(c)}\dot{\rho} = 0$ and $\nabla\cdot\vec{D} = 4\pi\rho$, [cf. p. 720],

$$\nabla\cdot\left(\vec{i} + \frac{1}{4\pi(c)}\dot{\vec{D}}\right) = 0.$$

Thus $\vec{i} + \frac{1}{4\pi(c)}\dot{\vec{D}}$ is a solenoidal vector known as the ***total current density***, and Maxwell proceeded on the hypothesis that the equation

$$\nabla \times \vec{H} = 4\pi\vec{i},$$

which is only true for steady currents, becomes

$$\nabla \times \vec{H} = 4\pi\left\{\vec{i} + \frac{1}{4\pi(c)}\dot{\vec{D}}\right\},$$

---

* We have often made use of the fact that $(c) = 3 \times 10^{10}$. In this chapter this will be proved, so that at this stage it is necessary to look upon (c) as a conversion factor whose value is to be discovered.

when dielectric media are present and the value of $\vec{D}$ varies with time. This equation may be written

$$\nabla \times \vec{H} = 4\pi\vec{i} + \frac{1}{(c)}\dot{\vec{D}},$$

and is known as ***Maxwell's first law of electromagnetics.***

If ρ is the density (e.s.u.) of the electrification at any point and $\vec{u}$ is the velocity with which it is moving,

$$\vec{i} = \rho\frac{\vec{u}}{(c)},$$

so that

$$\nabla \times \vec{H} = 4\pi\frac{\rho}{(c)}\vec{u} + \frac{1}{(c)}\dot{\vec{D}}.$$

If $\rho = 0$,

$$\nabla \times \vec{H} = \frac{1}{(c)}\dot{\vec{D}}$$

$$= \frac{\kappa}{(c)}\dot{\vec{E}}, \qquad [\because \vec{D} = \kappa\vec{E}]$$

for isotropic homogeneous media.

Maxwell's second law of electromagnetics.—If N is the number of turns in a coil, Φ the flux of magnetic induction, i.e.

$$\Phi = \int \vec{B}\cdot\hat{n}\,dS,$$

then

$$\Psi = N\Phi,$$

is the number of linkages associated with the coil.

Since $e = -\dfrac{\partial\Psi}{\partial t}$, we have for a closed path in any medium,

$$\oint \vec{E}\cdot\vec{ds} = -\frac{1}{(c)}\frac{\partial}{\partial t}\int \vec{B}\cdot\hat{n}\,dS \quad \text{[cf. footnote*.]}$$

By Stokes' theorem,

$$\oint \vec{E}\cdot\vec{ds} = \int \hat{n}\cdot[\nabla \times \vec{E}]\,dS$$

$$\therefore\ [\nabla \times \vec{E}] = -\frac{1}{(c)}\dot{\vec{B}} = -\frac{\mu}{(c)}\dot{\vec{H}}.$$

This is ***Maxwell's second law of electromagnetics.***

The force on a moving charge.—In an earlier chapter, cf. p. 136, it was shown that the force on an element of a conductor, $\vec{\delta s}$, in a field with induction $\vec{B}$ is given by

$$\vec{\delta F} = i[\vec{\delta s} \times \vec{B}].$$

---

* Since (c) e.s.u. of charge = 1 e.m.u. of charge, and energy is ½ QV, it is easily shown that (c) e.m.u. of potential = 1 e.s.u. of potential.

If a regular succession of point charges q e.s.u., moving with velocity $\vec{u}$ constitute the current, then $i = \frac{nqu}{(c)}$, where n is the number of charges per unit length of path, and the force on the charges within a length $\vec{\delta s}$ is

$$\frac{nqu}{(c)}[\vec{\delta s} \times \vec{\mathrm{B}}] = \frac{nq\,\delta s}{(c)}[\vec{u} \times \vec{\mathrm{B}}].$$

Since the number of point charges is $n\,\delta s$, the force on a single point charge q is

$$\frac{q}{(c)}[\vec{u} \times \vec{\mathrm{B}}].$$

Poynting's theorem.—If energy exists in an electromagnetic field it must move from one place to another as the field changes. Consider a closed surface S which, as usual, must have a unique outward drawn normal at every point. Then the rate at which the electric and magnetic fields are doing work on the charges within S, plus the rate at which energy is flowing outwards across S, is equal to the rate at which the total field energy within S is decreasing.

Now the force per unit charge on any small charge in the electromagnetic field is the vector sum of the forces per unit charge due to the electric and to the magnetic field, viz.

$$\vec{\mathrm{E}} + \frac{1}{(c)}[\vec{u} \times \vec{\mathrm{B}}],$$

where $\vec{u}$ is the velocity with which the charge is moving.

Hence if $\rho\,\delta v$ (e.s.u.) is the elementary charge moving with velocity $\vec{u}$, the force on it will be

$$\vec{\delta \mathrm{F}} = \rho\,\delta v\left\{\vec{\mathrm{E}} + \frac{1}{(c)}[\vec{u} \times \vec{\mathrm{B}}]\right\}.$$

The rate at which this force does work will be

$$\vec{\delta \mathrm{F}}\cdot\vec{u} = \rho\,\delta v\left\{\vec{\mathrm{E}} + \frac{1}{(c)}[\vec{u} \times \vec{\mathrm{B}}]\right\}\cdot\vec{u},$$

and this, when integrated throughout the volume enclosed by S, is the rate at which the forces are doing work on the charges within S. Call this P. Then

$$\begin{aligned}\mathrm{P} &= \int\left\{\vec{\mathrm{E}} + \frac{1}{(c)}[\vec{u} \times \vec{\mathrm{B}}]\right\}\cdot\vec{u}\,\rho dv\\ &= \int \vec{\mathrm{E}}\cdot\vec{u}\,\rho dv,\end{aligned}$$

the second term vanishing since $\vec{u}$ is normal to $[\vec{u} \times \vec{\mathrm{B}}]$.

But the first law of electromagnetics gives

$$\nabla \times \vec{\mathrm{H}} = \frac{4\pi\rho\vec{u}}{(c)} + \frac{1}{(c)}\frac{\partial\vec{\mathrm{D}}}{\partial t}.$$

$$\therefore\ \rho\vec{u} = \frac{(c)}{4\pi}\left[\nabla \times \vec{\mathrm{H}} - \frac{\kappa}{(c)}\frac{\partial\vec{\mathrm{E}}}{\partial t}\right].$$

$$\therefore\ \mathrm{P} = \int \rho\vec{\mathrm{E}}\cdot\vec{u}\,dv = \frac{(c)}{4\pi}\int[\nabla \times \vec{\mathrm{H}}\cdot\vec{\mathrm{E}} - \frac{\kappa}{(c)}\vec{\mathrm{E}}\cdot\dot{\vec{\mathrm{E}}}]\,dv,$$

and since $\nabla\cdot\vec{\mathrm{E}} \times \vec{\mathrm{H}} = \vec{\mathrm{H}}\cdot\nabla \times \vec{\mathrm{E}} - \vec{\mathrm{E}}\cdot\nabla \times \vec{\mathrm{H}}$

the above equation may be written

$$\mathrm{P} = \frac{(c)}{4\pi}\int[\vec{\mathrm{H}}\cdot\nabla \times \vec{\mathrm{E}} - \nabla\cdot\vec{\mathrm{E}} \times \vec{\mathrm{H}} - \frac{\kappa}{c}\vec{\mathrm{E}}\cdot\dot{\vec{\mathrm{E}}}]dv.$$

Using Maxwell's second law of electromagnetics, viz.

$$\nabla \times \vec{\mathrm{E}} = -\frac{\mu}{(c)}\dot{\vec{\mathrm{H}}},$$

we have

$$\mathrm{P} = -\frac{(c)}{4\pi}\int\left\{\frac{\mu}{(c)}\vec{\mathrm{H}}\cdot\dot{\vec{\mathrm{H}}} + \nabla\cdot\vec{\mathrm{E}} \times \vec{\mathrm{H}} + \frac{\kappa}{(c)}\vec{\mathrm{E}}\cdot\dot{\vec{\mathrm{E}}}\right\}dv$$

$$= -\frac{(c)}{4\pi}\int\left\{\frac{1}{(c)}\vec{\mathrm{B}}\cdot\dot{\vec{\mathrm{H}}} + \frac{1}{(c)}\vec{\mathrm{D}}\cdot\dot{\vec{\mathrm{E}}}\right\}dv - \frac{(c)}{4\pi}\int[\vec{\mathrm{E}} \times \vec{\mathrm{H}}]\cdot\hat{n}\,d\mathrm{S}$$

$$= -\frac{\partial}{\partial t}\int\frac{\vec{\mathrm{B}}\cdot\vec{\mathrm{H}} + \vec{\mathrm{D}}\cdot\vec{\mathrm{E}}}{8\pi}dv - \frac{(c)}{4\pi}\int[\vec{\mathrm{E}} \times \vec{\mathrm{H}}]\cdot\hat{n}\,d\mathrm{S},$$

because, in general, if $\vec{\mathrm{P}}$ and $\vec{\mathrm{Q}}$ are two vectors which vary with time,

$$\delta(\vec{\mathrm{Q}}\cdot\vec{\mathrm{R}}) = (\vec{\mathrm{Q}} + \delta\vec{\mathrm{Q}})\cdot(\vec{\mathrm{R}} + \delta\vec{\mathrm{R}}) - \vec{\mathrm{Q}}\cdot\vec{\mathrm{R}}$$

$$= \delta\vec{\mathrm{Q}}\cdot\vec{\mathrm{R}} + \vec{\mathrm{Q}}\cdot\delta\vec{\mathrm{R}} \quad \text{if } \delta\vec{\mathrm{Q}}\cdot\delta\vec{\mathrm{R}} \rightarrow 0.$$

Proceeding to the limit

$$\frac{\partial}{\partial t}(\vec{\mathrm{Q}}\cdot\vec{\mathrm{R}}) = \frac{\partial\vec{\mathrm{Q}}}{\partial t}\cdot\vec{\mathrm{R}} + \vec{\mathrm{Q}}\cdot\frac{\partial\vec{\mathrm{R}}}{\partial t} = \dot{\vec{\mathrm{Q}}}\cdot\vec{\mathrm{R}} + \vec{\mathrm{Q}}\cdot\dot{\vec{\mathrm{R}}}.$$

When $\vec{\mathrm{B}} = \mu\vec{\mathrm{H}}$, then $\vec{\mathrm{B}}\cdot\dot{\vec{\mathrm{H}}} = \mu\vec{\mathrm{H}}\cdot\dot{\vec{\mathrm{H}}} = \vec{\mathrm{H}}\cdot\mu\dot{\vec{\mathrm{H}}} = \vec{\mathrm{H}}\cdot\dot{\vec{\mathrm{B}}}$

$$\therefore\ \dot{\vec{\mathrm{B}}}\cdot\vec{\mathrm{H}} = \tfrac{1}{2}[\dot{\vec{\mathrm{B}}}\cdot\vec{\mathrm{H}} + \vec{\mathrm{B}}\cdot\dot{\vec{\mathrm{H}}}] = \tfrac{1}{2}\frac{\partial}{\partial t}(\vec{\mathrm{B}}\cdot\vec{\mathrm{H}}).$$

Thus it has been proved that

$$\mathrm{P} + \frac{(c)}{4\pi}\int(\vec{\mathrm{E}} \times \vec{\mathrm{H}})\cdot\hat{n}\,d\mathrm{S}$$

$$= -\frac{\partial}{\partial t}\int\frac{\vec{\mathrm{B}}\cdot\vec{\mathrm{H}} + \vec{\mathrm{D}}\cdot\vec{\mathrm{E}}}{8\pi}\,dv$$

= the rate at which the total energy stored in the medium is decreasing.

Thus $\frac{(c)}{4\pi}\int(\vec{\mathrm{E}} \times \vec{\mathrm{H}})\cdot\hat{n}\,d\mathrm{S}$ must be the rate at which energy is flowing outwards across S.

If $\int\vec{\sigma}\cdot\hat{n}\,d\mathrm{S}$ is the rate at which the energy flows outwards we cannot, strictly speaking, say

$$\vec{\sigma} = \frac{(c)}{4\pi}[\vec{\mathrm{E}} \times \vec{\mathrm{H}}],$$

i.e. $\vec{\sigma}$ is not uniquely defined, for, without affecting the result, we may add to the r.h.s. any vector whose flux over a closed surface is zero.

Hence one should write

$$\vec{\sigma} = \frac{(c)}{4\pi}[\vec{\mathrm{E}} \times \vec{\mathrm{H}}] + \vec{\mathrm{U}}$$

where $\nabla\cdot\vec{\mathrm{U}} = 0$. This arbitrariness in the expression for $\vec{\sigma}$, however, is seldom important in practice for in actual applications it is generally with closed surfaces that one has to deal, so that the arbitrary function $\vec{\mathrm{U}}$ automatically disappears. The vector $\vec{\sigma}$ given by

$$\vec{\sigma} = \frac{(c)}{4\pi}[\vec{\mathrm{E}} \times \vec{\mathrm{H}}],$$

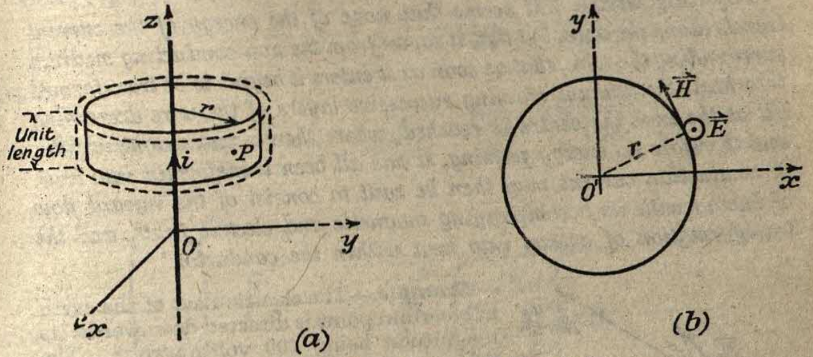
is known as ***Poynting's vector.***

The Joule effect in a wire carrying a current.—Suppose that the axis of the wire lies along the z-axis, is infinitely long, of radius r, and carries a current i (e.s.u.). Consider unit length of the wire and a cylindrical surface just outside the wire and with its axis along Oz, as indicated in Fig. 20·08 (a).

Then the magnetic field at any point P on this surface has magnitude given by $\mathrm{H} = \frac{2i}{(c)r}$, and it is directed along the tangent to the surface at P, being parallel to the plane $x\mathrm{O}y$ as indicated in

Fig. 20·08 (*b*). The components of the magnetic field are

$$H_x = -\frac{2i}{(c)r}\frac{y}{r},\quad H_y = \frac{2i}{(c)r}\frac{x}{r},\quad H_z = 0.$$



(*a*) (*b*)

FIG. 20·08.—The Joule effect in a wire.

Let ρ be the resistance (e.s.u.) per unit length of the wire. Then the electric field at the surface considered has components

$$E_x = 0,\quad E_y = 0,\quad E_z = \rho i.$$

$$\therefore\ \vec{\sigma} = \frac{(c)}{4\pi}[\vec{E}\times\vec{H}] = \frac{(c)}{4\pi}\begin{vmatrix} \hat{\imath}, & \hat{\jmath}, & \hat{k} \\ 0, & 0, & \rho i \\ -\frac{y}{r}, & \frac{x}{r}, & 0 \end{vmatrix}\frac{2i}{(c)r}$$

$$= \frac{i}{2\pi r}\left[\hat{\imath}\left(-\rho i\frac{x}{r}\right) + \hat{\jmath}\left(-\rho i\frac{y}{r}\right) + \hat{k}(0)\right]$$

$$= \frac{\rho i^2}{2\pi r}\left[-\frac{\hat{\imath}x + \hat{\jmath}y}{r}\right] = -\frac{\rho i^2}{2\pi r}\frac{\vec{r}}{r}.$$

$\therefore\ \vec{\sigma}$ is of magnitude $\frac{\rho i^2}{2\pi r}$ and is actually directed towards O′.

Hence the rate of passage of energy from the surrounding medium across the cylindrical surface is $\int \vec{\sigma}\cdot\hat{n}\, dS$, where $\hat{n}$ is a unit vector along the normal to δS and directed outwards from the region considered, i.e. $\hat{n} = -\frac{\vec{r}}{r}$.

The above integral is $\int -\frac{\rho i^2}{2\pi r}\frac{\vec{r}}{r}\left(-\frac{\vec{r}}{r}\right) dS$

$$= 2\pi r\frac{\rho i^2}{2\pi r} = \rho i^2 \text{ erg.sec.}^{-1}$$

In the electromagnetic system of units the above rate becomes

$$\rho_m(c)^2 \times \frac{i_m{}^2}{(c)^2} = \rho_m i_m{}^2 \text{ erg.sec.}^{-1}$$

Poynting writes, '*It seems that none of the energy of the current travels along the wire, but that it comes from the non-conducting medium surrounding the wire, that as soon as it enters it begins to be transformed into heat, the amount crossing successive layers of the wire decreasing till at the time the centre is reached, where there is no magnetic field, and therefore no energy passing, it has all been transformed into heat. A conduction current may then be said to consist of the inward flow of energy with its accompanying magnetic and electric fields, and the transformation of energy into heat within the conductor.*'

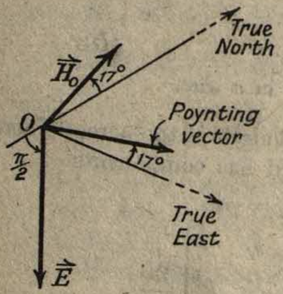


FIG. 20·09.—Illustration of Poynting's theorem.

Example.—The electric field of the earth at a certain point is directed downwards, its magnitude being 200 volt.metre.$^{-1}$. The horizontal component of the magnetic field is 0·30 oersted and is directed 17° to the west of true north. What is the Poynting vector?

The relative positions of $\vec{E}$, $\vec{H}$, true north and true east are shown in Fig. 20·09. Then

$$\vec{\sigma} = \frac{(c)}{4\pi}[\vec{E} \times \vec{H}]$$

$$= \left(\frac{3 \times 10^{10}}{4\pi} \times \frac{2}{300} \times 0{\cdot}3\right) \text{ erg.sec.}^{-1}\text{cm.}^{-2}$$

$$= 0{\cdot}48 \text{ watt.cm.}^{-2}$$

Its direction is 17° north of true east.

Example.—A circular loop of wire is placed in a uniform magnetic field. If the field is changing, find the energy flow into the wire.

It may be assumed that $\vec{B}$ is normal to the plane of the circular loop, since it is only the 'normal component' of $\vec{B}$ which is effective. Let a be the radius of the circle while a_0 is the radius of the cross-section of the wire—cf. Fig. 20·10 (*a*) and (*b*).

The flux of magnetic induction, Φ, through the loop is given by $\Phi = \pi a^2 B$; hence the induced e.m.f. e is given by $e = -\dot{\Phi}$.

$$\therefore\ i_m = \frac{e}{r_m} = -\frac{\pi a^2}{r_m}\frac{\partial B}{\partial t},$$

where r_m is the resistance (e.m.u.) of the complete loop of wire.

Hence at a point P on the surface of the wire, if this is thin compared with the dimensions of the loop, the magnetic field strength (e.m.u.) is

$$H_m = \frac{2i_m}{a_0}.$$

Also, $E_s = R_s i_s$, where R_s (e.s.u.) is the resistance per unit length.

∴ Energy per second entering the cylindrical surface of length δs is

$$2\pi a_0\,\delta s\,\frac{(c)}{4\pi}\left[\mathrm{R}_s i_s \times \frac{2i_m}{a_0}\right] = \mathrm{R}_m i_m^2\,\delta s.$$

∴ Energy per second entering the wire $= 2\pi a\,\mathrm{R}_m i_m^2$

$$= r_m i_m^2 = \frac{\pi a^4}{r_m}\left|\frac{\partial \mathrm{B}}{\partial t}\right|^2.$$

[Check:

$$\text{Rate of working} = e_m i_m = \frac{e_m^2}{2\pi a\,\mathrm{R}_m} = \frac{e_m^2}{r_m} = \frac{\pi a^4}{r_m}\left|\frac{\partial \mathrm{B}}{\partial t}\right|^2 \Big]$$

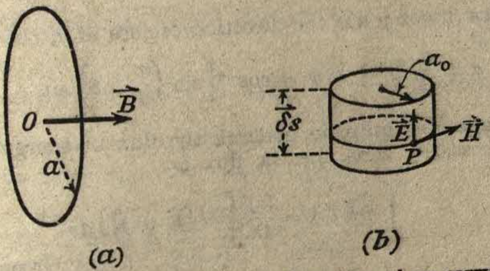


FIG. 20·10.—Flow of energy into a circular wire when carrying a current and situated in a magnetic field.

Example.—Calculate the Poynting vector in the region surrounding a uniformly charged sphere placed in a uniform magnetic field.

Without any loss of generality it may be assumed that $\vec{\mathrm{H}}$ is everywhere parallel to Ox, as in Fig. 20·11 (a). Let P be the point (r, θ, ϕ).

Let $\vec{\mathrm{E}} = \hat{r}\mathrm{E}$. Then, cf. Fig. 20·11 ($b$),

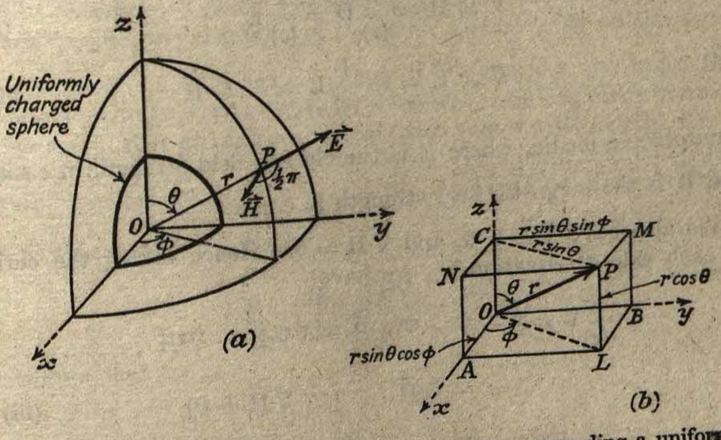


FIG. 20·11.—On the Poynting vector in the region surrounding a uniformly charged sphere when placed in a uniform magnetic field.

$$E_x = E \sin\theta \cos\phi$$
$$E_y = E \sin\theta \sin\phi$$
$$E_z = E \cos\theta.$$

Hence $$\vec{\sigma} = \frac{(c)}{4\pi}[\vec{E} \times \vec{H}]$$

$$= \frac{(c)}{4\pi}\begin{vmatrix} \hat{\imath}, & \hat{\jmath}, & \hat{k} \\ E \sin\theta\cos\phi, & E\sin\theta\sin\phi, & E\cos\theta \\ H, & 0, & 0 \end{vmatrix}$$

$$= \frac{(c)EH}{4\pi}[\hat{\imath}(0) + \hat{\jmath}(\cos\theta) - \hat{k}(\sin\theta\sin\phi)].$$

If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\vec{\sigma}$, then

$$\alpha = \frac{\pi}{2},\ \beta = 0,\ \gamma = \cos^{-1}\left[\sin\left(\frac{\pi}{2} + \theta\right)\sin\phi\right].$$

Now although $\vec{\sigma}$ is finite, in general, the flux of energy across any surface is necessarily zero, for the flux is

$$\int \vec{\sigma}\cdot\hat{n}\, dS = \frac{(c)}{4\pi}\int \nabla\cdot[\vec{E} \times \vec{H}]\, dv$$

[by Green's theorem]

$$= 0.$$ [Apart from an integration constant, which is of no consequence here.]

Therefore there is no flux of energy across any closed surface in the combined steady fields.

Electromagnetic waves.—If $\rho = 0$, Maxwell's electromagnetic equations become

$$\nabla \times \vec{H} = \frac{1}{(c)}\dot{\vec{D}} = \frac{\kappa}{(c)}\dot{\vec{E}}, \quad . \quad . \quad . \quad \text{(i)}$$

and

$$\nabla \times \vec{E} = -\frac{1}{(c)}\dot{\vec{B}} = -\frac{\mu}{(c)}\dot{\vec{H}}, \quad . \quad . \quad \text{(ii)}$$

for isotropic media, where $\vec{E}$ is the electric field strength in e.s.u. and $\vec{H}$ is the magnetic field strength in e.m.u.

We also have $\nabla\cdot\vec{E} = 0$, and $\nabla\cdot\vec{H} = 0$. Hence taking the curl of each side of equation (i), we have

$$\frac{\kappa}{(c)}\nabla \times \dot{\vec{E}} = \nabla \times \nabla \times \vec{H} = \nabla\nabla\cdot\vec{H} - \nabla^2\vec{H}$$

$$= -\nabla^2\vec{H}, \quad [\because \nabla\cdot\vec{H} = 0] \quad . \quad . \quad \text{(iii)}$$

where $$\nabla^2\vec{H} = \hat{\imath}\nabla^2 H_x + \hat{\jmath}\nabla^2 H_y + \hat{k}\nabla^2 H_z.$$ [cf. p. 725.]

From equation (iii) we have

$$\nabla^2\vec{\mathrm{H}} = -\frac{\kappa}{(c)}\nabla\times\dot{\vec{\mathrm{E}}} = -\frac{\kappa}{(c)}\frac{\partial}{\partial t}[\nabla\times\vec{\mathrm{E}}] = \frac{\mu\kappa}{(c)^2}\frac{\partial^2\vec{\mathrm{H}}}{\partial t^2} \qquad \text{(iv)}$$

Similarly,

$$\nabla^2\vec{\mathrm{E}} = \frac{\mu\kappa}{(c)^2}\frac{\partial^2\vec{\mathrm{E}}}{\partial t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

The equation of wave-motion for the propagation of electromagnetic fields in isotropic dielectrics.—Since $\frac{1}{\sqrt{\mu\kappa}}$ has the dimensions of velocity, we may write $\frac{\mu\kappa}{(c)^2} = \frac{1}{c_m{}^2}$, where c_m will be shown to be the velocity of the disturbance in a medium and it is convenient to write $c_m = u$. Accordingly

$$\nabla^2\vec{\mathrm{E}} = \frac{\mu\kappa}{(c)^2}\frac{\partial^2\vec{\mathrm{E}}}{\partial t^2} = \frac{1}{u^2}\frac{\partial^2\vec{\mathrm{E}}}{\partial t^2},$$

and

$$\nabla^2\vec{\mathrm{H}} = \frac{\mu\kappa}{(c)^2}\frac{\partial^2\vec{\mathrm{H}}}{\partial t^2} = \frac{1}{u^2}\frac{\partial^2\vec{\mathrm{H}}}{\partial t^2};$$

it must be remembered that

$$\nabla^2\vec{\mathrm{A}} = \hat{\imath}\nabla^2\mathrm{A}_x + \hat{\jmath}\nabla^2\mathrm{A}_y + \hat{k}\nabla^2\mathrm{A}_z. \qquad \text{[Cf. p. 725.]}$$

Of the tremendous manifold of phenomena described by either of these equations only the simplest case, which is that of a plane wave, will be considered. Now a plane wave is one for which any quantity which specifies the character of the wave depends only on the time and one coordinate in space. Thus, if we consider the vector $\vec{\mathrm{E}}$ associated with an electromagnetic plane wave with its plane normal to the z-axis, then at any given instant this vector will have the same value at all points in the plane considered, viz. a plane normal to the z-axis; mathematically this means the vanishing of all derivatives involving $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Since

$$\nabla\cdot\vec{\mathrm{E}} = \frac{\partial\mathrm{E}_x}{\partial x} + \frac{\partial\mathrm{E}_y}{\partial y} + \frac{\partial\mathrm{E}_z}{\partial z},$$

it follows that

$$\frac{\partial\mathrm{E}_z}{\partial z} = 0.$$

This means that the electric field can only have a constant component in the direction of the z-axis, but since we are dealing with a wave-motion, a superimposed field which has the same value at all points is of no interest. Accordingly we write $E_z = 0$.

In the same way $H_z = 0$; the vanishing of $\nabla \cdot \vec{E}$ and $\nabla \cdot \vec{H}$ implies the transversality of the plane wave.

From this discussion it follows that for a plane wave travelling with its wave-front parallel to the xOy plane, the equation

$$\nabla^2 \vec{E} = \frac{1}{u^2} \frac{\partial^2 \vec{E}}{\partial t^2},$$

gives

$$\nabla^2 E_x = \frac{1}{u^2} \frac{\partial^2 E_x}{\partial t^2}, \ \nabla^2 E_y = \frac{1}{u^2} \frac{\partial^2 E_y}{\partial t^2} \text{ and } \nabla^2 E_z = \frac{1}{u^2} \frac{\partial^2 E_z}{\partial t^2}.$$

The first of these equations gives

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 E_x}{\partial t^2},$$

or

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 E_x}{\partial t^2},$$

since E_x is independent of x and y. Similarly

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 E_y}{\partial t^2}.$$

Each of these equations is of the type

$$\frac{\partial^2 \xi}{\partial t^2} - u^2 \frac{\partial^2 \xi}{\partial z^2} = 0.$$

so that, cf. Vol. 1, p. 38, for a disturbance propagated with velocity u in the positive direction of the z-axis

$$\xi = \phi(z - ut).$$

The physical significance of this equation is that, at time $t = 0$, $\xi = \phi(z)$, where $\phi(z)$ is the ***wave profile***. Thus $\phi(z)$ may be regarded as an instantaneous photograph of the disturbance at $t = 0$. It is now assumed that the wave is propagated without change of shape, so that an instantaneous photograph taken at a later time, t, will be identical with that taken at $t = 0$, except that the profile will have moved forward a distance ut. If a new origin of coordinates is taken at the point $z = ut$ and if Z is the distance at time t of a point from this new origin, i.e. $z = Z + ut$, the wave profile would be represented by

$$\xi = \phi(Z),$$

i.e.

$$\xi = \phi(z - ut).$$

If $\omega = \dfrac{2\pi}{T}$ and $\beta = \dfrac{2\pi}{\lambda}$, where T and λ are respectively the period and the wavelength of a sinusoidal disturbance, we may write

$$\xi = f(\omega t - \beta z)$$
$$= \hat{A} \cos (\omega t - \beta z),$$

where $\hat{A}$ is the amplitude, i.e. the maximum value of the disturbance.

Hence for a plane wave advancing in the z-direction

$$E_x = \hat{A}_x \cos (\omega t - \beta z),\ E_y = \hat{A}_y \cos (\omega t - bz + \psi),\ E_z = 0.$$

This, cf. Vol. III, p. 12, is an elliptical vibration in the plane of the wave.

Now suppose that a plane wave is propagated with its wave-front normal to a fixed direction OR; let r be a distance measured from O along OR. Then for a sinusoidal displacement we have $\xi = \hat{A} \cos (\omega t - \beta r)$.

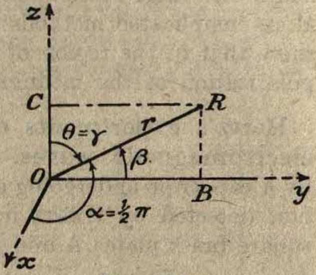


FIG. 20·12.

If, cf. Fig. 20·12, OR lies in the plane yOz and makes an angle θ with Oz, we have, if B and C are the projections of OR on Oy and Oz respectively,

$$r = \text{OB} \cos \beta + \text{BR} \cos \gamma \qquad [\alpha = \tfrac{1}{2}\pi,\ \beta \text{ and } \gamma \text{ define the d.c.s of OR.}]$$
$$= y \sin \theta + z \cos \theta$$
$$\therefore\ \xi = \hat{A} \cos [\omega t - (0x + y \sin \theta + z \cos \theta)],$$

and this equation will be required in the sequel.

The refractive index of a material medium.—It has just been shown that an electromagnetic disturbance moves with a velocity $u = \frac{(c)}{\sqrt{\mu\kappa}}$, through a medium with permittivity κ and permeability μ. Now for a vacuum $(\mu_0) = (\kappa_0) = 1$, so that (c) is the numerical value of the velocity of electromagnetic radiation (light) across empty space. Thus the significance of (c) has been revealed; earlier in this chapter, cf. p. 732, it was introduced as a conversion factor.

Now from the science of optics, the velocity of light, u, in a material medium is inversely proportional to its refractive index n, i.e. $u = \frac{c}{n}$, since c is the velocity in a medium of refractive index unity. But it has just been proved that

$$u = \frac{(c)}{\sqrt{\mu\kappa}} = \frac{c}{n}.$$

$$\therefore\ \frac{n}{\sqrt{\mu\kappa}} = \frac{c}{(c)} = 1 \text{ cm.sec.}^{-1}$$

Hence $n^2 = (\mu\kappa)$.

For most materials $\mu = 1$, so that $n^2 = (\kappa)$.

This relation is not always satisfied. For example, for water $(\kappa) = 80$, $n = \frac{4}{3}$. But if (κ) and n are measured *at the same frequency*, the relation is more nearly true. On the other hand, if the relation $n^2 = (\kappa)$ were true for very short waves, then such a fact would imply that the refractive index did not vary with the wavelength, i.e. there could be no such phenomenon as that of dispersion. More complicated methods than those which can be discussed here show that in the region of very high frequency fields, the electric polarization of the medium varies with frequency. [Cf. p. 762.]

Hertz's experiments on the generation and detection of electromagnetic waves.—The electromagnetic waves radiated by a capacitor undergoing an oscillatory discharge, cf. p. 480, were first detected by HERTZ in 1886. The vibrator consisted of two square brass plates A and B, Fig. 20·13 (*a*), to each of which was attached a stiff brass wire terminating in a gilded brass knob; approximate dimensions are indicated. The plates were connected to the terminals of an induction coil and when an oscillatory discharge took place between the knobs induced currents were set up in neighbouring conductors. To detect these currents Hertz used a simple resonator in the form of a circular wire, Fig. 20·13 (*b*), terminating in brass knobs S and separated by a small adjustable distance; the length of wire was carefully selected until the resonance sparking occurs between the knobs at S.

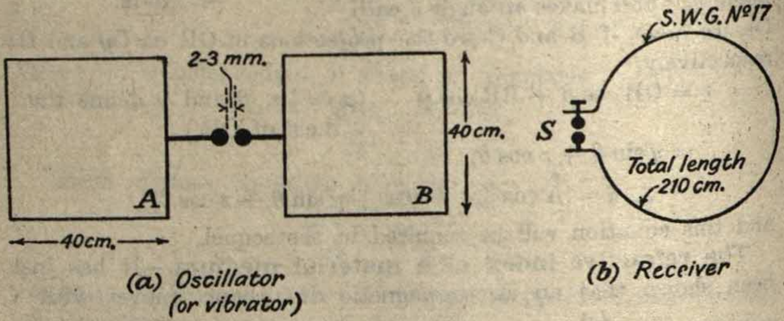


FIG. 20·13.—Hertz's first apparatus for the generation and detection of electromagnetic waves.

To observe the reflexion and interference of electromagnetic waves a large zinc plate, two metres square, was placed immediately behind the resonator; the sparking at S increased in brilliancy. When the metal plate was moved back a distance z, say, no sparking occurred at S even when the gap was considerably reduced in length. At a distance $2z$ the sparking was more vigorous than when the screen was removed. Hertz also observed that no sparking occurred at S when the zinc plate was placed before this gap; the plate then acted as a screen.

The interpretation of these observations is that the electromagnetic waves set up by the oscillator are reflected from the zinc plate, and when this is suitably placed interference between the direct and reflected waves takes place.

Hertz then modified the apparatus in order to examine the laws of reflexion and refraction for electromagnetic waves produced in this way. The essential features of the apparatus are shown in Fig. 20·14.

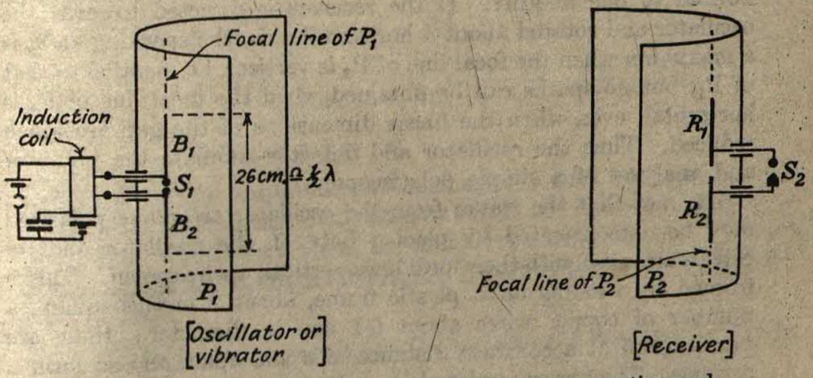


FIG. 20·14.—Hertz's second apparatus for the generation and detection of electromagnetic waves.

The electromagnetic waves were excited by a discharge across the gap S_1; this was at the centre of a system of two brass cylinders B_1 and B_2, each about 13 cm. long, 3 cm. in diameter and terminating in rounded knobs at the sparking ends. This gap S_1 lay on the focal line of a parabolic reflector P_1.

The receiver consisted of two wires R_1 and R_2, each about 15 cm. long and lying along the focal line of the metal paraboloid P_2. From R_1 and R_2 thin wires passed through the reflector to a spark gap S_2 which consisted of an upper brass sphere and a lower pointed copper rod. Any sparking which took place across S_2 could be observed without disturbing the waves falling upon the reflector. The waves had a wavelength of about one metre.

With this apparatus Hertz detected electromagnetic waves reflected from the walls of the room. He also constructed a large prism of pitch with an apical angle of 30°; experiment showed that the deviation of waves from the oscillator and passing through the prism had a minimum value of 22°; the material of the prism therefore had a refractive index of 1·69 for these long electromagnetic waves whereas its value in the visible region, as deduced from observations on the Brewsterian angle, cf. Vol. III, p. 719, varies between 1·5 and 1·6. The disparity between these values is not to be wondered at, for it would be very remarkable if the refractive index of a material did not vary as the wavelength of the radiations concerned cover the range from one metre to 5×10^{-5} cm.

In the wave proceeding from the oscillator the electric vector is parallel to the axis of the spark gap while the lines of magnetic induction are circular and in a plane normal to the above axis. The directions of $\vec{E}$ and $\vec{H}$ are therefore in the wave front and, of course, normal to one another. If the receiver is directed towards the oscillator and rotated about a horizontal axis, the sparking at S_2 is a maximum when the focal line of P_2 is vertical, i.e. parallel to that of P_1, but no sparks can be obtained when the focal line of P_2 is horizontal, even when the linear dimensions of the gap are much reduced. Thus the oscillator and detector simulate the polarizer and analyzer of a simple polariscope.

The fact that the waves from the oscillator are plane polarized may be demonstrated by placing between the oscillator and receiver, arranged with their focal lines vertical, a wire screen. This is formed by winding on a plastic frame, about 2 metres square, a number of copper wires about 0·1 cm. in diameter; these are parallel and at a constant distance of 3 cm. apart and so form a grating. The screen is placed in a plane normal to S_1S_2 and when the wires are parallel to the focal line of P_1 the screen completely intercepts the radiation from S_1. On the other hand, when the wires are horizontal the waves are transmitted freely and sparks appear across S_2. The system of wires behaves like a Nicol prism.

These classical experiments are so elegant and convincing that one almost forgets the formidable difficulties which had to be overcome before HERTZ and then LODGE and FITZGERALD brought them to a successful conclusion. In recent years the work of ESSEN has shown that visible radiations from caesium atoms may be detected by purely electrical methods; these experiments identify visible and long wavelength radiations as part of one continuous spectrum and provide us with the most accurate standard of time yet realized.

Plane polarized waves.—Let such a wave be propagated in the z-direction and suppose that $\vec{E}$ is parallel to Ox, i.e.

$$\vec{E} = \imath E_x + \jmath 0 + k0 ;$$

this in no way limits the generality of the analysis which follows. We have

$$E_x = \hat{E} \cos (\omega t - \beta z).$$

Also
$$\nabla \times \vec{\mathrm{E}} = -\frac{\mu}{(c)}\frac{\partial \vec{\mathrm{H}}}{\partial t}.$$

$$\therefore\ -\frac{\mu}{(c)}\frac{\partial \mathrm{H}_x}{\partial t} = \frac{\partial \mathrm{E}_z}{\partial y} - \frac{\partial \mathrm{E}_y}{\partial z} = 0, \qquad [\because\ \mathrm{E}_y = \mathrm{E}_z = 0]$$

$$-\frac{\mu}{(c)}\frac{\partial \mathrm{H}_y}{\partial t} = \frac{\partial \mathrm{E}_x}{\partial z} - \frac{\partial \mathrm{E}_z}{\partial x} = \hat{\mathrm{E}}\beta \sin(\omega t - \beta z),$$

and
$$-\frac{\mu}{(c)}\frac{\partial \mathrm{H}_z}{\partial t} = \frac{\partial \mathrm{E}_y}{\partial x} - \frac{\partial \mathrm{E}_x}{\partial y} = 0. \quad [\because\ \mathrm{E}_y = 0 \text{ and } \mathrm{E}_x \text{ is independent of } y.]$$

$\therefore\ \mathrm{H}_x = \mathrm{H}_z = 0$, and

$$\begin{aligned}
\mathrm{H}_y &= \frac{(c)}{\mu}\hat{\mathrm{E}}.\frac{\beta}{\omega}\cos(\omega t - \beta z)\\
&= \sqrt{\frac{\kappa}{\mu}}\hat{\mathrm{E}}\cos(\omega t - \beta z), \quad \left[\because\ \frac{\beta}{\omega} = \frac{\mathrm{T}}{\lambda} = \frac{1}{c_m}\right]\\
&= \hat{\mathrm{H}}\cos(\omega t - \beta z),
\end{aligned}$$

where $\hat{\mathrm{H}}$ is the amplitude of the magnetic field strength.

[A constant of integration could be added but this would imply the superimposition of a state of things independent of time. This can have no relation to wave-motion and so the constants of integration are omitted.]

Hence $\vec{\mathrm{H}}$ is always parallel to the y-axis, and since for all media in which light travels $\mu = \mu_0$, we have $\sqrt{\mu_0}\,\hat{\mathrm{H}} = \sqrt{\kappa}\,\hat{\mathrm{E}}$.

Thus the vectors $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$ lie in the plane of the wave and are mutually perpendicular; they are in the same phase and their amplitudes are numerically equal when the waves are propagated in free space. [In making this last statement it must not be forgotten that $\vec{\mathrm{E}}$ is expressed in e.s.u. whereas $\vec{\mathrm{H}}$ is expressed in e.m.u.]

Solar energy.—The amount of solar energy received by the earth is about 8 joule.min.$^{-1}$cm.$^{-2}$. What are the amplitudes of the mean electric and magnetic field strengths characteristic of this radiation?

The waves are plane and, by convention, fall normally on the unit area considered. Hence we have

$$\vec{\mathrm{E}} = \hat{\imath}\hat{\mathrm{E}}\cos(\omega t - \beta z) + \hat{\jmath}0 + \hat{k}0,$$

and
$$\vec{\mathrm{H}} = \hat{\imath}0 + \hat{\jmath}\hat{\mathrm{H}}\cos(\omega t - \beta z) + \hat{k}0,$$

so that the antaneous value of the Poynting vector is given by

$$\vec{\sigma} = \frac{(c)}{4\pi}[\vec{\mathrm{E}}_{\wedge}\vec{\mathrm{H}}] = \frac{(c)}{4\pi}\begin{bmatrix} \hat{\imath}, & \hat{\jmath}, & \hat{k}\\ \hat{\mathrm{E}}\cos(\omega t - \beta z), & 0, & 0\\ 0, & \hat{\mathrm{H}}\cos(\omega t - \beta z), & 0 \end{bmatrix}$$

$$= \frac{(c)}{4\pi}\hat{k}\,\{\hat{\mathrm{E}}\hat{\mathrm{H}}\}\cos^2(\omega t - \beta z).$$

∴ Flow of energy per second across unit area normal to Oz is

$$\vec{\sigma}\cdot\vec{k} = \frac{(c)}{4\pi}\{\hat{E}\hat{H}\}^2 \cos^2 (\omega t - \beta z).$$

Since, for propagation of electromagnetic waves in free space $|\hat{E}| = |\hat{H}| = \hat{A}$ say, the mean rate of flow of energy

$$= \frac{1}{T}\cdot\frac{(c)}{4\pi}\{\hat{A}\}^2 \int_0^T \cos^2 (\omega t - \beta z)\, dt$$

$$= \frac{(c)}{8\pi}\{\hat{A}\}^2.$$

$$\frac{(c)}{8\pi}\{\hat{A}\}^2 = \frac{8 \times 10^7}{60} \text{ erg.sec.}^{-1}\text{cm.}^{-2}$$

$$\therefore\ \hat{A} = \hat{E} = \frac{8}{3}\sqrt{\frac{\pi}{2}} \times 10^{-2} \text{ e.s.u.}$$

$$= 8\sqrt{\frac{\pi}{2}} \text{ volt-cm.}^{-1} = 10 \text{ volt.cm.}^{-1}$$

$$\therefore\ \hat{H} = \frac{8}{3}\sqrt{\frac{\pi}{2}} \times 10^{-2} \text{ e.m.u.} = 3{\cdot}3 \times 10^{-2} \text{ oersted.}$$

The boundary conditions.—When a plane wave is incident upon the boundary between two dielectrics, the following well-known boundary conditions must be satisfied :—

(*a*) The tangential components of both $\vec{E}$ and $\vec{H}$ must be continuous.*

(*b*) The normal components of both $\vec{D}$ and $\vec{B}$ must be continuous.

If therefore xOy is the interface and suffixes (1) and (2) refer to the insulating media on either side of this plane, we have

$$E_{1x} = E_{2x},\ E_{1y} = E_{2y},\ \kappa_1 E_{1z} = \kappa_2 E_{2z},$$

and

$$H_{1x} = H_{2x},\ H_{1y} = H_{2y},\ \mu_1 H_{1z} = \mu_2 H_{2z},$$

where, in general, κ and μ are the permittivity and permeability respectively of the medium concerned.

Of these six equations only four are independent, since

$$-\frac{\mu}{(c)}\frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \quad \left[\because\ \nabla \times \vec{E} = -\frac{1}{(c)}\frac{\partial \vec{B}}{\partial t}\right]$$

i.e.

$$\mu_1 H_{1z} = \mu_2 H_{2z}.$$

---

[*] A quantity is said to be continuous when its values at two neighbouring points, [here one on each side of the boundary concerned], are equal.

Similarly,

$$\frac{\kappa}{(c)}\frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \qquad \left[\because\ \nabla \times \vec{H} = \frac{1}{(c)}\frac{\partial \vec{D}}{\partial t}\right]$$

i.e. $$\kappa_1 E_{1z} = \kappa_2 E_{2z}.$$

The refraction and reflexion of a plane polarized wave incident normally on a surface separating two dielectrics.—Let us consider a plane polarized wave travelling in the positive

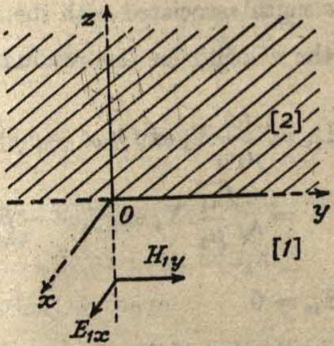


FIG. 20·15.—A plane polarized wave incident normally on a surface.

direction of the z-axis and, cf. Fig. 20·15, passing from a medium (1) to a medium (2). Let the electric vector for the incident wave be

$$\hat{\imath}E_{1x} + \hat{\jmath}0 + \hat{k}0,$$

and let $$E_{1x} = X_1 \cos(\omega t - \beta z)$$

$$= \text{the real part of } X_1 \exp j(\omega t - \beta z),$$

where X_1 is written for convenience for the amplitude instead of $\hat{E}$, and the other symbols have their usual meanings. Since, in general,

$$\nabla \times \vec{E} = -\frac{\mu}{(c)}\dot{\vec{H}},$$

we have

$$\frac{\partial E_{1z}}{\partial y} - \frac{\partial E_{1y}}{\partial z} = -\frac{\mu_1}{(c)}\frac{\partial H_{1x}}{\partial t} \quad . \quad . \quad . \quad \text{(i)}$$

$$\therefore\ H_{1x} = 0, \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

no constant being added for reasons already given.

Also

$$\frac{\partial E_{1x}}{\partial z} - \frac{\partial E_{1z}}{\partial x} = -\frac{\mu_1}{(c)}\frac{\partial H_{1y}}{\partial t}.$$

$$\therefore -j\beta_1 X_1 \exp j(\omega t - \beta_1 z) - 0 = -\frac{\mu}{(c)}\frac{\partial H_{1y}}{\partial t}.$$

$$\therefore \frac{\partial H_{1y}}{\partial t} = \frac{j(c)}{\mu_1}\beta_1 X_1 \exp j(\omega t - \beta_1 z).$$

$$\therefore H_{1y} = \frac{(c)}{\mu_1 \omega}\beta_1 X_1 \exp j(\omega t - \beta_1 z). \quad . \quad . \quad \text{(iii)}$$

But
$$\frac{(c)}{\mu_1\omega}\beta_1 = \frac{(c)}{\mu_1}.\frac{T}{\lambda_1} = \frac{(c)}{\mu_1 c_1},$$
where λ_1 is the wavelength associated with the disturbance in the first medium; c_1 is the velocity for this medium, viz. $\frac{(c)}{\sqrt{\mu_1\kappa_1}}$.

$$\therefore H_{1y} = \frac{(c)}{\mu_1 c_1}X_1 \exp j(\omega t - \beta_1 z) \quad . \quad . \quad \text{(iv)}$$

$$= \sqrt{\frac{\kappa_1}{\mu_1}}.X_1 \exp j(\omega t - \beta_1 z),$$

and finally

$$H_{1z} = 0.$$

Equations (i)–(iv) imply that if the electric vector, $\vec{E}$, is normal to the plane yOz, then the magnetic vector, $\vec{H}$, is normal to the plane xOz.

For the refracted wave

$$E_{2x} = X_2 \exp j(\omega t - \beta_2 z),\ E_{2y} = 0,\ E_{2z} = 0,$$

while for the reflected wave

$$E_{3x} = X_3 \exp j(\omega t + \beta_1 z),\ E_{3y} = 0,\ E_{3z} = 0,$$

since the wavelength remains λ_1, but the wave travels in the negative direction of the z-axis.

$$\therefore H_{2y} = \sqrt{\frac{\kappa_2}{\mu_2}}X_2 \exp j(\omega t - \beta_2 z),$$

and
$$H_{3z} = \sqrt{\frac{\kappa_1}{\mu_1}}X_3 \exp j(\omega t + \beta_1 z),$$

and other components of the magnetic field strengths concerned being each zero.

Now the magnetic and electric field strengths in the second medium are due solely to the waves in that medium; in the first medium, however, the effects due to the incident and reflected waves are superimposed. At the interface, therefore,

$$E_{1x} + E_{3x} = E_{2x};\ H_{1y} + H_{3y} = H_{2y}.$$

By considering the expressions for the components of the various field strengths at $z = 0$, we find

$$X_1 + X_2 = X_3, \text{ and } \sqrt{\frac{\kappa_1}{\mu_1}}X_1 - \sqrt{\frac{\kappa_1}{\mu_1}}X_3 = \sqrt{\frac{\kappa_2}{\mu_2}}X_2.$$

But, in general, the velocity c_m with which an electromagnetic disturbance travels in a medium is given by $c_m = \frac{(c)}{\sqrt{\mu\kappa}} = \frac{c}{n}$, where n is the absolute refractive index of the medium Since,.

$$n = \frac{c}{c_m} = \frac{c}{(c)}\sqrt{\mu\kappa} \text{ and } \sqrt{\frac{\kappa}{\mu}} = \frac{\sqrt{\mu\kappa}}{\mu},$$

we have

$$\frac{(c)}{c}\frac{n_1}{\mu_1}X_1 - \frac{(c)}{c}\frac{n_1}{\mu_1}X_3 = \frac{(c)}{c}\frac{n_2}{\mu_2}X_2.$$

Now for all media through which light travels we may write $\mu_1 = \mu_2 = 1$, so that

$$n_1X_1 - n_1X_3 = n_2X_2.$$

From these equations we have

$$X_2 = \frac{2n_1}{n_1 + n_2}.X_1 = \frac{2}{1+n}.X_1,$$

and

$$X_3 = -\frac{n-1}{n+1}.X_1,$$

where $n = \frac{n_2}{n_1}$, the relative refractive index of the two media.

If $n > 1$, $\frac{X_3}{X_1} < 0$, i.e. for normal reflexion there is a change in sign equivalent to a change in phase of π. This occurs when the first medium is less dense (optically) than in the second medium.

Reflecting power.—This is equal to the ratio of the intensities of the reflected and incident waves, or to the square of the ratio of their amplitudes. Thus, if R is the reflecting power,

$$R = \left(\frac{X_3}{X_1}\right)^2 = \left(\frac{n-1}{n+1}\right)^2,$$

and this is independent of whether we deal with the electric or magnetic quantities concerned.

Moreover, if we put $n = n^{-1}$, the above expression for R is not altered so that the proportion of light reflected from a surface is independent of whether the waves pass from (1) to (2) or vice-versa. It is on account of this fact that the minima in fringes formed by reflexion from thin films are perfectly black when the light is monochromatic.

As $n_1 \rightarrow n_2$, i.e. $n \rightarrow 1$, so does the reflecting power become vanishingly small.

The refraction and reflexion of a plane wave polarized in the plane of incidence (*i.e.* the electric vector is normal to the plane of incidence).—In physical optics, when it is stated that a wave is plane polarized it is implied that the vibrations are normal to the plane of incidence, which is the plane containing the 'ray' and the normal at the point of incidence. According to the electro magnetic theory, cf. p. 750, the two vectors $\vec{E}$ and $\vec{H}$ are mutually perpendicular so that either $\vec{E}$ or $\vec{H}$ may be set normal to the plane of incidence. If $\vec{E}$ is selected to be the appropriate vector, then the equations obtained are consistent with experimental results for waves polarized in the plane of incidence.

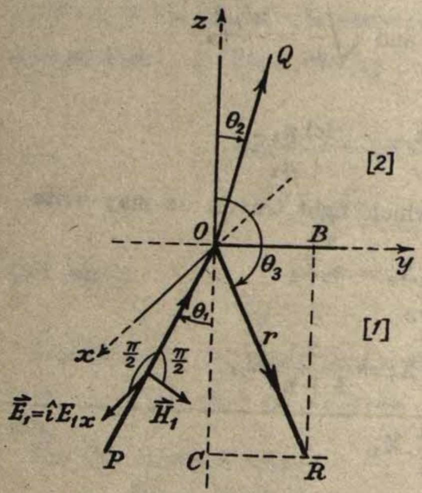


FIG. 20·16.—Plane wave with the electric vector normal to the plane of incidence.

Let us therefore consider a plane wave propagated along PO, Fig. 20·16, and for which $\vec{E} = \hat{\imath}E_x + \hat{\jmath}0 + \hat{k}0$, i.e. this vector is normal to the plane of incidence, which is the plane of the paper. If xOy is the surface of separation between two dielectric media (1) and (2), assumed to be loss-free, and θ_1 is the angle of incidence, we have

$$E_{1x} = X_1 \cos \{\omega t - \beta_1(x\,0 + y \sin \theta_1 + z \cos \theta_1)\}$$
$$\equiv X_1 \exp j\{\omega t - \beta_1(y \sin \theta_1 + z \cos \theta_1)\}.$$

Since
$$\nabla \times \vec{E} = -\frac{\mu}{(c)}\frac{\partial \vec{H}}{\partial t},$$
$$\frac{\partial E_{1z}}{\partial y} - \frac{\partial E_{1y}}{\partial z} = -\frac{\mu_1}{(c)}\frac{\partial H_{1x}}{\partial t},$$
and since E_{1z} and E_{1y} are each zero, we write, as usual,
$$H_{1x} = 0.$$
Also
$$\frac{\partial E_{1x}}{\partial z} - \frac{\partial E_{1z}}{\partial x} = -\frac{\mu_1}{(c)}\frac{\partial H_{1y}}{\partial t},$$

which yields

$$-j\beta_1 \cos\theta_1 \mathrm{X}_1 \exp j\{\ldots\} = -\frac{\mu_1 \partial \mathrm{H}_{1y}}{(c)\,\partial t},$$

i.e. $$\mathrm{H}_{1y} = \frac{(c)}{\mu_1}.\frac{\mathrm{X}_1}{c_1} \cos\theta_1 \exp j\{\ldots\}.$$

Similarly $$\mathrm{H}_{1z} = -\frac{(c)}{\mu_1}.\frac{\mathrm{X}_1}{c_1} \sin\theta_1 \exp j\{\ldots\}.$$

The simplest way in which we can hope to satisfy the boundary conditions of continuity is to make the y and z components of $\overleftrightarrow{\mathrm{E}}$ vanish for the refracted and reflected waves, as they do in the incident wave. The refracted wave will therefore be given by

$\mathrm{E}_{2x} = \mathrm{X}_2 \exp j\{\omega t - \beta_2(y \sin\theta_2 + z\cos\theta_2)\}$, $\mathrm{E}_{2y} = 0$, $\mathrm{E}_{2z} = 0$,

where θ_2 is the angle of refraction.

$$\therefore\ \mathrm{H}_{2x} = 0,\ \mathrm{H}_{2y} = \frac{(c)}{\mu_2}.\frac{\mathrm{X}_2}{c_2}\cos\theta_2.\exp j\{\ldots\},$$
$$\mathrm{H}_{2z} = -\frac{(c)}{\mu_2}.\frac{\mathrm{X}_2}{c_2}\sin\theta_2.\exp j\{\ldots\}.$$

If $\theta_3 = z\hat{\mathrm{O}}\mathrm{R}$, where OR is the reflected ray,

$\mathrm{E}_{3x} = \mathrm{X}_3 \exp j\{\omega t - \beta_1(y\sin\theta_3 + z\cos\theta_3)\}$, $\mathrm{E}_{3y} = 0$, $\mathrm{E}_{3z} = 0$,

since, with the notation indicated on the diagram,

$$r = \mathrm{OR} = \mathrm{OC}\cos(\pi - \theta_3) + \mathrm{OB}\cos(\tfrac{1}{2}\pi - \overline{\pi - \theta_3})$$
$$= x(0) + y\sin\theta_3 + z\cos\theta_3. \quad [\because\ \mathrm{OC} = -z]$$

$$\therefore\ \mathrm{H}_{3x} = 0,\ \mathrm{H}_{3y} = \frac{(c)}{\mu_1}.\frac{\mathrm{X}_3}{c_1}\cos\theta_3.\exp j\{\ldots\},$$
$$\mathrm{H}_{3z} = -\frac{(c)}{\mu_1}.\frac{\mathrm{X}_3}{c_1}\sin\theta_3.\exp j\{\ldots\}$$

Since the time factor is the same in all these expressions the boundary conditions are satisfied for all values of t, provided that at the interface, where $z = 0$, the same conditions are satisfied for all values of y; this means that the exponential factor in y must be common to each component of the vectors. Thus, cf. p. 754,

$$\beta_1 \sin\theta_1 = \beta_2 \sin\theta_2 = \beta_1 \sin\theta_3.$$

Hence $\sin\theta_1 = \sin\theta_3$, and since from the conditions of the problem, $\theta_1 \neq \theta_3$, we have $\theta_1 = \pi - \theta_3$. Also

$$\frac{\sin\theta_1}{\lambda_1} = \frac{\sin\theta_2}{\lambda_2},$$

i.e. $$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where n_1 and n_2 are the absolute refractive indices of the two media. This equation is Snell's second law of refraction.

Since $\theta_3 = \pi - \theta_1$, $\cos\theta_3 = -\cos\theta_1$, so that the boundary conditions will be satisfied if

$$\begin{aligned} \mathrm{X}_1 \exp j\{\omega t - \beta_1 y \sin\theta_1\} + \mathrm{X}_3 \exp j\{\omega t - \beta_1 y \sin\theta_3\} \\ = \mathrm{X}_2 \exp j\{\omega t - \beta_2 y \sin\theta_2\} \quad . \quad . \quad . \quad \text{(i)} \end{aligned}$$

or, since $\beta\sin\theta$ is an optical invariant,

$$\mathrm{X}_1 + \mathrm{X}_3 = \mathrm{X}_2 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Using the fact that the tangential components of $\vec{\mathrm{H}}$ are continuous at a boundary,

$$n_1 \cos\theta_1(\mathrm{X}_1 - \mathrm{X}_3) = n_2 \cos\theta_2 . \mathrm{X}_2 \quad . \quad . \quad \text{(iii)}$$

Eliminating X_2 from equations (i) and (ii) we find

$$\begin{aligned} \mathrm{X}_3 &= \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} . \mathrm{X}_1 \\ &= \frac{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1}{\sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1} . \mathrm{X}_1 . \end{aligned}$$

$$\therefore \frac{\mathrm{X}_3}{\mathrm{X}_1} = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 + \theta_1)} .$$

Similarly

$$\frac{\mathrm{X}_2}{\mathrm{X}_1} = \frac{2\tan\theta_2}{\tan\theta_2 + \tan\theta_1} .$$

The squares of the ratios $\dfrac{\mathrm{X}_3}{\mathrm{X}_1}$ and $\dfrac{\mathrm{X}_2}{\mathrm{X}_1}$ give the ratio of the intensities of the reflected and refracted rays in terms of that of the incident ray and agree satisfactorily with those found experimentally when the light vector is normal to the plane of incidence. Thus Fresnel's vector is the electric vector $\vec{\mathrm{E}}$; the plane of polarization contains the magnetic vector $\vec{\mathrm{H}}$.

[The conclusion that $\beta\sin\theta$ is an invariant seems intuitively obvious but a more satisfactory proof is as follows. Equation (i) may be written

$$\mathrm{Z}_1 \exp jy\beta_1\sin\theta_1 + \mathrm{Z}_3 \exp jy\beta_1\sin\theta_3 = \mathrm{Z}_2 \exp jy\beta_2\sin\theta_2, \quad \text{(iv)}$$

where the Z's are abbreviations for functions of z, t, etc. Let $\exp jy_0\beta_1\sin\theta_1 = \alpha_1$, $\exp jy_0\beta_2\sin\theta_2 = \alpha_2$, and $\exp jy_0\beta_1\sin\theta_3 = \alpha_3$, and put $y = 0$, $y = y_0$ and $y = 2y_0$ in equation (iv), giving

$$\mathrm{Z}_1 + \mathrm{Z}_3 = \mathrm{Z}_2, \quad . \quad . \quad . \quad . \quad \text{(v)}$$

$$\mathrm{Z}_1\alpha_1 + \mathrm{Z}_3\alpha_3 = \mathrm{Z}_2\alpha_2, \quad . \quad . \quad . \quad \text{(vi)}$$

and

$$\mathrm{Z}_1\alpha_1^2 + \mathrm{Z}_3\alpha_3^2 = \mathrm{Z}_2\alpha_2^2 \quad . \quad . \quad . \quad \text{(vii)}$$

Multiplying (v) by α_2 and subtracting the result from (vi), we get

$$\mathrm{Z}_1(\alpha_1 - \alpha_2) + \mathrm{Z}_3(\alpha_3 - \alpha_2) = 0 \quad . \quad . \quad . \quad \text{(viii)}$$

Similarly from (v) and (vii)

$$Z_1(\alpha_1^2 - \alpha_2^2) + Z_3(\alpha_3^2 - \alpha_2^2) = 0 \qquad . \quad . \quad \text{(ix)}$$

$$\therefore\ (\alpha_1 - \alpha_2)(\alpha_3^2 - \alpha_2^2) = (\alpha_1^2 - \alpha_2^2)(\alpha_3 - \alpha_2), \quad [Z_1 \neq 0, \ Z_3 \neq 0]$$

or

$$(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)(\alpha_3 + \alpha_2 - \alpha_1 - \alpha_2) = 0.$$

$$\therefore\ \alpha_1 = \alpha_2, \text{ or } \alpha_2 = \alpha_3, \text{ or } \alpha_3 = \alpha_1.$$

But if $\alpha_1 = \alpha_2$, then from (viii) $\alpha_2 = \alpha_3$, otherwise $Z_3 = 0$.

Hence

$$\alpha_1 = \alpha_2 = \alpha_3,$$

or

$$\beta_1 \sin\theta_1 = \beta_2 \sin\theta_2 = \beta_1 \sin\theta_3.$$

Strictly speaking, the coefficients of 'time' should be treated in the same way.]

The refraction and reflexion of a wave polarized in a plane normal to the plane of incidence (*i.e.* the magnetic vector is normal to the plane of incidence).—In this instance the magnetic vector associated with the wave is

$$\vec{\vec{H}}_1 = \hat{\imath}H_{1x} + \hat{\jmath}0 + \hat{k}0,$$

where, cf. Fig. 20·17,

$$H_{1x} = \hat{H}_1 \exp j\{\omega t - \beta_1(y \sin\theta_1 + z\cos\theta_1)\}.$$

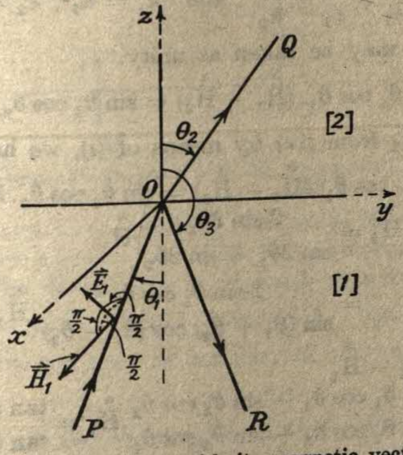


FIG. 20·17.—A plane wave with its magnetic vector normal to the plane of incidence.

Now

$$\nabla \times \vec{\vec{H}} = \frac{\kappa}{(c)}\frac{\partial \vec{\vec{E}}}{\partial t}.$$

$$\therefore\ \frac{\partial H_{1z}}{\partial y} - \frac{\partial H_{1y}}{\partial z} = 0 = \frac{\kappa_1}{(c)}\frac{\partial E_{1x}}{\partial t},$$

so that $E_{1x} = 0$. Also

$$\frac{\partial H_{1x}}{\partial z} - \frac{\partial H_{1z}}{\partial x} = \frac{\kappa_1}{(c)}\frac{\partial E_{1y}}{\partial t},$$

from which we obtain

$$E_{1y} = -\frac{(c)}{\kappa_1}\cdot\frac{\beta_1}{\omega}\cos\theta_1 . \hat{H}_1 \exp j\{\omega t - \beta_1(y\sin\theta_1 + z\cos\theta_1)\}$$

$$= -\sqrt{\frac{\mu_1}{\kappa_1}}\cos\theta_1 . \hat{H}_1 \exp j\{\omega t - \beta_1(y\sin\theta_1 + z\cos\theta_1)\}.$$

Similarly,

$$E_{1z} = \sqrt{\frac{\mu_1}{\kappa_1}}\sin\theta_1 . \hat{H}_1 \exp j\{\omega t - \beta_1(y\sin\theta_1 + z\cos\theta_1)\}.$$

Proceeding as before we find $\theta_1 = \pi - \theta_3$, and the boundary conditions are satisfied if $\hat{H}_1 + \hat{H}_3 = \hat{H}_2$. . . . (i)

$$\frac{n_1\cos\theta_1}{\kappa_1}.\hat{H}_1 - \frac{n_1\cos\theta_1}{\kappa_1}.\hat{H}_3 = \frac{n_2}{\kappa_2}\cos\theta_2 . \hat{H}_2, \quad . \quad \text{(ii)}$$

and

$$n_1\sin\theta_1 . (\hat{H}_1 + \hat{H}_3) = n_2\sin\theta_2 . \hat{H}_2 \quad . \quad . \quad \text{(iii)}$$

Equations (i) and (iii) are equivalent.

Now $\frac{\beta_1}{\beta_2} = \frac{\lambda_2}{\lambda_1} = \frac{c_2}{c_1} = \frac{n_1}{n_2}$, and $\kappa_1 = n_1{}^2$, $\kappa_2 = n_2{}^2$, since μ_1 and μ_2 may be taken as unity.

$$\therefore \sin\theta_1\cos\theta_1 . (\hat{H}_1 - \hat{H}_3) = \sin\theta_2\cos\theta_2 . \hat{H}_2 \quad . \quad \text{(iv)}$$

Eliminating $\hat{H}_3$ from (iv) by means of (i), we have

$$\sin\theta_1\cos\theta_1(2\hat{H}_1 - \hat{H}_2) = \sin\theta_2\cos\theta_2 . \hat{H}_2$$

$$\therefore \hat{H}_2 = \frac{2\sin 2\theta_1}{\sin 2\theta_1 + \sin 2\theta_2}.\hat{H}_1 \quad . \quad . \quad . \quad \text{(v)}$$

$$= \frac{2\sin\theta_1\cos\theta_1}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}.\hat{H}_1 \quad . \quad . \quad \text{(vi)}$$

Also

$$\hat{H}_3 = \hat{H}_2 - \hat{H}_1$$

$$= \frac{\sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2}{\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2}.\hat{H}_1 = \frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}.\hat{H}_1$$

Expressions for the ratio of the intensities are therefore easily written down.

When $\theta_1 + \theta_2 = \frac{1}{2}\pi$, the intensity of the reflected wave is zero. If θ_p is the value of θ_1 when this occurs, θ_p is known as the ***Brewsterian*** or ***polarizing angle***. When θ_1 exceeds θ_p, $\hat{H}_3$ becomes opposite to $\hat{H}_1$ so that there is a sudden change of phase equal to π at the polarizing angle.

The wave theory of the pressure due to electromagnetic radiation.—When a beam of electromagnetic radiation is incident on a metallic surface some of the radiation is absorbed and some reflected; eddy currents are set up in the surface and the magnetic field associated with the radiation exerts a force on these currents. In a similar way a pressure arises in a dielectric which absorbs electromagnetic radiations; it is due to the action between the magnetic field and the displacement currents. An indirect proof, first given by LARMOR, to obtain an expression for the pressure due to electromagnetic radiation is as follows.

When a wave-train is incident normally on an ideal reflector, let the displacements due to the incident and reflected waves be represented by

$$\xi_1 = \hat{a}_1 \cos(\omega_1 t + \beta_1 x) \quad \text{and} \quad \xi_2 = \hat{a}_2 \cos(\omega_2 t - \beta_2 x).$$

Thus the wave is advancing in the negative direction of the x-axis, as shown in Fig. 20·18. Let the reflector M be moving with constant velocity u in the positive direction of the x-axis; at time t its position is given by $x = ut$. Since the disturbance does not enter the reflector, the disturbance at M must be zero, i.e.

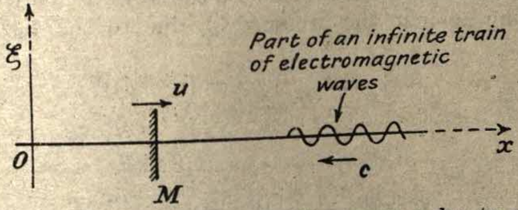


FIG. 20·18.—Larmor's theorem and the pressure due to radiation.

$$\hat{a}_1 \cos(\omega_1 t + \beta_1 ut) + \hat{a}_2 \cos(\omega_2 t - \beta_2 ut) = 0.$$

This condition can only be satisfied at all times if

$$\hat{a}_1 = -\hat{a}_2, \quad \text{and} \quad \omega_1 + \beta_1 u = \omega_2 - \beta_2 u.$$

If λ_1 and λ_2 are the wavelengths of the incident and reflected wave-trains respectively,.

$$\frac{\lambda_1}{\tau_1} = \frac{\omega_1}{\beta_1},$$

which is also $\frac{\omega_2}{\beta_2}$. This velocity may, for all practical purposes, be taken as c.

Hence

$$\beta_1(c + u) = \beta_2(c - u), \quad \text{or} \quad \lambda_2(c + u) = \lambda_1(c - u).$$

Hence, by reflexion at a moving mirror, the wavelength is shortened in the ratio $\frac{c - u}{c + u}$.

Now the average energy, $\overline{W}$, of a system of plane waves per unit area of a wave-front per unit length of the train may be shown, cf. Vol. IV, to be in general $\frac{1}{2}\rho_0(\hat{a}\omega)^2$, where ρ_0 is the undisturbed density of the medium ; the expression obtained measures the mean energy density of the wave system.

$$\therefore \frac{\overline{W}_1}{\overline{W}_2} = \left(\frac{\omega_1}{\omega_2}\right)^2 = \left(\frac{\lambda_2}{\lambda_1}\right)^2 = \left(\frac{c-u}{c+u}\right)^2.$$

Since, in unit time, a length $(c + u)$ of the incident wave-train moves up to the reflector, the energy arriving in unit time is $(c + u)\overline{W}_1$; that moving away is $(c - u)\overline{W}_2$. Thus the energy moving away exceeds that moving towards the reflector in the ratio

$$\frac{(c-u)\overline{W}_2}{(c+u)\overline{W}_1} = \frac{c+u}{c-u}.$$

Larmor attributed this increase to the work done by the reflector in advancing against the pressure, p, due to the electromagnetic radiation.

Hence
$$(c+u)\overline{W}_1 + pu = (c-u)\overline{W}_2$$
$$= \frac{(c+u)^2}{c-u}.\overline{W}_1.$$

$$\therefore p = 2.\frac{c+u}{c-u}.\overline{W}_1 = 2.\frac{c-u}{c+u}.\overline{W}_2.$$

$$\therefore \tfrac{1}{2}p\left(\frac{c-u}{c+u}\right) + \tfrac{1}{2}p\left(\frac{c+u}{c-u}\right) = \overline{W}_1 + \overline{W}_2,$$

i.e.
$$p = \frac{c^2-u^2}{c^2+u^2}(\overline{W}_1 + \overline{W}_2).$$

When u is zero, we have

$$p = \overline{W}_1 + \overline{W}_2,$$

i.e. the pressure on a stationary reflector is equal to the sum of the mean energy densities due to the incident and reflected wave-trains.

If a plane wave falls normally on a surface which absorbs all the incident radiation, $p = \overline{W}_1$.

The energy in light waves.—For a plane wave of light propagated parallel to the Oz direction and having its electric vector parallel to Ox,

$$E_x = \hat{E}\cos(\omega t - \beta z),\ E_y = 0,\ E_z = 0,$$

and
$$H_x = 0,\ H_y = \hat{H}\cos(\omega t - \beta z),\ H_z = 0,$$

for these satisfy the electromagnetic equations provided

$$\frac{\hat{H}}{\hat{E}} = \frac{\kappa u}{c} = \frac{c}{\mu u} = \sqrt{\frac{\kappa}{\mu}}, \quad . \quad . \quad . \quad \text{(i)}$$

where u is the velocity of propagation, cf. pp. 741–3. The total energy per unit volume is

$$\frac{1}{8\pi}[\kappa E_x^2 + \mu H_y^2] = \frac{1}{8\pi}[\kappa(\hat{E})^2 + \mu(\hat{H})^2]\cos^2(\omega t - \beta z),$$

and on account of equation (i) the density of the 'electric energy' is equal to that of the 'magnetic energy'. Also, since the mean value of $\cos^2\theta$ over a complete period is 0·5, the average energy per unit volume $= \dfrac{\kappa(\hat{E})^2}{8\pi} = \dfrac{\mu(\hat{H})^2}{8\pi}$.

Experimental work on the pressure due to radiation.—For electromagnetic radiation

$$\overline{W}_1 = \frac{\kappa(\hat{E})^2}{8\pi} = \frac{\mu(\hat{H})^2}{8\pi},$$

and for an ideal reflector $\overline{W}_2 = \overline{W}_1$. When radiation falls on such a surface

$$p = 2\overline{W}_1,$$

while for an ideal absorber the pressure is one half this value.

Early attempts to verify the existence of this pressure in a given instance were marred by the inability of the experimenters to separate the effect from the so-called 'radiometer effect'. Nicholls and Hull (1903) employed a reflecting surface so that the pressure is nearly twice that exerted on an ideally blackened surface while simultaneously the heating is reduced to a low value. The radiation fell upon a silver-coated vane and by suspending this in an enclosure where the pressure was about 16 mm. of mercury, it was maintained that the radiometer effect had been eliminated. Unfortunately they appear to have made the elementary mistake of assuming 1 gm.-wt. = 1,000 dynes, so that when the necessary correction is applied the measured pressure differs by ten per cent from that calculated from the measured energy densities in the incident and reflected beams. The most complete work on this subject is probably that of Mary Bell and Green (1933) who worked under the direction of Callendar. The essential basis of their work is as follows.

When a vane is suspended in an ideal vacuum its motion is given by

$$I\ddot{\theta} + b\theta = 0,$$

where I is the moment of inertia of the suspended system about its axis of rotation and b is the torsional constant of the fibre. A solution of this equation is

$$\theta = A\cos\left(\sqrt{\frac{b}{I}}.t - \alpha\right),$$

where A and α are constants. When radiation falls asymmetrically on the vane let it give rise to a constant couple G, so that

$$I\ddot{\theta} + b\theta = G.$$

Since the radiation exerts a uniform pressure, p, over a circular area whose centre is at a distance r from the axis of rotation, it is easily proved that $G = pr$; it is only necessary to consider the forces on elements of area symmetrically placed with reference to the vertical diameter of the circle. The solution of the above equation is

$$\theta = \frac{G}{b} + B\cos\left(\sqrt{\frac{b}{I}}.t - \beta\right),$$

where B and β are constants, and this equation represents an undamped harmonic motion about the mean displaced position $\frac{G}{b}$. In an ideal vacuum the vibrations of the vane are perfectly regular and the periodic time is constant. When the radiation is removed these workers found that the vane oscillated about its zero position.

When the vacuum is imperfect the time of swing is found to be irregular but at pressures below 10^{-6} mm. of mercury such irregularities disappeared and the radiometer effect was taken to be zero; the experiments then yielded the true radiation pressure.

The apparatus used is indicated in plan in Fig. 20·19 (*a*). The lens system L_1, L_2 projects a beam of light from a lamp L on to the vane V suspended in the flask F which, in turn, is connected to an efficient pumping system. [Actually the lens L_2 casts an image of the aperture of L_1 on to the vane.] The vane, cf. Fig. 20·19 (*b*), was suspended by means of a fine quartz fibre attached at its upper end to a collar which fitted into the neck of the flask. A concave mirror M, normal to the plane of the vane, enabled the deflexions of the latter to be observed by the lamp and scale method.

In all this work it was essential for the vane to be normal to the incident radiation and that its position in the flask should be such that no radiation was reflected back from the inner surface of the flask to the vane. The image produced in this way of the focused spot on the vane was readily seen and the vane arranged so that this image fell just below it.

If k is the reflecting power of the vane, and θ its mean angular deflexion, the pressure p due to the radiation is given by

$$(1 + k)pAr = b\theta,$$

and since

$$T = 2\pi\sqrt{\frac{I}{b}}$$

we have
$$p = \frac{2\pi^2 Id}{T^2 rA(1+k)D},$$
where d is the scale reading and D the distance of the scale from the suspension.

The reflecting power was determined in a manner similar to that used by NICHOLLS and HULL, a CALLENDAR radio-balance * being used to measure energy. The essential optical features of the method are shown in Fig. 20·19 (*c*). A concave mirror M_1 focused, on the radio-balance, R.B., an image of the 'spot' on the vane; the vane was then removed and the turntable rotated so that the radio-balance received the incident beam after reflexion from the concave mirror M_2.

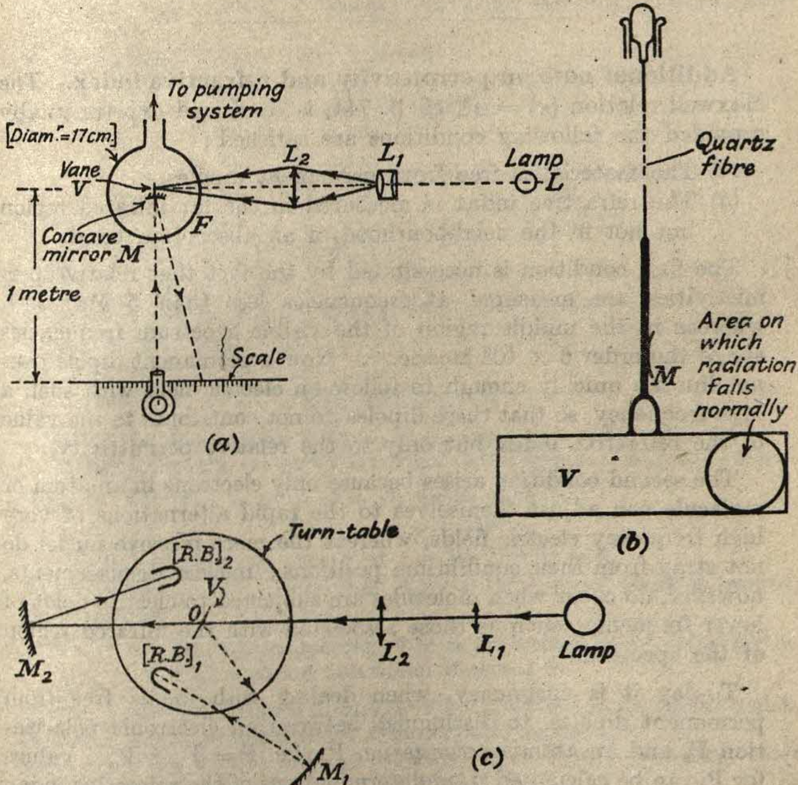


FIG. 20·19.—The pressure due to radiation.

The energy density of the beam used in the main part of the experiment was also measured with the aid of the radio-balance.

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* Cf. *Intermediate Physics*, Chap. XVII.

Bell and **Green** found

$$p = 20{\cdot}1 \times 10^{-6} \text{ dyne.cm.}^{-2},$$

while the total energy density of the incident and reflected beams was $18{\cdot}8 \times 10^{-6}$ erg.cm.$^{-3}$. Thus, with an error not exceeding seven per cent, these experiments establish in a quantitative manner the existence of the pressure of electromagnetic radiation and its equality with the energy density in the incident and reflected beams.

Additional note on permittivity and refractive index.—The Maxwell relation $(\kappa) = n^2$, cf. p. 744, is confirmed experimentally provided the following conditions are satisfied:

(*a*) The material is free from permanent dipoles.

(*b*) The refractive index is measured in the far infrared region but not in the neighbourhood of an absorption band.

The first condition is necessitated by the fact that relative permittivities are measured at frequencies less than 5 Mc.sec.$^{-1}$, whereas in the middle region of the visible spectrum frequencies are of the order 6×10^8 Mc.sec.$^{-1}$. Now a permanent dipole cannot line up quickly enough to follow an electric field with such a high frequency, so that these dipoles do not contribute to the value of the refractive index but only to the relative permittivity.

The second condition arises because only electrons in an atom or molecule can adjust themselves to the rapid alternations of very high frequency electric fields, whereas the more massive nuclei do not stray from their equilibrium positions; nuclear displacements, however, do occur when molecules are subjected to electric fields of lower frequency, such as those associated with the infrared region of the spectrum.

To-day it is customary, when dealing with media free from permanent dipoles, to distinguish between an electronic polarization P_e and an atomic polarization P_a, i.e. $P = P_e + P_a$. Values for P_e can be calculated from determinations of the refractive index in the visible region, while those for P_a can be derived from relative permittivity measurements or from infrared values of the refractive index. In general, it is found that $P_a < 0{\cdot}1\, P_e$.

Current density.—Let i be the current across an element of area δA, drawn at right angles to the direction of the current. The current density $\vec{i}$ is defined as $\dfrac{di}{dA}\hat{\imath}$, where $\hat{\imath}$ is a unit vector normal to δA. Thus if $\hat{n}$ is a unit vector normal to a surface δS, where $\delta S\, \hat{\imath}\cdot\hat{n} = \delta S \cos\theta = \delta A$,

$$\vec{i}\cdot\hat{n} = \frac{di}{dA}\,\hat{i}\cdot\hat{n} = \lim\left(\frac{\delta i}{\delta\mathrm{A} \div \cos\theta}\right) = \frac{di}{d\mathrm{S}}.$$

$$\therefore\ \delta i = \vec{i}\cdot\hat{n}\,\delta\mathrm{S}.$$

Hence the total current across a surface is given by

$$i = \int \vec{i}\cdot\hat{n}\,d\mathrm{S}.$$

If $\bar{\mathrm{N}}$ is the number of charged particles per unit volume, each with a charge q e.s.u. and velocity $\vec{u}$,

$$\delta i = \bar{\mathrm{N}}\frac{q}{(c)}\,\delta\mathrm{S}\,u\cos\theta.$$

$$\therefore\ \vec{i}\cdot\hat{n}\delta\mathrm{S} = \bar{\mathrm{N}}\frac{q}{(c)}u\delta\mathrm{S}\cos\theta = \bar{\mathrm{N}}\frac{q}{(c)}\vec{u}\cdot\hat{n}\,\delta\mathrm{S}.$$

$$\therefore\ \vec{i} = \bar{\mathrm{N}}\frac{q}{(c)}\vec{u} = \frac{\rho}{(c)}\vec{u},$$

where ρ e.s.u. is the density of the electrification.

EXAMPLES XX

20·1. (*a*) Prove that $\nabla\cdot\nabla(r^{-1}) = 0$, where $r^2 = x^2 + y^2 + z^2$.
(*b*) Show that the vector $\nabla\mathrm{V}$ is normal to the surface $\mathrm{V}(x, y, z) = \text{const.}$

20·2. (*a*) If $r^2 = x^2 + y^2 + z^2$, prove that

$$\nabla^2(r^n) = n(n+1)r^{n-2},$$

and hence show that, except at $r = 0$, $\nabla^2(r^{-1}) = 0$.

(*b*) If $\nabla^2\Phi(r) = 0$, prove that $\Phi(r) = a_1r^{-1} + a_2$, where a_1 and a_2 are constants.

20·3. Show that, in general, $r^{2\cdot n}r$ is an irrotational vector for all values of n, but becomes rotational if $n = 1{\cdot}5$.

20·4. Explain what is meant by $\nabla \times \vec{\mathrm{A}}$, where $\vec{\mathrm{A}}$ is a vector which, together with its derivatives, is finite and continuous.

Prove that $\nabla\cdot[\nabla\mathrm{V}_1 \times \nabla\mathrm{V}_2]$, where V_1 and V_2 are arbitrary scalar point functions of x, y, and z, is zero.

20·5. A current i (e.m.u.) flows in a long straight wire whose cross-section is a circle of radius a. Show that the energy associated with the magnetic field inside the wire is $\frac{1}{4}\mu i^2$ per unit length, where μ is the permeability of the material of the wire; this is assumed to be independent of the field strength. Hence show that the contribution to the self inductance arising from the energy stored inside the wire is $0{\cdot}5\mu$ per unit length.

20·6. The back of a plane transparent membrane is coated with an ideally opaque varnish. Both the material of the membrane and the varnish have an absolute refractive index n. Prove that if a beam of monochromate light of intensity I erg.sec.$^{-1}$cm.$^{-2}$ is incident normally on the exposed surface of the membrane, the pressure on the membrane is

$$2(1 + n^2)(1 + n)^{-2}Ic^{-1} \text{ dyne.cm.}^{-2},$$

where c is the velocity of light in a vacuum.

Verify that these units are correct.

20·7. Obtain an expression for the phase velocity of a plane sinusoidal electromagnetic wave in an isotropic dielectric. With the usual notation show that

$$\mu^{\frac{1}{2}}\overrightarrow{\mathrm{H}} = \kappa^{\frac{1}{2}}\overrightarrow{\mathrm{E}}.$$

20·8. Check the dimensions of the expression on p. 736 for the Poynting vector.

If the energy emitted by a 500 W. lamp is radiated uniformly, obtain a value for the electric field strength at a point in air and 250 cm. from the lamp.

[0·49 volt.cm.$^{-1}$.]

20·9. The earth receives from the sun energy at a rate of 8 × 10^4 joule.min.$^{-1}$metre^{-2}. Estimate the *maximum* values of $\overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{H}}$ in sunlight. Assuming that all the energy is absorbed, obtain a value for the *mean* pressure at normal incidence.

[10 volt.cm.$^{-1}$, 0·034 oersted, 4·5 × 10^{-5} dyne.cm.$^{-2}$.]

20·10. Standing waves are formed by two sets of progressive plane waves of the same amplitude and wavelength, but travelling in opposite directions. Two progressive electromagnetic waves in empty space are such that

$$(\mathrm{E}_y)_1 = \mathrm{A} \cos(\omega t - \beta x), \text{ and } (\mathrm{E}_y)_2 = \mathrm{A} \cos(\omega t + \beta x),$$

while $(\mathrm{E}_x)_1 = (\mathrm{E}_x)_2 = 0$, and $(\mathrm{E}_z)_1 = (\mathrm{E}_z)_2 = 0$. Prove that for the standing wave

$$\mathrm{E}_y = 2\mathrm{A} \cos \omega t \cos \beta x,$$

$$\mathrm{H}_z = \frac{(c)}{\mu_0}\cdot\frac{1}{c}\cdot 2\mathrm{A} \sin \omega t \sin \beta x.$$

Examine the dimensions of these equations.

Show that the mean flux of energy for the standing wave is zero.

20·11. Give a detailed account of the theory of the reflexion of plane electromagnetic waves incident obliquely on the interface between two isotropic dielectric media, which may be assumed to be loss-free. Describe and explain how you would investigate experimentally the validity of your theoretical investigation.

[Cf. Vol. III, Chap. XXIII.]

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